Two-person Bargaining Game

1. **Two-person Strategic Form Game**

$$(N = \{1, 2\}, (S_1 = \{s_1, ..., s_m\}, S_2 = \{t_1, ..., t_n\}), (g_1, g_2))$$
$$g_1(s_i, t_j) = a_{ij}, g_2(s_i, t_j) = b_{ij}$$

   (a) Correlated Strategy
   $r = (r_{11}, ..., r_{mn}),\ \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij} = 1,\ r_{ij} \geq 0,\ i = 1, ..., m, j = 1, ..., n$
   r_{ij}: probability that (s_i, t_j) is chosen

   (b) Expected Payoff
   $u_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}r_{ij},\ u_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}r_{ij}$

   (c) Feasible Set
   $R = \{u = (u_1, u_2) | u_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}r_{ij},\ u_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}r_{ij}\}$

   (d) Disagreement Point
   $u^0 = (u_1^0, u_2^0)$
   (e.g. maximin value, minimax value, Nash equilibrium outcome)

2. **Bargaining Problem** (R, u^0)

   (a) R: a convex and compact (closed and bounded) subset of $\Re^2$ (two-dimensional Euclidean space)

   (b) $u^0 \in R$

   (c) there is a $u = (u_1, u_2) \in R$ such that $u_1 > u_1^0, u_2 > u_2^0$

   Denote by $\mathcal{B}$ the set of all bargaining problems (R, u^0)

   - R is *convex* $\Leftrightarrow$ for any $u, v \in R$ and for any $\alpha (0 \leq \alpha \leq 1)$, $\alpha u + (1 - \alpha)v \in R$
   - R is *bounded* $\Leftrightarrow$ there exists $M \in \Re_+$ such that for any $u = (u_1, u_2) \in R$, $-M \leq u_1, u_2 \leq M$
   - R is *closed* $\Leftrightarrow$ for any sequence $u^1, u^2, ... \in R$ such that $lim_{n\to\infty} = u$, $u \in R$.

3. **Nash Bargaining Solution**

   A function $f : \mathcal{B} \to \Re^2$ that satisfies the following four axioms:

   (a) (Strong) Pareto optimality
   For every $(R, u^0) \in B$
   $f(R, u^0) = (f(R, u^0)_1, f(R, u^0)_2)$ must be a strong Pareto optimal alternative in R.

   (Definition of Strong Pareto Optimality)
   $u = (u_1, u_2)$ is (strong) Pareto optimal in $R \Leftrightarrow$
   if there is a $u' \in R$ with $u'_1 \geq u_1, u'_2 \geq u_2$, then $u' = u$

(b) Symmetry

If (R, u^0) is symmetric then $f(R, u^0)_1 = f(R, u^0)_2$

(Definition of Symmetry for (R, u^0))
(R, u^0) is symmetric $\Leftrightarrow$
(1)if $(u_1, u_2) \in R$, then $(u_2, u_1) \in R$
(2)$u_1^0 = u_2^0$

(c) Independence of Strictly Positive Affine Transformation

For (R, u^0) define (R', u'^0) as follows

$R' = \{u' = (u'_1, u'_2)|u'_1 = \alpha_1 u_1 + \beta_1, u'_2 = \alpha_2 u_2 + \beta_2, u = (u_1, u_2) \in R\}$
$u'^0_1 = \alpha_1 u_1^0 + \beta_1,$
$u'^0_2 = \alpha_2 u_2^0 + \beta_2$
$\alpha_1 > 0, \alpha_2 > 0, \beta_1, \beta_2$ are constants

$f(R', u'^0)_1 = \alpha_1 f(R, u^0)_1 + \beta_1,$
$f(R', u'^0)_2 = \alpha_2 f(R, u^0)_2 + \beta_2$

(d) Independence of Irrelevant Alternatives

For (R, u^0) if there exists $T \subseteq R$ such that $f(R, u^0) \in T, u^0 \in T$, then $f(T, u^0) = f(R, u^0)$

4. **Existence and Uniqueness of Nash Bargaining Solution**

There exists a unique $f : \mathcal{B} \to \Re^2$ that satisfies the above four axioms. Moreover, for any bargaining problem $(R, u^0) \in \mathcal{B}$ $f(R, u^0)$ solves

$$max\{(u_1 - u_1^0)(u_2 - u_2^0)|(u_1, u_2) \in R, u_1 \geq u_1^0, u_2 \geq u_2^0\}$$

This f is the Nash bargaining solution.

5. **Non-cooperative game theoretic approaches to the Nash bargaining solution**

"Introduction to Game Theory p.68, Problem 2"

Two players 1 and 2 negotiate on how to share 1,000,000 JPY. Any sharing the total of which is less than or equal to 1,000,000 JPY is allowed; and if they do not reach any agreement, they obtain nothing.

(a) Bargaining game

Feasible set $R = \{(u_1, u_2) \in \Re^2 | u_1 + u_2 \leq 100,\ 0 \leq u_1, u_2\}$
disagreement point $u^0 = (u_1^0, u_2^0) = (0, 0)$ From the Pareto optimality and the symmetry, the Nash bargaining solution should be $u^* = (u_1^*, u_2^*) = (50, 50)$

(b) Nash's idea

Each of the players 1 and 2 announces his/her own demand $u_1, u_2 \geq 0$ simultaneously and independently. If $u_1 + u_2 \leq 100$, then each player obtains his/her own demand.

If $u_1 + u_2 > 100$, then each of them obtains 0.
In this non-cooperative game, there are many Nash equilibria including the Nash bargainng solution $(50, 50)$ (Refer to "Introduction to Game Theory, p.235")

(c) Rubinstein's idea
The 1st period：Player 1 proposes (u_1^1, u_2^1). If player 2 accepts the proposal, player 1 and 2 obtain u_1^1 and u_2^1 respectively, and the game ends. If he/she rejects, they go to the 2nd period.
The second period：Player 2 proposes (u_1^2, u_2^2). If player 1 accepts the proposal, player 1 and 2 obtain u_1^2 and u_2^2 respectively, and the game ends. If he/she rejects, they go to the 3rd period.
The third period：Player 1 proposes (u_1^3, u_2^3), player 2 decides whether to accept or not.
Repeat the prcedure until one of the players accepts another player's proposal.
Let each player's payoff be 0 in case the negotiation never ends. Introduce the discount factor δ, $0 < \delta < 1$, and consider the dicounted payoff. Thus the payoffs in the 2nd period are $(\delta u_1^2, \delta u_2^2)$, and the payoffs in the third stage are は $(\delta^2 u_1^3, \delta^2 u_2^3)$.
The subgame perfect equilibrium of the game converges to the Nash bargaining solution $(50, 50)$ when $\delta \to 1$.