# Problem Set 3

1. Calculate the core of the 3-person game below and draw a diagram of the core.
   $v(\{1,2,3\}) = 100$,
   $v(\{1,2\}) = 70$, $v(\{1,3\}) = 60$, $v(\{2,3\}) = 50$,
   $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

2. A,B, and C's houses locate in the same direction from Ookayama station. Suppose A,B, and C have agreed to share a cab and have the furthest person pay for the cab, while the other two people will reimburse that person the next day. The cab fare to A's house is 1200 yen, the cab fare to B's house is 2000 yen, and the cab fare to C's house is 2500 yen. Find the core.

3. Suppose players 1 and 2 each has one left glove, while players 3 and 4 each has one right glove. Each pair (a left glove and a right glove) of gloves is sold at 10,000 yen.
   (a) Formulate this situation as a characteristic function form game (coalitional form game).
   (b) Calculate the core, Shapley value, and the nucleolus.
   (c) Suppose now that there is another player, player 5, who has a left glove. How will the characteristic function change? What is the core, nucleolus, and the Shapley value of the new game?
   (d) Suppose now that there are $m$ players, each with a left glove, and $n$ players, each with a right glove. Find the core of this game.

4. A game $(N, v)$ with a characteristic function $v$ such that $v(S) = 0$ or 1 for all $S \subseteq N$ and $v(N) = 1$ is called a **simple game**. If $v$ satisfies
   (1) if $S \subseteq T$, $v(S) \leq v(T)$,
   (2) if $v(S) = 1$, then $v(N \setminus S) = 0$,
   is called a **voting game.** A coaliton $S$ such that $v(S) = 1$is called a winning coalition, and let $\mathbf{W}$ denote the set of winning coalitions. A player who is contained in every winning coalition is called a **veto player.** Show that in a voting game, the core is nonempty if and only if there exists one veto player. Moreover, if $V$ is the set of all veto players, then show that the core of the voting game is given by
   $$C = \{x \in A | x_i = 0 \ \ \forall i \in N \setminus V\}$$

5. Condition for the existence of the core (3-person games)

- $x = (x_1, x_2, x_3) \in C \Longleftrightarrow$
$$\begin{aligned} x_1 + x_2 + x_3 &= v(\{1,2,3\}) \\ x_1 + x_2 &\geq v(\{1,2\}) \\ x_1 + x_3 &\geq v(\{1,3\}) \\ x_2 + x_3 &\geq v(\{2,3\}) \\ x_1 &\geq v(\{1\}) \\ x_2 &\geq v(\{2\}) \\ x_3 &\geq v(\{3\}) \end{aligned}$$

- $C \neq \emptyset \Longleftrightarrow$
Minimization LP problem
$$x_1 + x_2 + x_3 \longrightarrow min$$
subject to
$$\begin{aligned} x_1 + x_2 &\geq v(\{1,2\}) \\ x_1 + x_3 &\geq v(\{1,3\}) \\ x_2 + x_3 &\geq v(\{2,3\}) \\ x_1 &\geq v(\{1\}) \\ x_2 &\geq v(\{2\}) \\ x_3 &\geq v(\{3\}) \end{aligned}$$
Min of this problem $\leq v(\{1,2,3\})$

- Dual problem
Let dual variables be $\gamma_{12}, \gamma_{13}, \gamma_{23}, \gamma_1, \gamma_2, \gamma_3$.
Consider the dual of the LP problem above.
$$\gamma_{12}v(\{1,2\}) + \gamma_{13}v(\{1,3\}) + \gamma_{23}v(\{2,3\}) + \gamma_1 v(\{1\}) + \gamma_2 v(\{2\}) + \gamma_3 v(\{3\}) \longrightarrow max$$
$$\begin{aligned} \gamma_{12} + \gamma_{13} + \gamma_1 &= 1 \\ \gamma_{12} + \gamma_{23} + \gamma_2 &= 1 \\ \gamma_{13} + \gamma_{23} + \gamma_3 &= 1 \\ \gamma_{12}, \gamma_{13}, \gamma_{23}, \gamma_1, \gamma_2 . \gamma_3 &\geq 0 \end{aligned}$$
Since both of the primal and dual problems are feasible, by the duality theorem both problems have optimal solutions and optimal values are equal. Therefore max of the dual problem $\leq v(\{1,2,3\})$ is a necessary and suufficient condition for the existence of the core.

- (Problem) Taking account of the fact above, show that $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \leq 2v(\{1,2,3\})$ is a necessary and sufficient condition for the existence of the core in 3-person superadditive games with $v(\{i\}) = 0 \ \forall i \in \{1,2,3\}$

6. Condition for the existence of the core (n-person games)

- Core $\quad C = \{x = (x_1, x_2, ..., x_n) \in A | \sum_{i \in S} x_i \geq v(S) \ \forall S \subset N, \ S \neq N, \emptyset\}$
where $A = \{x = (x_1, x_2, ..., x_n) \in \Re^n | \sum_{i \in N} x_i = v(N), \ x_i \geq v(\{i\}) \ \forall i \in N\}$

- LP problem

$\sum_{i\in N} x_i \ \rightarrow\ \min$
subject to $\sum_{i\in S} x_i \geq v(S)\ \forall S \subseteq N\ S \neq N, \emptyset$

- $C \neq \emptyset \iff$ min of the LP problem $\leq v(N)$
- Dual problem
  $\sum_{S\subset N, S\neq N,\emptyset} \gamma_S v(S) \ \rightarrow\ \max$
  subject to $\sum_{S: i\in S\subset N, S\neq N,\emptyset} \gamma_S = 1\ \forall i \in N$
  $\gamma_S \geq 0\ \forall S \subset N, S \neq N, \emptyset$
- $C \neq \emptyset \iff$ max of the dual problem $\leq v(N)$
- A family of coalitions $\{S_1, ..., S_m\}$ is a balanced $\Leftrightarrow$
  There exist positive real numbers $\gamma_1, ..., \gamma_m$ such that $\sum_{j: i\in S_j} \gamma_j S_j = 1\ \forall i \in N$
  $\gamma_1, ..., \gamma_m$ : weight
- $(N, v)$ is a balanced game $\iff$
  For any balanced family of coalitions $\{S_1, ..., S_m\}$ and for any of its weights $\gamma_1, ..., \gamma_m$
  $\sum_{j=1}^{m} \gamma_j v(S_j) \leq v(N)$
- $C \neq \emptyset \iff (N, v)$ is a balanced game.

7. Relationship between the core $C$ and the dominance core $DC$

- (Problem) Show $C \subseteq DC$.
- If $(N, v)$ is superadditive, then it can be shown that $DC \subseteq C$ as follows
  Take any imputation $x \in DC$, and suppose $x \notin C$. By definition of $C$, there exists $S \subset N$ such that $\sum_{i\in S} x_i < v(S)$. Let $\epsilon = v(S) - \sum_{i\in S} x_i > 0$ and define $y = (y_1, \cdots, y_n)$ with

$$y_i = \begin{cases} x_i + \frac{\epsilon}{|S|} & i \in S \\ v(\{i\}) + \frac{v(N) - v(S) - \sum_{i\in N\setminus S} v(\{i\})}{|N\setminus S|} & i \in N \setminus S \end{cases}$$

  where $|S|, |N \setminus S|$ denote the number of players in $S$ and $N \setminus S$.
  (Problem) Show that $y$ satisfies efficiency.
  (Problem) Show that if $(N, v)$ is superadditive, then $y$ satisfies individual rationality.
  Therefore, if $(N, v)$ is superadditive, then $y$ is an imputation.
  (Problem) Show that $y$ dominates $x$ via coalition $S$.
  which contradicts $x \in DC$.