Problem Set 1

1. For the following bargaining games, find the Nash bargaining solution by (a) solving the maximization problem and by (b) using *only* the four axioms.

   (a) R is the closed region inside the triangle with vertices $(0,0),(9,0),(0,6)$ and the disagreement point is $u^0 = (0,0)$

   (b) R is the closed region inside the triangle with vertices $(0,0),(9,0),(0,6)$ and the disagreement point is $u^0 = (3,2)$

   (c) R is the closed region inside the triangle with vertices $(0,0),(8,0),(0,8)$ and the disagreement point is $u^0 = (2,1)$

   (d) R is the closed region inside the quadrilateral with vertices $(0,0),(0,6),(6,3),(8,0)$ and the disagreement point is $(0,0)$

   (e) R is the closed region inside the quadrilateral with vertices $(0,0),(0,6),(6,3),(8,0)$ and the disagreement point is $(2,2)$

2. Proof of the Existence and Uniqueness of the Nash Bargaining Solution
   Nash's Theorem
   There is only one solution $f : B \to \Re^2$ that satisfies Pareto optimality, Symmetry, Preservation under Strictly Increasing Affine Transformation and Independence of Irrelevant Alternatives. Moreover, for any (R,u^0), $f(R,u^0)$ solves

$$max\{(u_1 - u_1^0)(u_2 - u_2^0)|(u_1,u_2) \in R, u_1 \geq u_1^0, u_2 \geq u_2^0\}$$

   and the solution (u_1,u_2) to the above maximization problem is unique. This f is called the Nash bargaining solution
   Let B be the set of bargaining problems (R,u^0) such that

   - R is a convex and compact subset of $\Re^2$
   - $u^0 = (u_1^0, u_2^0) \in R$.
   - There is a $(u_1,u_2) \in R$ such that $u_1 > u_1^0, u_2 > u_2^0$

   (Proof)

   (a) Let f be a function such that for each (R,u^0), $f(R,u^0)$ is the solution to the maximization problem above. To show that f above is well-defined as a function (i.e. $f(R,u^0)$ is single-valued for each (R,u^0))
   Let $H(u_1,u_2) = (u_1 - u_1^0)(u_2 - u_2^0)$ and let $R' = \{u \in R | u_1 \geq u_1^0, u_2 \geq u_2^0\}$
   Because R is compact, R' is also compact
   Because H is a continuous function on R', H attains a maximum on R'
   (Problem) Prove the following statements.

      i. If $s* = (s*_1, s*_2)$ is a maximizer for H on R', then $s*_1 > u_1^0$ and $s*_2 > u_2^0$

      ii. R' is convex

      iii. There is only one such $s* = (s*_1, s*_2)$; therefore f is a well-defined function
      (Hint)Suppose there is another maximizer $t* = (t*_1, t*_2)$ in R', that is different from $s*$; define $r* = (r*_1, r*_2) = ((s*_1 + t*_1)/2, (s*_2 + t*_2)/2)$
      Show that $H(r*_1, r*_2) > H(s*_1, s*_2)$ and $(r*_1, r*_2) \in R'$, which contradicts the maximality of $s*$

(b) (Problem) Show that f satisfies Pareto optimality, Symmetry, Preservation under Strictly Increasing Affine Transformation and Independence of Irrelevant Alternatives.

(c) To show that f is the unique solution that satisfies the four axioms:

Let $g : B \to \Re^2$ be another solution that satisfies Pareto optimality, Symmetry, Preservation under Strictly Increasing Affine Transformation, and Independence of Irrelevant Alternatives.

It is sufficient to show that for each(R, u^0), $f(R, u^0) = g(R, u^0)$

Take any (R, u^0) and let $u* = f(R, u^0)$

i. Consider the following affine transformation and let R' be the set of (u'_1, u'_2) defined below $((u_1, u_2) \in R)$

$$u'_1 = \frac{u_1}{2(u*_1 - u_1^0)} - \frac{u_1^0}{2(u*_1 - u_1^0)}$$
$$u'_2 = \frac{u_2}{2(u*_2 - u_2^0)} - \frac{u_2^0}{2(u*_2 - u_2^0)}$$

ii. (Problem) Show that under the transformation defined above,
- $(u*_1, u*_2)$ is transformed to $(1/2, 1/2)$
- (u_1^0, u_2^0) is transformed to $(0, 0)$

iii. Therefore, $f(R', (0, 0)) = (1/2, 1/2)$ and by axiom 3 (Preservation under Strictly Increasing Affine Transformation), it is sufficient to show $g(R', (0, 0)) = (1/2, 1/2)$

iv. For each $u' = (u'_1, u'_2) \in R'$ · it can be shown that $u'_1 + u'_2 \leq 1$ has to hold.
- Suppose $u'_1 + u'_2 > 1$ for some (u'_1, u'_2)
- For a small $\epsilon, 0 \leq \epsilon \leq 1$, consider $(1 - \epsilon)(1/2, 1/2) + \epsilon(u'_1, u'_2)$
- (Problem) Show that this point lies in R'
- (Problem) Show that for sufficiently small ϵ · the product of the two coordinates of this point exceed $1/4$
- This contradicts $f(R', u^0) = (1/2, 1/2)$.

v. Let T be any triangle that is symmetric with respect to the 45^0 line and contains R' and that $(1/2, 1/2)$ is Pareto optimal within T. Because R is bounded, such T must exist. By Pareto optimality and symmetry, $g(T, (0, 0)) = (1/2, 1/2)$. $R' \subseteq T$ and $(0, 0), (1/2, 1/2) \in R'$, which implies (by independence of irrelevant alternatives $g(R', (0, 0)) = (1/2, 1/2)$.

3. A Non-cooperative Approach to the Nash Bargaining Solution
(Refer to Problem 2 on page 68 of "Introduction to Game theory".)

(a) Bargaining Game

Feasible set $R = \{(u_1, u_2) \in \Re^2 | u_1 + u_2 \leq 100\}$

Disagreement point $u^0 = (u_1^0, u_2^0) = (0, 0)$

The Pareto optimality and the symmetry give the Nash bargaining solution $u^* = (u_1^*, u_2^*) = (50, 50)$

(b) Nash's idea

Players 1 and 2 simulataneously and independently announce their demands, $u_1, u_2 \geq 0$, respectively. If $u_1 + u_2 \leq 100$, they obtain their demands. If $u_1 + u_2 > 100$, then they get their payoffs at the disagreement point.

In this game, there are many Nash equilibria including the Nash bargaining solution.
(Problem) Find all Nash equilibria of this game.

(c) Rubinstein's idea

i. bargaining process

1st period

One player (player 1 in the following) offers two players' payoffs (u_1^1, u_2^1). Then player 2 decides whether to accept 1's offer or not. If player 2 accepts the offer, the game ends, and players 1 and 2 obtain u_1^1 and u_2^1, respectively. If player 2 rejects, they go into the 2nd period.

2ne period

Player 2 offers (u_1^2, u_2^2). If player 1 accepts the offer, the game ends, and players 1 and 2 obtain u_1^2 and u_2^2, respectively. If player 1 rejects, they go into the 3rd period.

3rd period

Player 1 offers (u_1^3, u_2^3). Then player 2 decides whether to accept 1's offer or not. Repeat this procedure until one of the players accepts another player's offer. Both players' payoffs are 0 when the game never ends. Introduce the discount factor δ, $0 < \delta < 1$, and consider the discounted payoffs. Thus the discounted payoffs when player 1 accepts the offer in the second period are $(\delta u_1^2, \delta u_2^2)$, and the payoffs when the game ends in the third period are $(\delta^2 u_1^3, \delta^2 u_2^3)$

ii. Claim

The subgame perfect equilibrium outcome of the game converges to the Nash bargaining solution $(50, 50)$ when $\delta \to 1$ converges to 1.

iii. (Proof)

Suppose there exists a subgame perfect equilibrium and let the payoffs of the two players be u_1^*, u_2^* The whole game and the subgame starting from the third period have the same structure except the payoffs. The payoffs of the subgame are given by the payoffs of the whole game multiplied by δ^2. Thus we focus on the subgame perfect equilibrium in which the equilibrium stategies of the whole game and the subgame starting from the third period are the same.

If the two players obtain u_1^* and u_2^* in the subgame starting from the third stage, their dicounted payofs in the whole game are $\delta^2 u_1^*, \delta^2 u_2^*$. Hereafter all the payoffs are discounted ones.

The subgame starting from the period 2:If player 1 accepts player 2's offer (u_1^2, u_2^2), then his payoff is δu_1^2; and if he rejects, they go into the third stage and his payoff is $\delta^2 u_1^*$. Thus if $\delta u_1^2 \geq \delta^2 u_1^*$, i.e., $u_1^2 \geq \delta u_1^*$, then player 1 accepts the offer: the minimum offer that player 1 accepts is δu_1^*. Therefore the maximum amount that player 2 can obtain is $100 - \delta u_1^*$.

If player 1 rejects, then they go into the third period; thus player 2 obtains $\delta^2 u_2^*$. Since u_2^* is at most $100 - u_1^*$, tha maximum payoff that player 2 can get when player 1 rejects his offer is $\delta^2(100 - u_1^*)$. Since $0 < \delta < 1$, we have $\delta(100 - \delta u_1^*) > \delta^2(100 - u_1^*)$. Thus in the subgame perfect equilibrium in the subgame starting from the second period, player 2 offers $(\delta u_1^*, 100 - \delta u_1^*)$, and player 1 accepts 2's offer.

The whole game：

(Problem) Read carefully the analysis above in the subgame starting from the second period; and show the following.

A. The minimum amount u_2^1 that player 2 accepts 1's offer is $\delta(100 - \delta u_1^*)$.
B. Show $u_1^* \to 5$.

4. An application : Negotiation between management and labor
A manager and a labor union negotiates about wages w and the number of employees k. The labor union represents K workers. Each worker earns w^0 when not employed by this firm. This firm produces $f(k)$ unist of the good when it employes k workers. One unit of the good is sold at a price p. $f(k)$ satisfies the following three properties.

- $f(k)$ is strictly concave. That is, for any s, t and any α, $0 < \alpha < 1$,
$$f(\alpha s + (1-\alpha)t) > \alpha f(s) + (1-\alpha) f(t).$$
- $f(0) = 0$.
- There exists $k, 1 \leq k \leq K$, such that $pf(k) > wk$.

For each pair (w, k), payoffs to the manager and the labor union are given by
$$u_1(w,k) = pf(k) - wk, u_2(w,k) = wk + w^0(K-k)$$
respectively. the disagreement point is, $(0, w^0 K)$.
The sum of the payoffs to the manager and the labor union is
$$u_1(w,k) + u_2(w,k) = pf(k) - wk + wk + w^0(K-k) = pf(k) + w^0(K-k)$$
and thus depends only on k.

(a) Find the Preto optimal k. That is, k maximizing the total profit of manager and labor union.
(b) Find the Nash bargaining solution.