Structural Dynamics
構造動力学
(13)

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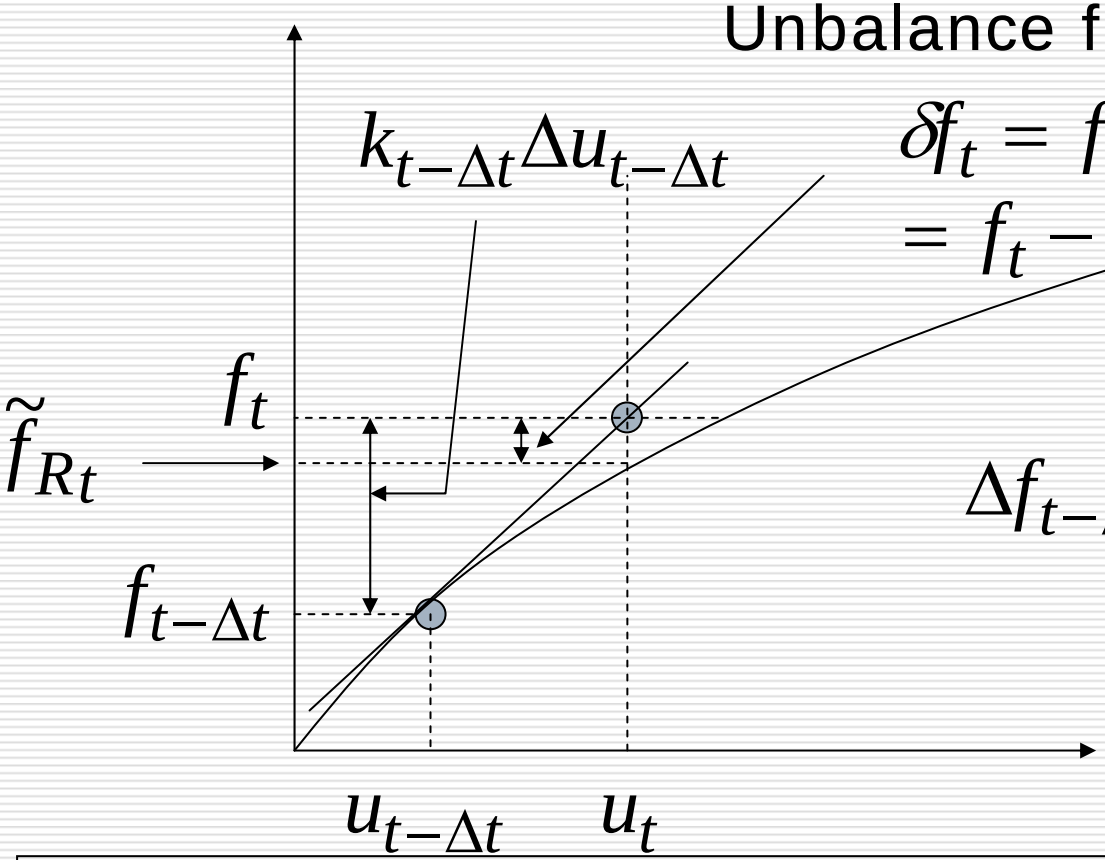
- α and β are determined as

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \frac{2\omega_i \omega_j}{\omega_j^2 - \omega_i^2} \begin{bmatrix} \omega_j & -\omega_i \\ -1/\omega_j & 1/\omega_i \end{bmatrix} \begin{Bmatrix} \xi_i \\ \xi_j \end{Bmatrix} \qquad (7.25)$$

7.4 Accuracy of Computed Responses (解析結果の精度)

●Approximation of the restoring force by Eq. (10.20) is sometimes insufficient to compute reliable response of a structure.



Unbalance force at time t

$$\delta f_t = f_t - \tilde{f}_{Rt}$$
$$= f_t - (m\ddot{u}_t + c\dot{u}_t + f_{Rt}) \quad (7.26)$$

$$\Delta f_{t-\Delta t} = f_t - f_{t-\Delta t}$$
$$= k_{t-\Delta t}\Delta u_{t-\Delta t}$$
$$= k_{t-\Delta t}(u_t - u_{t-\Delta t})$$

$$\{F_R(u_t)\} = \{F_R(u_{t-\Delta t})\} + [K_{t-\Delta t}]\{\Delta u_{t-\Delta t}\} \quad (7.20)$$

● By extending the unbalance force of SDOF system to that of MDOF system, accuracy of the computed response at time t is generally represented in terms of unbalance force (**不つり合い力**)

$$\{\delta F_t\} = \{F_t\} - ([M]\{\ddot{u}_t\} + [C]\{\dot{u}_t\} + \{F_{Rt}\}) \quad (7.27)$$

● Amount of the unbalance force can be represented in various ways. One of the expressions may be to define a ratio between the norm of the unbalance force and the norm of the external force as

$$\Delta_P = \frac{\|\delta F_t\|}{\|F_t\|} < \Delta_{PS} \quad (7.28)$$

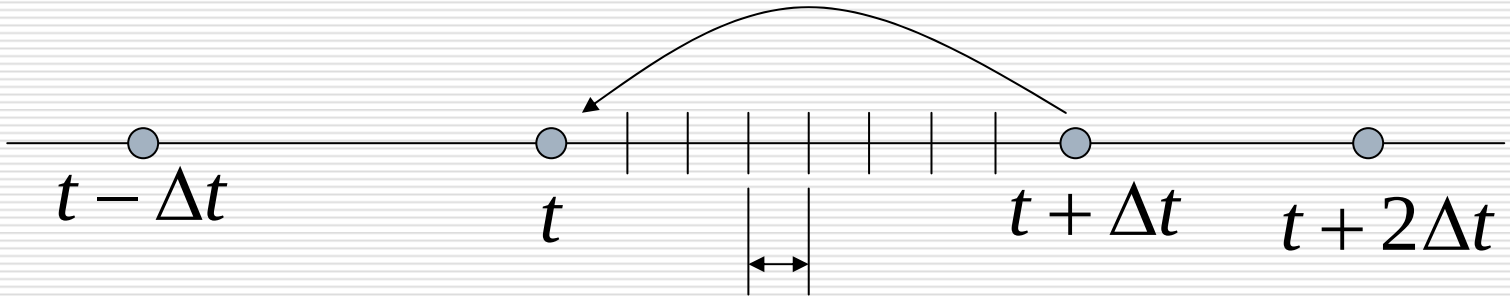
where Δ_{PS} is the threshold value (acceptable error)

$$\boxed{\delta f_t = f_t - \tilde{f}_{Rt} = f_t - (m\ddot{u}_t + c\dot{u}_t + f_{Rt}) \quad (7.26)}$$

7.5 Improvement of Accuracy of Solutions

1) Use of smaller time interval for numerical integration

- It is always effective to use smaller time interval for numerical integration although computer time required increases.
- If strong nonlinearity exists only at several time intervals, it is useful to subdivide time interval only where the accuracy by Eq. (7.27) is insufficient.



$$\Delta\tilde{t} = \frac{\Delta t}{n}$$

2) Add unbalance force to incremental external force at the next time step

- It is often used to add the unbalance force by Eq. (7.27) to the incremental load in the next time step. Rewriting Eq. (7.27) at time t, the unbalance force at time t is

$$\{\delta F_t\} = \{F_t\} - ([M]\{\ddot{u}_t\} + [C]\{\dot{u}_t\} + \{F_{Rt}\}) \qquad (7.29)$$

and adding this unbalanced force to the incremental load in Eq. (10.12), one obtains

$$[M]\{\Delta\ddot{u}_t\} + [C]\{\Delta\dot{u}_t\} + [K_t]\{\Delta u_t\} = \{\Delta F_t\} + \{\partial F_t\} \qquad (7.30)$$

$$\{\delta F_{t+\Delta t}\} = \{F_{t+\Delta t}\} - ([M]\{\ddot{u}_{t+\Delta t}\} + [C]\{\dot{u}_{t+\Delta t}\} + \{F_{Rt+\Delta t}\}) \qquad (7.27)$$

$$[M]\{\Delta\ddot{u}_t\} + [C]\{\Delta\dot{u}_t\} + [K_t]\{\Delta u_t\} = \{\Delta F_t\} \qquad (7.12)$$

- Based on Eq. (7.10), the right hand side of Eq. (7.30) becomes

$$\{\Delta F_t\} + \{\partial F_t\} = \{F_{t+\Delta t}\} - \{F_t\} + \{\partial F_t\}$$

$$= \{F_{t+\Delta t}\} - \{F_t\} - \{\{F_t\} - ([M]\{\ddot{u}_t\} + [C]\{\dot{u}_t\} + \{F_{Rt}\})\}$$

$$= \{F_{t+\Delta t}\} + ([M]\{\ddot{u}_t\} + [C]\{\dot{u}_t\} + \{F_{Rt}\})$$

- Thus, Eq. (7.30) becomes

$$[M]\{\Delta\ddot{u}_t\} + [C]\{\Delta\dot{u}_t\} + [K_t]\{\Delta u_t\}$$
$$= \{F_{t+\Delta t}\} + ([M]\{\ddot{u}_t\} + [C]\{\dot{u}_t\} + \{F_{Rt}\}) \quad (7.31)$$

$$\{\Delta F_t\} = \{F(t+\Delta t)\} - \{F(t)\} \quad (7.10)$$

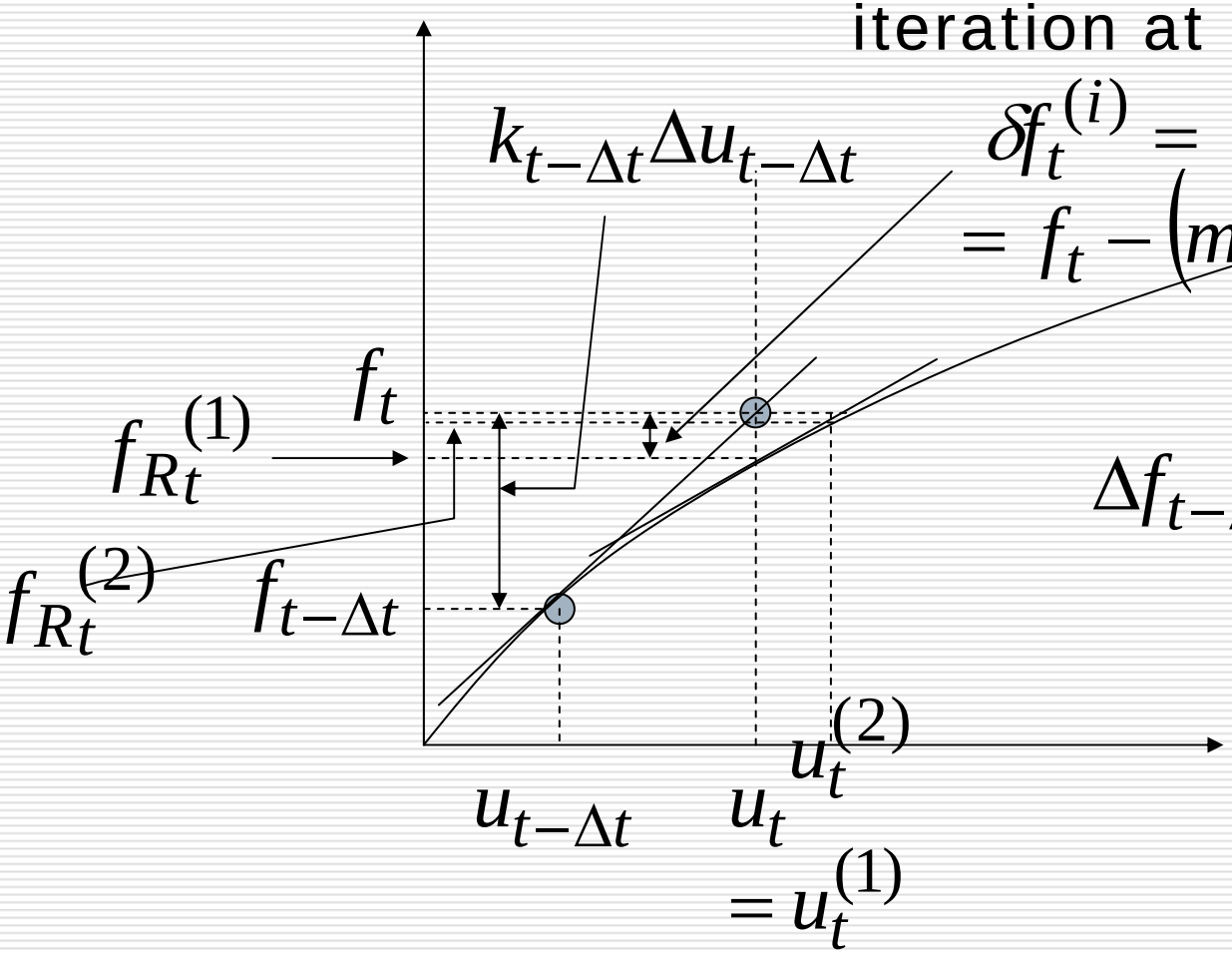
$$[M]\{\Delta\ddot{u}_t\} + [C]\{\Delta\dot{u}_t\} + [K_t]\{\Delta u_t\} = \{\Delta F_t\} + \{\partial F_t\} \quad (7.30)$$

- By adding unbalanced force to the incremental force, accuracy of the solution of Eq. (7.12) is generally improved.

3) Numerical Iteration

Unbalance force for the i-th iteration at time t

$$\delta f_t^{(i)} = f_t - \tilde{f}_{Rt}^{(i)}$$
$$= f_t - \left(m\ddot{u}_t^{(i)} + c\dot{u}_t^{(i)} + f_{Rt}^{(i)}\right)$$



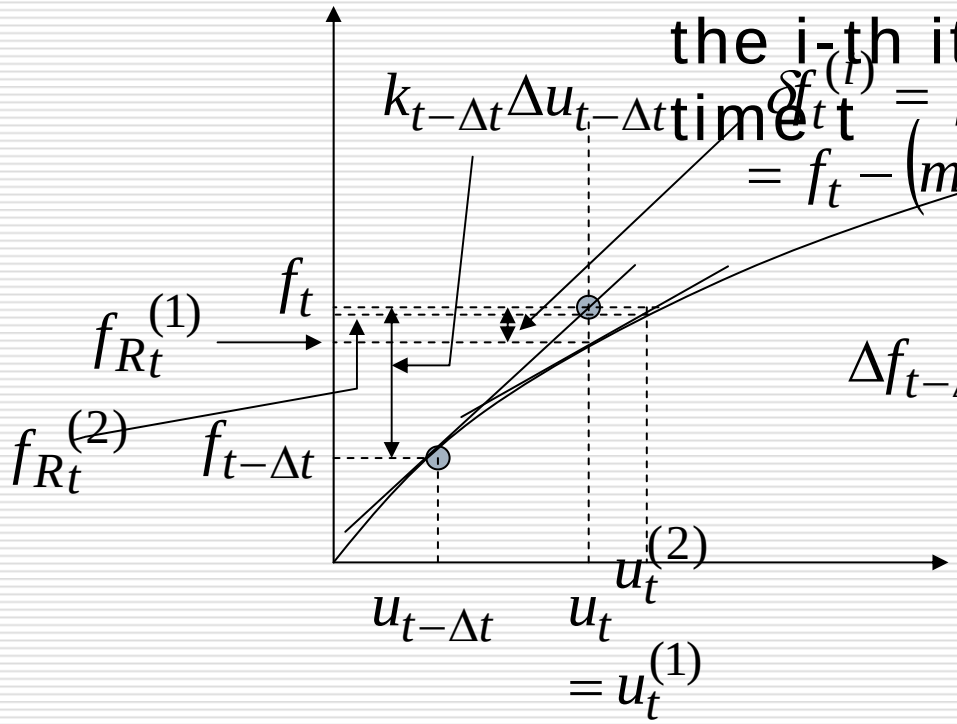
$$\Delta f_{t-\Delta t} = f_t - f_{t-\Delta t}$$
$$= k_{t-\Delta t}\Delta u_{t-\Delta t}$$
$$= k_{t-\Delta t}(u_t - u_{t-\Delta t})$$

The equation of motion for the i-th equilibrium iteration

$$m\delta\ddot{u}_t^{(i)} + c\delta\dot{u}_t^{(i)} + k_t\delta u_t^{(i)} = \delta f_t^{(i)} \qquad (7.32)$$

Unbalance force for the i-th iteration at time t

$$\delta f_t^{(i)} = f_t - f_{Rt}^{(i)}$$
$$= f_t - \left(m\ddot{u}_t^{(i)} + c\dot{u}_t^{(i)} + f_{Rt}^{(i)}\right)$$



$$\Delta f_{t-\Delta t} = f_t - f_{t-\Delta t}$$
$$= k_{t-\Delta t}\Delta u_{t-\Delta t}$$
$$= k_{t-\Delta t}(u_t - u_{t-\Delta t})$$

$$\delta u_t^{(i)} = u_t^{(i+1)} - u_t^{(i)}$$
$$\delta \dot{u}_t^{(i)} = \dot{u}_t^{(i+1)} - \dot{u}_t^{(i)}$$
$$\delta \ddot{u}_t^{(i)} = \ddot{u}_t^{(i+1)} - \ddot{u}_t^{(i)}$$

$$\delta f_t^{(i)} = f_t - (m\ddot{u}_t^{(i)} + c\dot{u}_t^{(i)} + f_{Rt}^{(i)})$$

(7.33)

$$\boxed{m\delta\ddot{u}_t^{(i)} + c\delta\dot{u}_t^{(i)} + k_t\delta u_t^{(i)} = \delta f_t^{(i)} \qquad (7.32)}$$

- Extending Eq. (7.32) to MDOF system, The equations of motion for the i-th equilibrium iteration at time t are expressed as

$$[M]\{\delta\ddot{u}_t^{(i)}\}+[C]\{\delta\dot{u}_t^{(i)}\}+[K_t]\{\delta u_t^{(i)}\}=\{\delta F_t^{(i)}\} \quad (7.34)$$

where, $\{\delta\ddot{u}_t^{(i)}\}$, $\{\delta\dot{u}_t^{(i)}\}$ and $\{\delta u_t^{(i)}\}$ are the corrective accelerations, velocities and displacements, respectively, for the i-th iteration as defined by

$$\begin{aligned}\{\delta\ddot{u}_t^{(i)}\}&=\{\ddot{u}_t^{(i+1)}\}-\{\ddot{u}_t^{(i)}\}\\ \{\delta\dot{u}_t^{(i)}\}&=\{\dot{u}_t^{(i+1)}\}-\{\dot{u}_t^{(i)}\}\\ \{\delta u_t^{(i)}\}&=\{u_t^{(i+1)}\}-\{u_t^{(i)}\}\end{aligned} \quad (7.35)$$

$$\boxed{m\delta\ddot{u}_t^{(i)}+c\delta\dot{u}_t^{(i)}+k_t\delta u_t^{(i)}=\delta f_t^{(i)} \qquad (7.32)}$$

and where $\left\{\delta F_t^{(i)}\right\}$ represents the unbalance forces for the i-th iteration given as

$$\left\{\delta F_t^{(i)}\right\} = \{F_t\} - \left([M]\left\{\ddot{u}_t^{(i)}\right\} + [C]\left\{\dot{u}_t^{(i)}\right\} + \left\{F_{Rt}^{(i)}\right\}\right) \quad (7.36)$$

- From Eq. (7.13), one obtains

$$\left\{\Delta\ddot{u}_i^{(i)}\right\} = c_1\left\{\Delta u_t^{(i)}\right\} - c_3\left\{\dot{u}_t^{(i)}\right\} - c_4\left\{\ddot{u}_t^{(i)}\right\} \quad (7.37a)$$

$$\left\{\Delta\dot{u}_t^{(i)}\right\} = c_2\left\{\Delta u_t^{(i)}\right\} - c_4\left\{\dot{u}_t^{(i)}\right\} - c_5\left\{\ddot{u}_t^{(i)}\right\} \quad (7.37b)$$

$$[M]\left\{\delta\ddot{u}_t^{(i)}\right\} + [C]\left\{\delta\dot{u}_t^{(i)}\right\} + [K_t]\left\{\delta u_t^{(i)}\right\} = \left\{\delta F_t^{(i)}\right\} \quad (7.34)$$

$$\{\Delta\ddot{u}_t\} = c_1\{\Delta u_t\} - c_3\{\dot{u}_t\} - c_4\{\ddot{u}_t\} \quad (7.13a)$$

$$\{\Delta\dot{u}_t\} = c_2\{\Delta u_t\} - c_4\{\dot{u}_t\} - c_5\{\ddot{u}_t\} \quad (7.13b)$$

- Similar to Eqs. (7.14), (7.15), and (7.16), substituting Eq. (7.35) into Eq. (7.34), we obtain

$$\left[\tilde{K}_t^{(i)}\right]\left\{\delta u_t^{(i)}\right\} = \left\{\delta F_t^{(i)}\right\} \qquad (7.38)$$

where,

$$\left[\tilde{K}_t^{(i)}\right] = \left[K_t^{(i)}\right] + c_1[M] + c_2[C] \qquad (7.39)$$

$$\left\{\Delta\tilde{F}_t^{(i)}\right\} = \left\{\Delta F_t^{(i)}\right\} + \left[c_3[M] + c_4[C]\right]\left\{\dot{u}_t^{(i)}\right\} + \left[c_4[M] + c_5[C]\right]\left\{\ddot{u}_t^{(i)}\right\} \qquad (7.40)$$

$$\left[\tilde{K}_t\right]\left\{\Delta u_t\right\} = \left\{\Delta\tilde{F}_t\right\} \qquad (7.14)$$

$$\left[\tilde{K}_t\right] = \left[K_t\right] + c_1[M] + c_2[C] \qquad (7.15)$$

$$\left\{\Delta\tilde{F}_t\right\} = \left\{\Delta F_t\right\} + \left[c_3[M] + c_4[C]\right]\left\{\dot{u}_t\right\} + \left[c_4[M] + c_5[C]\right]\left\{\ddot{u}_t\right\} \qquad (7.16)$$

- If convergence occurs, the iteration can be continued until the dynamic equilibrium of the motion is satisfied within the specified accuracy.
- The improved solutions can be obtained from Eq. (7.35).

$$\begin{aligned}
\{\delta\ddot{u}_t^{(i)}\} &= \{\ddot{u}_t^{(i+1)}\} - \{\ddot{u}_t^{(i)}\} \\
\{\delta\dot{u}_t^{(i)}\} &= \{\dot{u}_t^{(i+1)}\} - \{\dot{u}_t^{(i)}\} \\
\{\delta u_t^{(i)}\} &= \{u_t^{(i+1)}\} - \{u_t^{(i)}\}
\end{aligned} \qquad (7.35)$$

Example 7.1 Compute nonlinear response of the 3 DOFS structure of Example 9.? under JMA Kobe ground acceleration which was recorded during the 1995 Kobe earthquake.

- Mass matrix

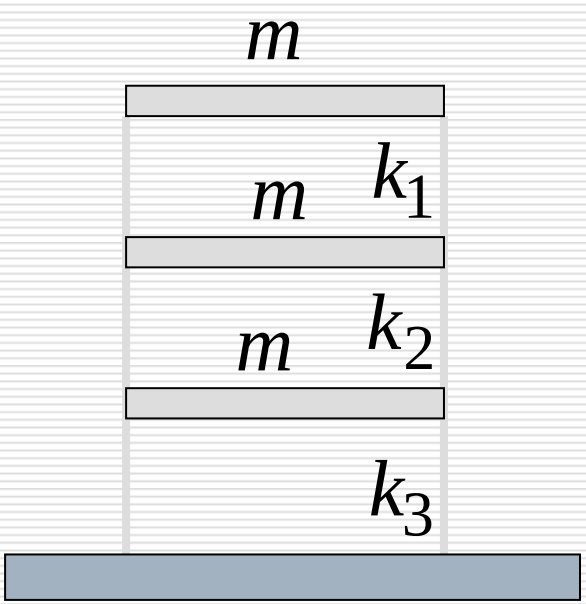
$$[M]=\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

- Stiffness matrix

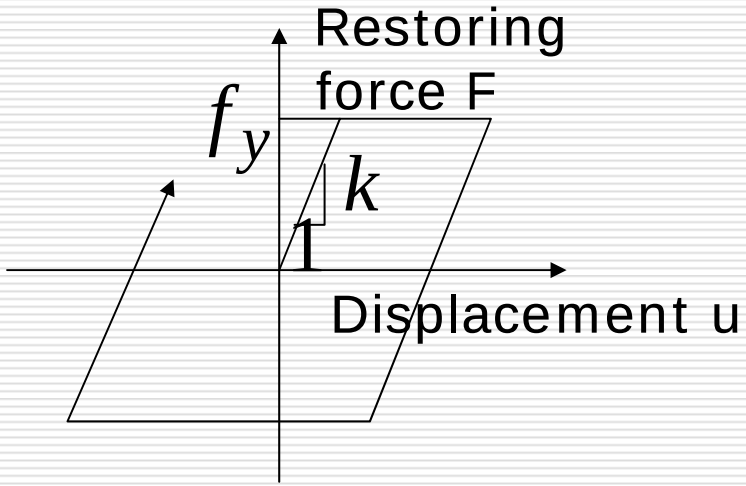
$$[K]=\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2+k_3 \end{bmatrix}$$

- Assume that

$$m=150\text{kN}/g=150/9.8$$



- The restoring forces of three springs are assumed to be perfect elasto-plastic shown below



$$k_1 = k_2 = k_3 = k = 3050.9\text{kN/m}$$

$$f_{y1} = f_{y2} = f_{y3} = f_y$$

$$f_y = 500\text{kN}$$

- Eigen value analysis

$$\begin{vmatrix} k_1 - \omega^2 m & -k_1 & 0 \\ -k_1 & k_1 + k_2 - \omega^2 m & -k_2 \\ 0 & -k_2 & k_2 + k_3 - \omega^2 m \end{vmatrix} = 0$$

$$k_1 = k_2 = k_3 = k = 3050.9\text{kN/m} \qquad m = 150/9.8\text{ton}$$

$$\Rightarrow \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} 6.28\text{rad/s} \\ 17.6\text{rad/s} \\ 25.4\text{rad/s} \end{Bmatrix} \qquad \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1.00\text{s} \\ 0.357\text{s} \\ 0.247\text{s} \end{Bmatrix}$$

- Rayleigh damping is assumed. Based on Eq. (7.25), α and β can be determined as

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \frac{2\omega_i\omega_j}{\omega_j^2 - \omega_i^2}\begin{bmatrix} \omega_j & -\omega_i \\ -1/\omega_j & 1/\omega_i \end{bmatrix}\begin{Bmatrix} \xi_i \\ \xi_j \end{Bmatrix} \qquad (7.25)$$

1st and 2nd mode is selected

$$\boxed{\begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix} = \begin{Bmatrix} 6.28\text{rad/s} \\ 17.6\text{rad/s} \end{Bmatrix}} \qquad \begin{Bmatrix} \omega_3 \end{Bmatrix} = \begin{Bmatrix} 25.4\text{rad/s} \end{Bmatrix}$$

$$\cancel{\xi_1 = \xi_2 = \xi_3 = 0.05}$$

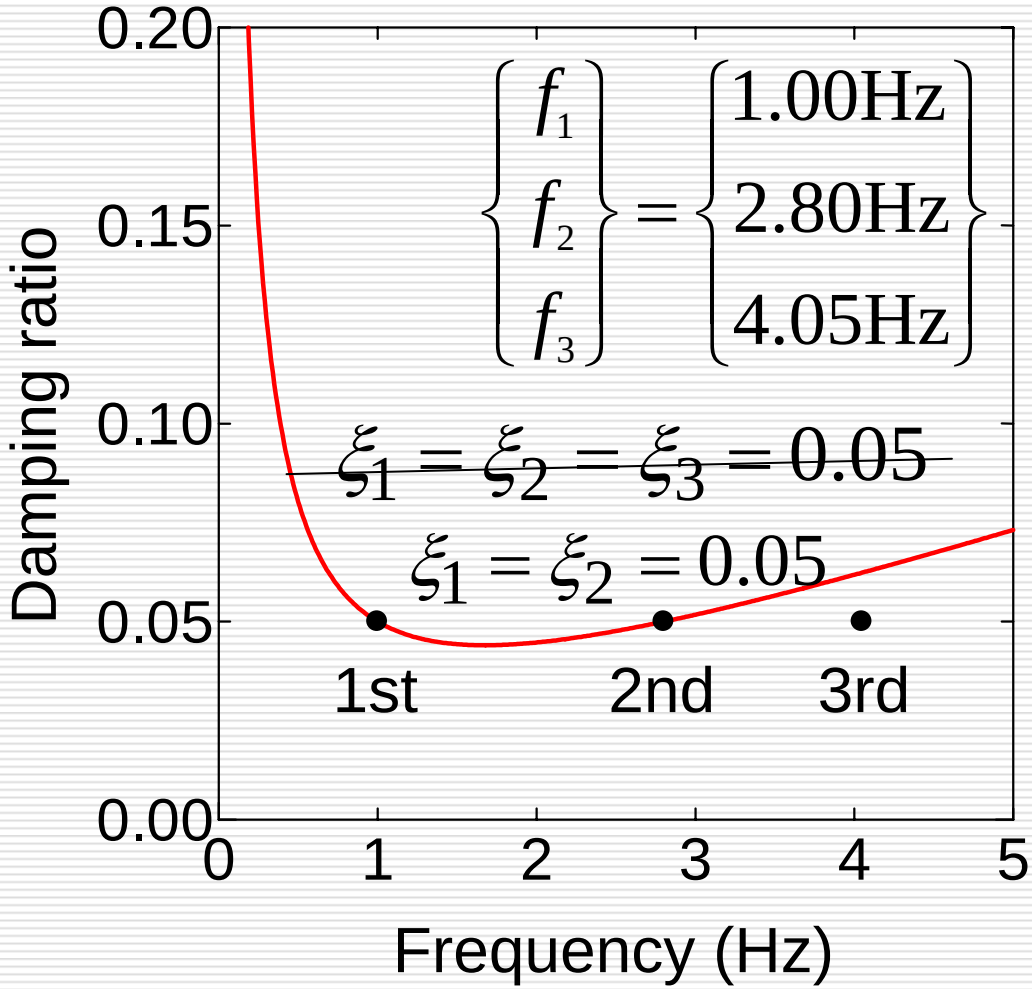
$$\xi_1 = \xi_2 = 0.05$$

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \frac{2\times 6.28\times 17.6}{17.6^2 - 6.28^2}\begin{bmatrix} 17.6 & -6.28 \\ -1/17.6 & 1/6.28 \end{bmatrix}\begin{Bmatrix} 0.05 \\ 0.05 \end{Bmatrix} = \begin{Bmatrix} 0.463 \\ 4.19\times 10^{-3} \end{Bmatrix}$$

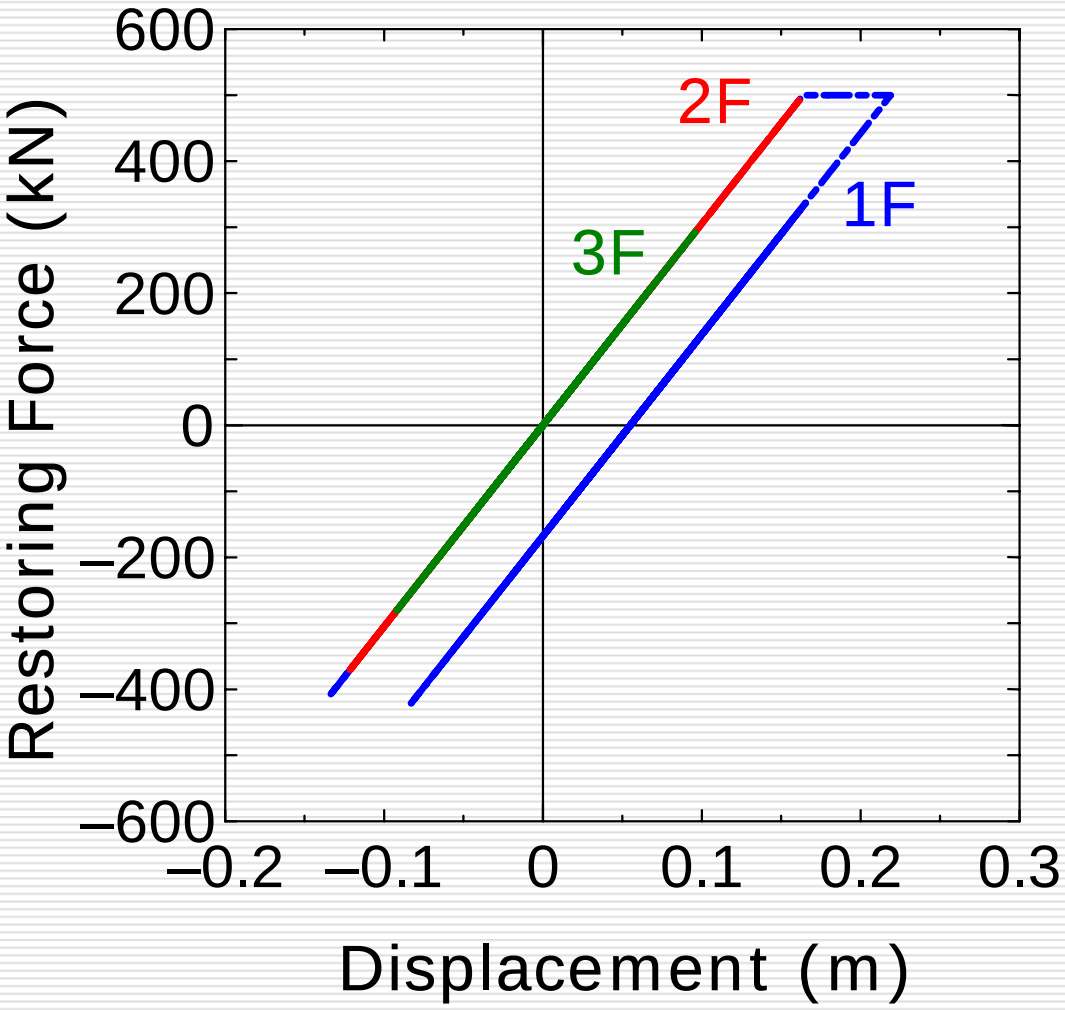
- Rayleigh damping is assumed

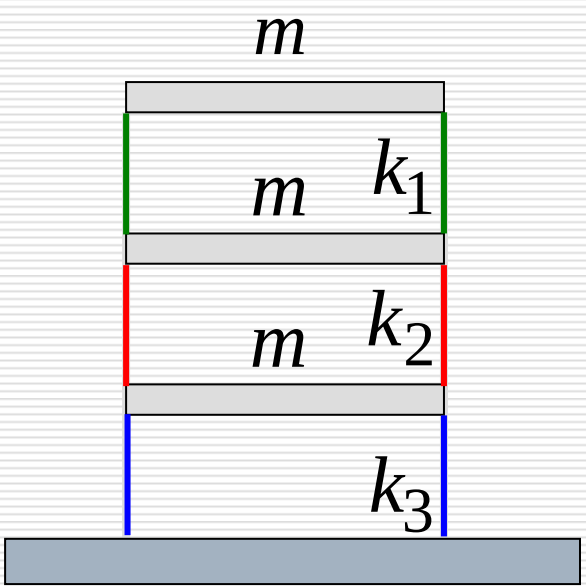
$$\xi(\omega) = \frac{\alpha}{2\omega} + \frac{\beta\omega}{2}$$

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} 0.463 \\ 4.19 \times 10^{-3} \end{Bmatrix}$$



● restoring forces vs. displacement

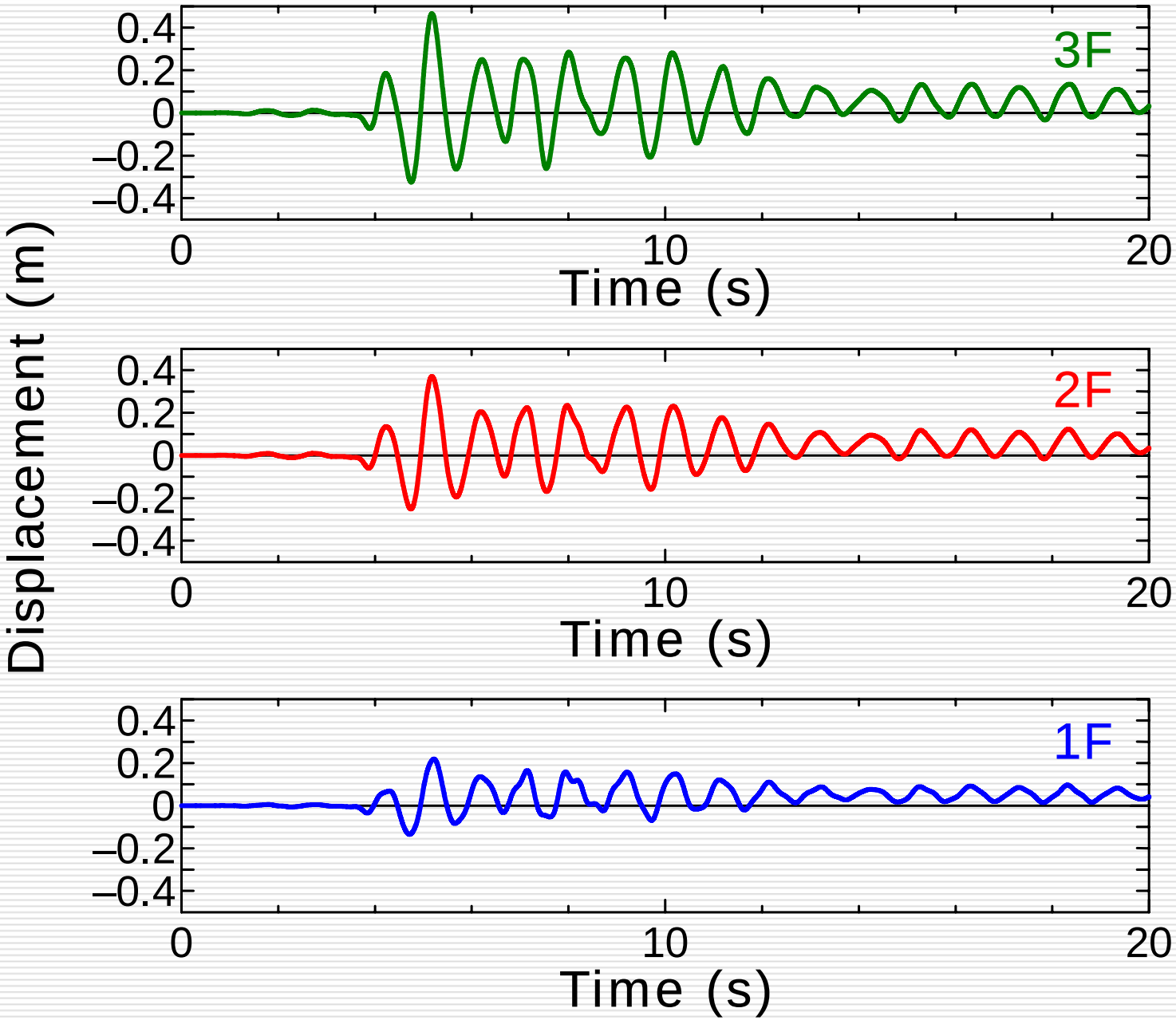




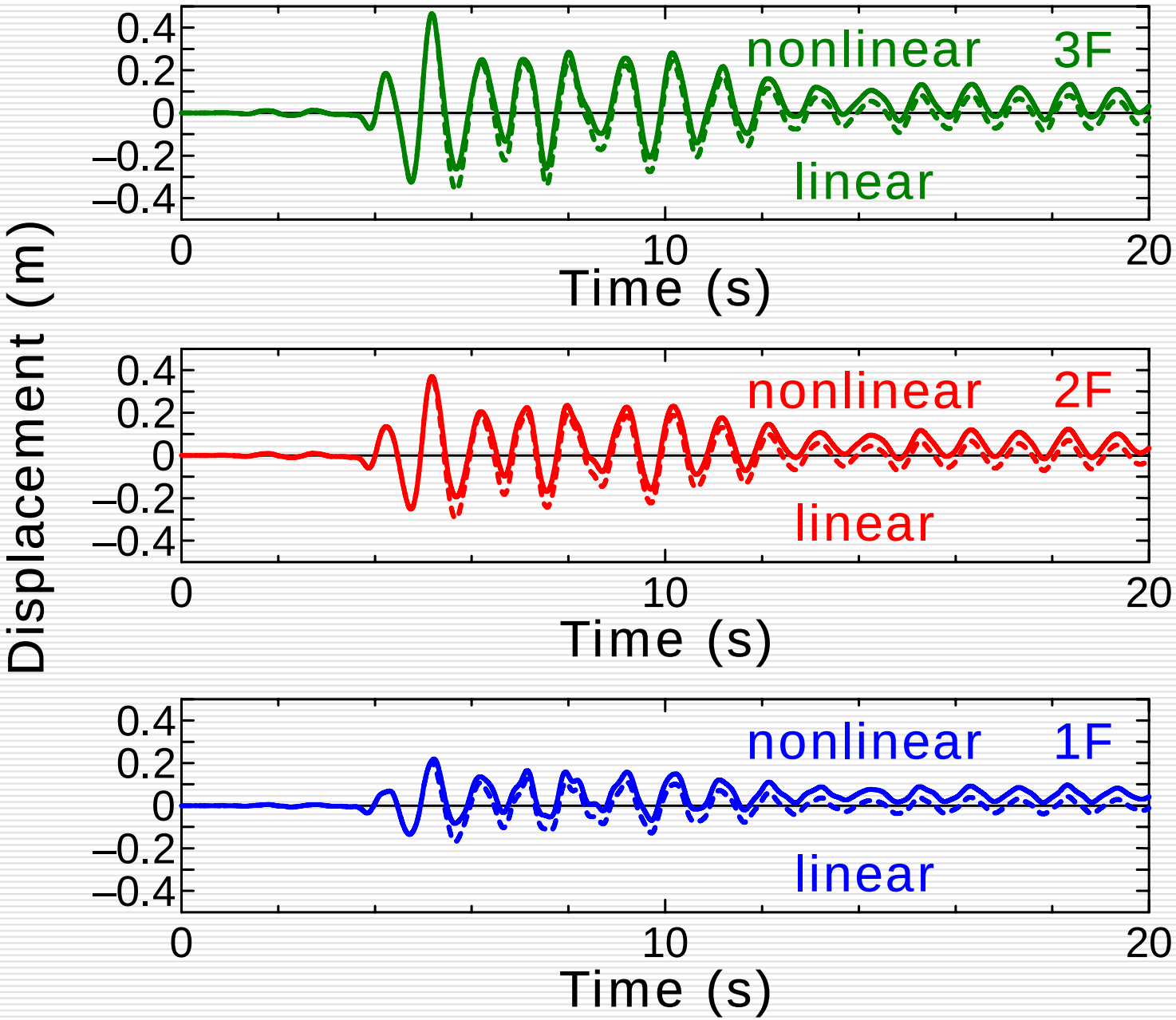
$$f_{y1} = f_{y2} = f_{y3} = f_y$$

$$f_y = 500\text{kN}$$

●response displacement



●response displacement



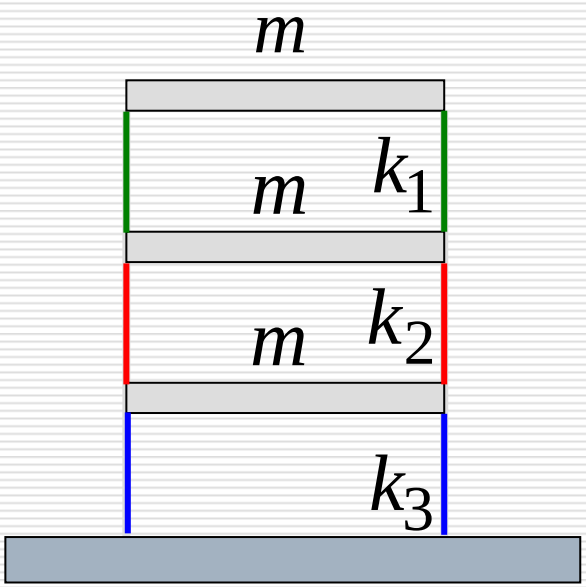
Example 7.2

$$f_y = 500\text{kN}$$

$$f_{y1} = \frac{2}{5} f_y$$

$$f_{y2} = \frac{4}{5} f_y$$

$$f_{y3} = f_y$$

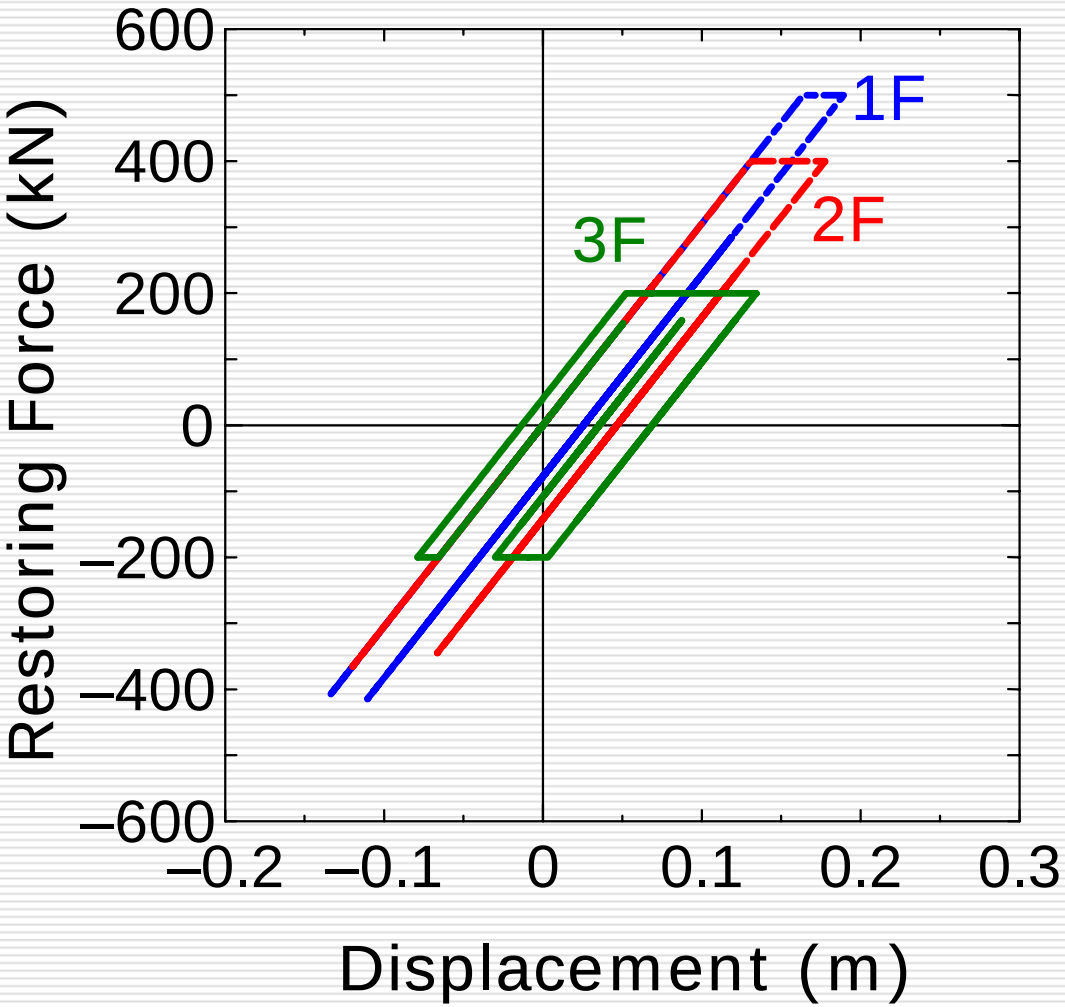


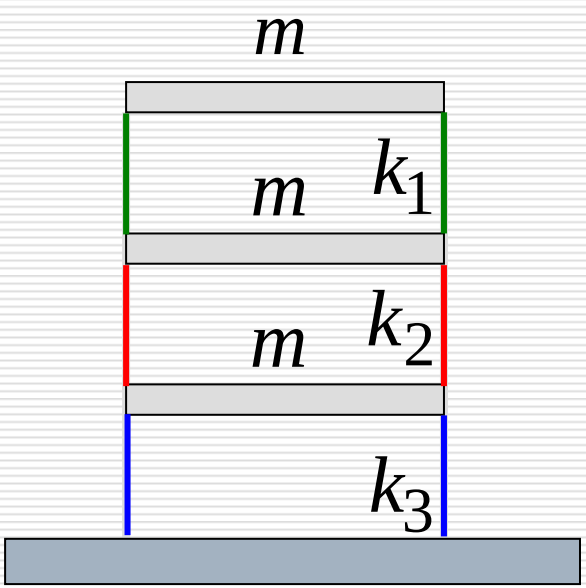
$$f_{y1} = 200\text{kN}$$

$$f_{y2} = 400\text{kN}$$

$$f_{y3} = 500\text{kN}$$

● restoring forces vs. displacement



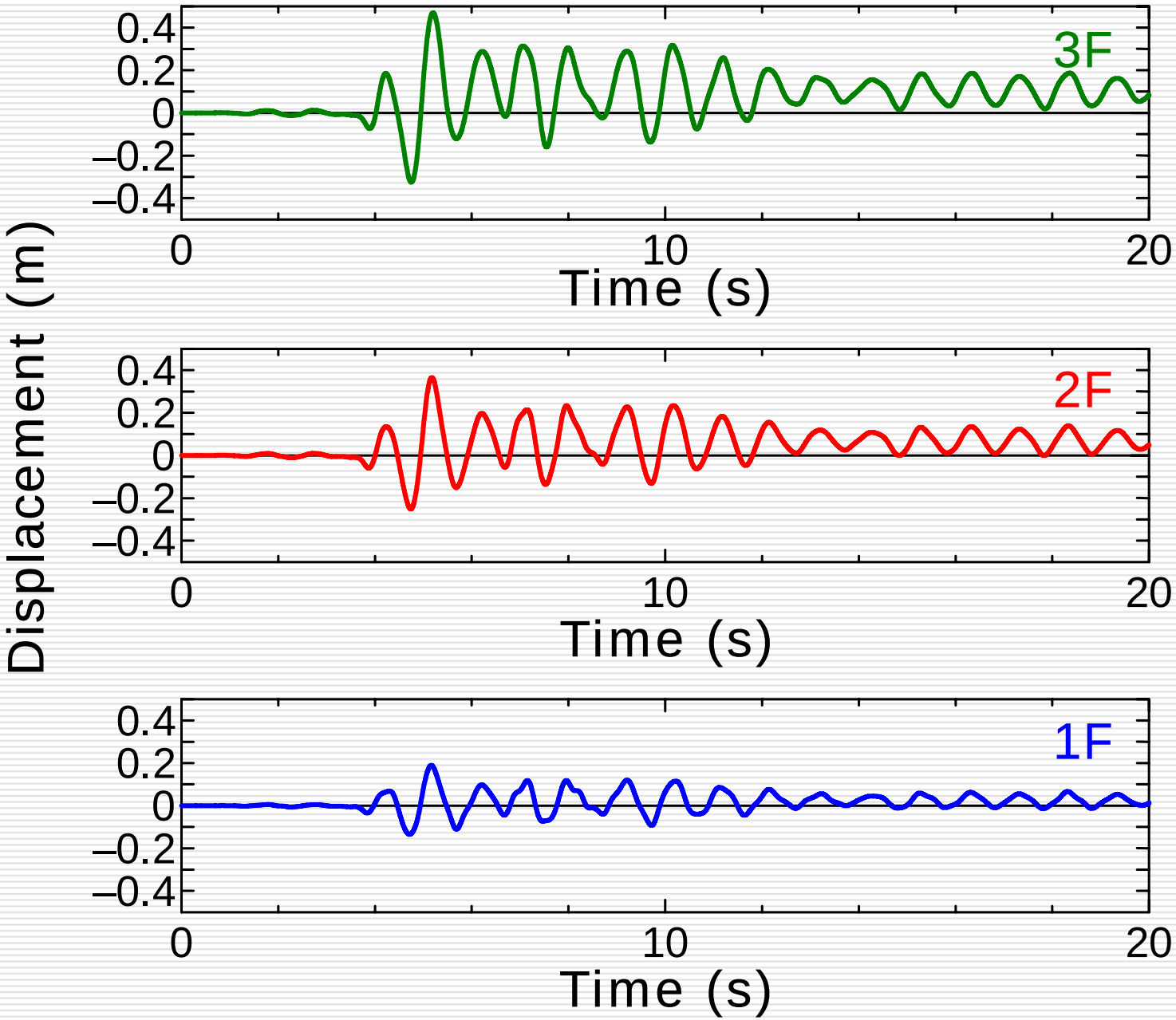


$$f_{y1} = 200\text{kN}$$

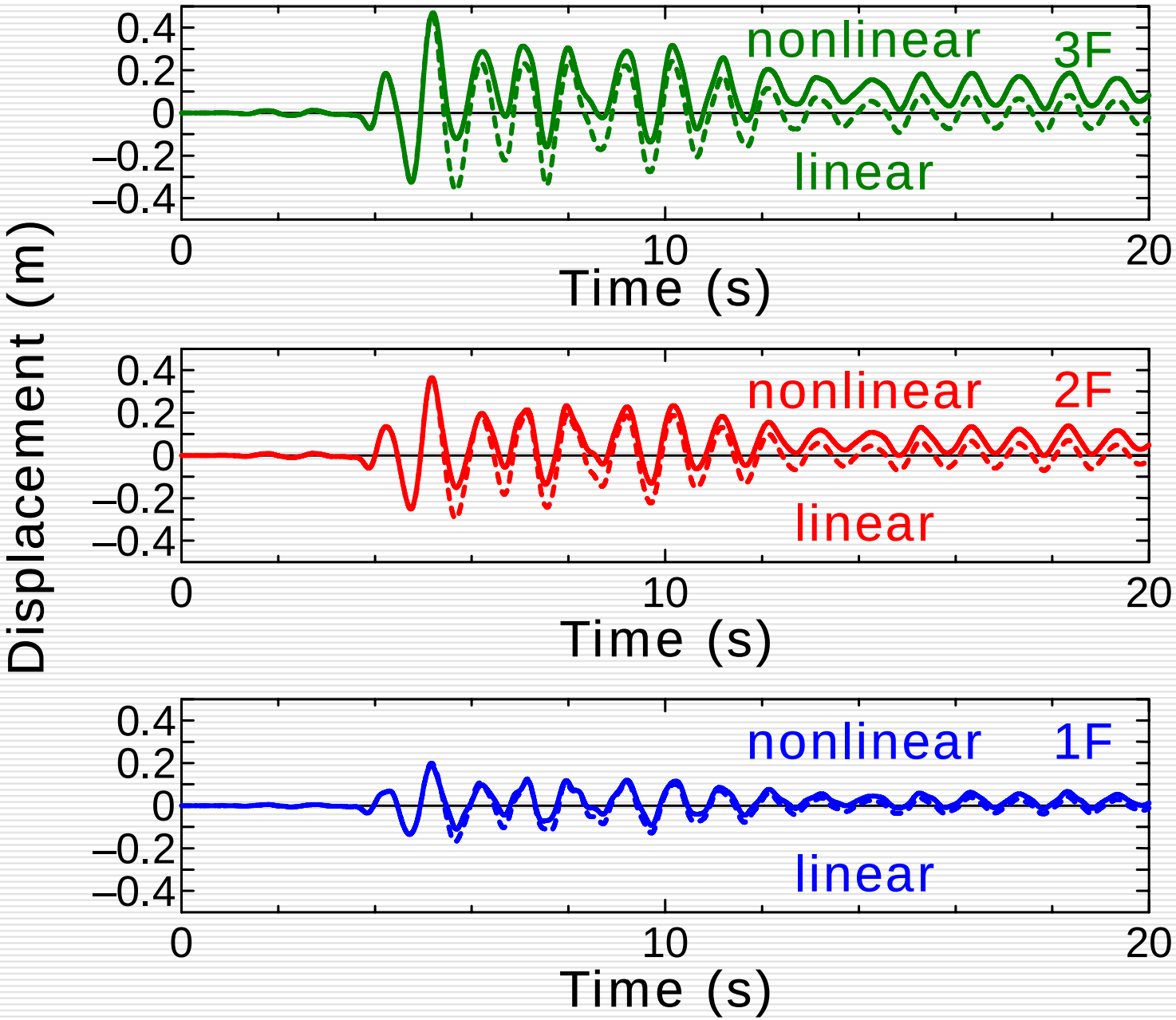
$$f_{y2} = 400\text{kN}$$

$$f_{y3} = 500\text{kN}$$

●response displacement



●response displacement



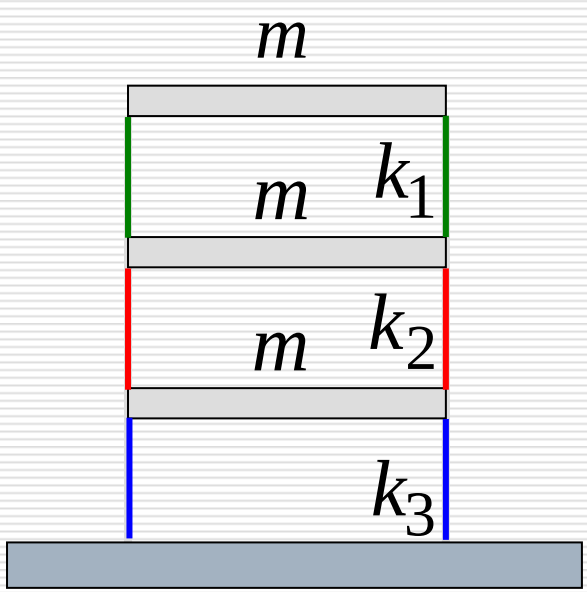
Example 7.3

$$f_y = 500\text{kN}$$

$$f_{y1} = \frac{2}{5} f_y$$

$$f_{y2} = \frac{4}{5} f_y$$

$$f_{y3} = \frac{3}{5} f_y$$

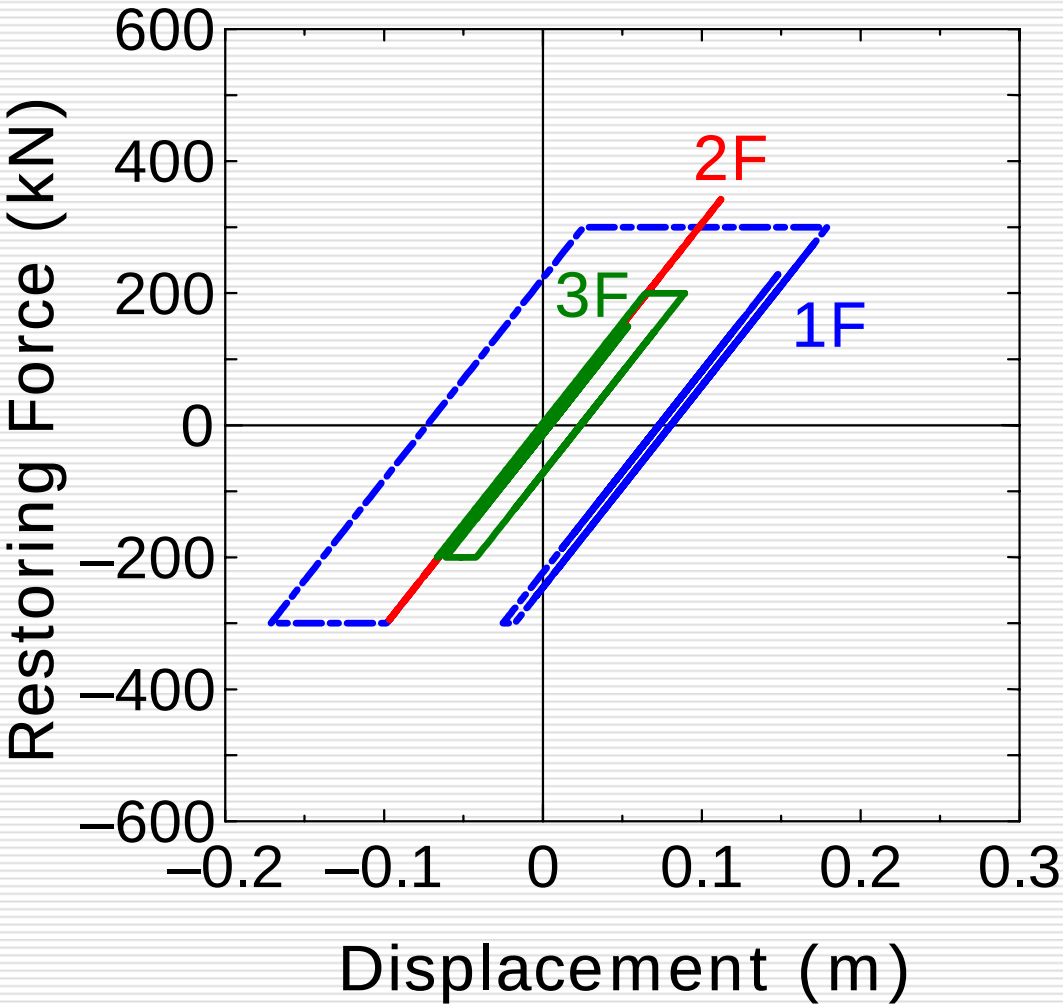


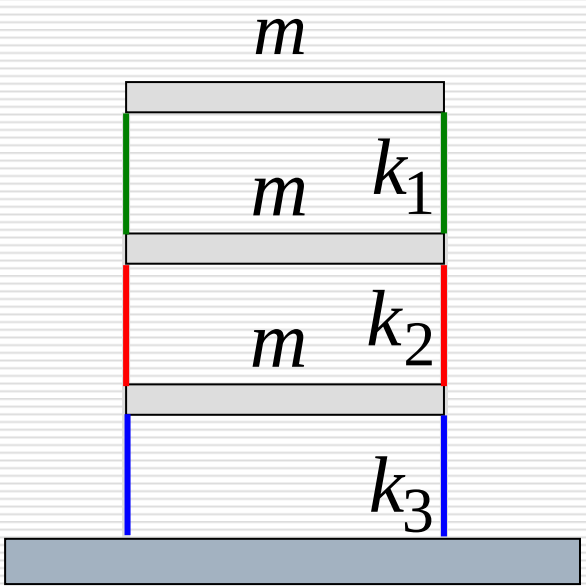
$$f_{y1} = 200\text{kN}$$

$$f_{y2} = 400\text{kN}$$

$$f_{y3} = 300\text{kN}$$

● restoring forces vs. displacement



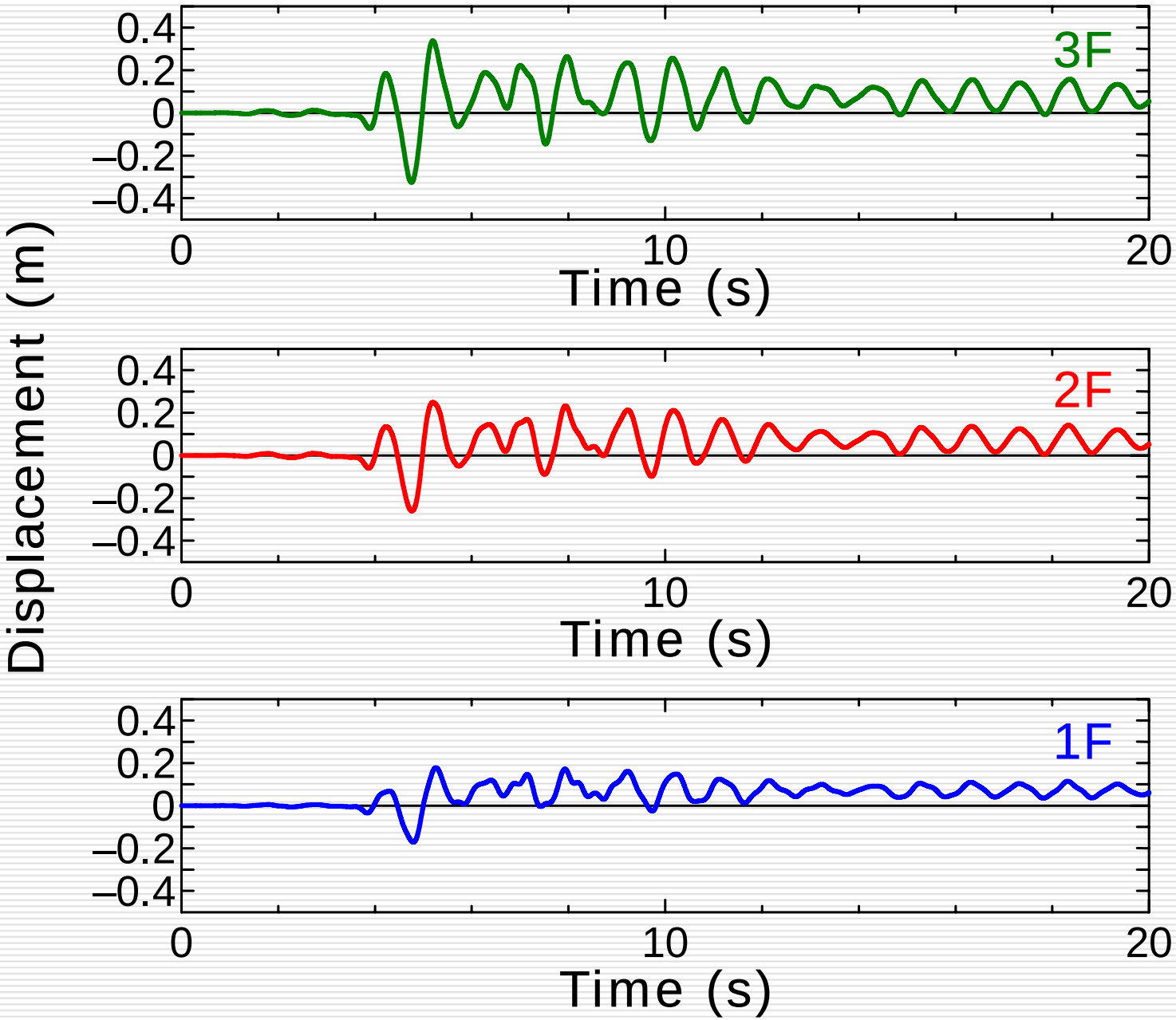
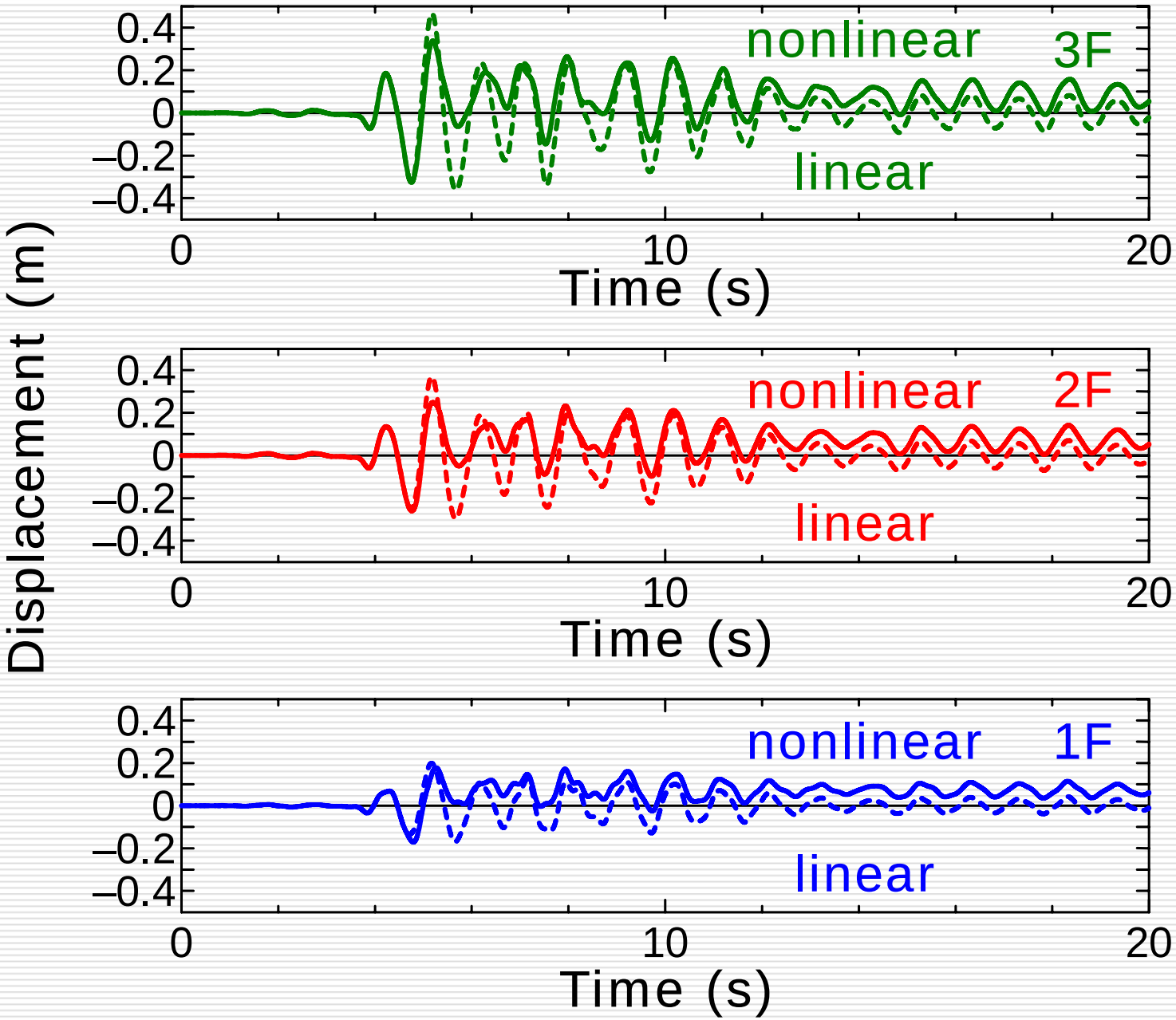


$$f_{y1} = 200\text{kN}$$

$$f_{y2} = 400\text{kN}$$

$$f_{y3} = 300\text{kN}$$

●response displacement



●response displacement