Problem Set 4

1. Find the nucleolus in 1 and 2 of Problem set 3.

2. Existence of the Nucleolus

   - (Problem) Show that for every $x \in A$, $\theta_i(x),\ i = 1, ..., 2^n - 2$ can be rewritten as

   $$\theta_i(x) = max_{\mathbf{S} \subseteq \mathbf{2^N} \setminus \{\mathbf{N}, \emptyset\}, |\mathbf{S}| = \mathbf{i}}(min_{s \in \mathbf{S}}\ e(S, x))$$

   - $e(S, x) = v(S) - \sum_{i \in S}\ x_i$ is a continuous function of x, since $\theta_i(x)$ is a max, min of a finite number of functions, θ_i is a continuous function of x.

   - Define

     $A^1 = \{x \in A | \theta_1(x) = min_{y \in A}\theta_1(y)\}$

     $A^i = \{x \in A^{i-1} | \theta_i(x) = min_{y \in A^{i-1}}\theta_i(y)\} \quad i = 2, ..., 2^n - 2$

     A is compact, and $\theta_1(x)$ is continuous; thus, a minimizer for θ_i exists, and $A^1 \neq \emptyset$ and A^1 is compact. By the same argument, since $\theta_2(x)$ is continuous $A^2 \neq \emptyset$ and A^2 is compact. Repeating this process, $A^{2^n-2} \neq \emptyset$.

     (Problem) Show that A^{2^n-2} coincides with the nucleolus.

3. Uniqueness of the nucleolus

   - Suppose that L has multiple imputations. (Show contradicition.).
     Pick two imputations $x, y \in L$ and let $z = (x + y)/2$.

   - Define $\theta(x), \theta(y)$ in the following manner.

     $\theta(x) = (\theta_1(x), ..., \theta_{2^n-2}(x)) = (e(S_1, x), ..., e(S_{2^n-2}, x))$

     $\theta(y) = (\theta_1(y), ..., \theta_{2^n-2}(y)) = (e(T_1, y), ..., e(T_{2^n-2}, y))$

     Since x, y belong to L, $\theta_i(x) = \theta_i(y)\ \forall i = 1, ..., 2^n - 2$.
     Let $\theta(z) = (\theta_1(z), ..., \theta_{2^n-2}(z))$.

   - (Problem) Show that if $S_i = T_i\ \forall i = 1, ..., 2^n - 2$, then $x = y$.

   - Since $x \neq y$,
     there exist $k, 0 \leq k \leq 2^n - 4$ such that
     $S_i = T_i\ \forall i = 1, ..., k,\ \ S_{k+1} \neq T_{k+1}$ Let k be as large as possible.

   - (Problem) Show that $\theta_i(x) = \theta_i(y) = \theta_i(z)\ \forall i = 1, ..., k$.

   - (Problem) Let $S = S_{k+1} = T_j$.
     Show that $j \geq k + 2$ and $\theta_{k+1}(y) > \theta_j(y)$.

   - (Problem) Show that $\theta_{k+1}(z) < \theta_{k+1}(x) (= \theta_{k+1}(y))$.

   - This contradicts $x, y \in L$. (QED)