Games in Characteristic Function Form

1. Chataceristic Function Form Games

   - $(N = \{1, 2, ..., n\}, v)$
     $N = \{1, 2, ..., n\}$: set of players
     $v : 2^N \to \Re$: characteristic function
     2^N: collection of subsets of N, $S \subseteq N$: coalition
   - $v(S)$: the maximum payoff that a coalition S can guarantee
   - (N, v) is a **superadditive** game $\Leftrightarrow$
     for every $S, T \subseteq N, S \cap T = \emptyset,$, $v(S) + v(T) \leq v(S \cup T)$
   - V^N:set of all superadditive functions on N
   - For $v, v' \in V^N$, v' is strategically equivalent to v if there exist $c > 0$ and $a_1, ..., a_n$ such that
     $$v'(S) = cv(S) + \sum_{i \in S} a_i$$
     Denote it by $v' \sim v$.
   - $\sim$ satisfies
     - (symmetry) For any $v \in V^N$, $v \sim V$.
     - (reflexivity) For any $v, v' \in V^N$, $v \sim v' \Leftrightarrow v' \sim v$.
     - (transitivity) For any $v, v', v'' \in V^N$, $v \sim v', v' \sim v'' \to v \sim v''$
   - By reflexivity, we simply say v and v' are strategically equivalent, in case $v \sim v'$.
   - For $v \in V^N$,
     $$v'(S) = v(S) - \sum_{i \in S} v(\{i\})$$
     is called 0-normalization ov v.

2. Imputation

   - $x = (x_1, x_2, ..., x_n)$: payoff vector
     payoff vector $x = (x_1, x_2, ..., x_n)$ is an **imputation** of (N, v) $\Leftrightarrow$
     $\sum_{i=1}^n x_i = v(N)$ (efficiency, group rationality)
     $x_i \geq v(\{i\})\ \forall i = 1, ..., n$ (individual rationality)
   - Set of imputations of (N, v), $A(v)$, can be expressed as
     $A(v) = \{x = (x_1, x_2, ..., x_n) \in \Re^n | \sum_{i=1}^n x_i = v(N), x_i \geq v(\{i\})\ \forall i = 1, ..., n\}$
   - Set of Imputations are not affected by strategic equivalence.
     Let $v'(S) = cv(S) + \sum_{i \in S} a_i\ \forall S \in N$. Then
     $$A(v') = \{y = (y_1, ..., y_n) \in \Re^n | y_i = cx_i + a_i\ \forall i \in N\ for some x \in A(v)\}$$
     is an imputation set of (N, v').

3. Examples

(a) Players 1, 2 and 3 earned 1 million JPY. A majority group (i.e. a coalition with two players or more) can get the whole amount.

(b) Players 1, 2 and 3 earned 1 million JPY. Player 1 has a veto. Thus a majority group with player 1 can get the whole amount.

(c) Each of three towns 1,2 and 3 plans to increase water supply. It will cost 140 million JPY for town 1, 160 million JPY for town 2 and 200million JPY for town 3. If neiboring towns 1 and 2 cooperate, the cost will be down to 240 million JPY. Similarly, the cost will be 280 million JPY for cooperation of town 2 and 3. Towns 1 and 3 cannot reduce the cost even if they cooperate because of the geographical reason. If all of the three towns cooperate, the total cost will be down to 300 million JPY.