Structural Dynamics
構造動力学
(12)

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Chapter 10 Formulation of the Nonlinear Multi-Degree-of-Freedom Equations of Motion (**非線形多自由度系の運動方程式**)

10.1 Incremental Equations of Motion of Nonlinear MDOF System (非線形多自由度系に対する増分系運動方程式)

●Equilibrium of the inertia force, damping force, restoring force and external force for SDOF system is given by Eq. (2.1). By extending Eq. (2.1) to MDOF system, one obtains

$$\begin{Bmatrix} f_{I1} \\ f_{I2} \\ . \\ f_{Ii} \\ . \\ f_{In} \end{Bmatrix} + \begin{Bmatrix} f_{D1} \\ f_{D2} \\ . \\ f_{Di} \\ . \\ f_{Dn} \end{Bmatrix} + \begin{Bmatrix} f_{R1} \\ f_{R2} \\ . \\ f_{Ri} \\ . \\ f_{Rn} \end{Bmatrix} = \begin{Bmatrix} f_{1} \\ f_{2} \\ . \\ f_{i} \\ . \\ f_{n} \end{Bmatrix} \qquad (7.1)$$

$$\boxed{f_I(t) + f_D(t) + f_S(t) = p(t) \qquad (2.1)}$$

- Eq. (10.1) can be written in matrix form by

$$\{F_I(t)\}+\{F_D(t)\}+\{F_R(t)\}=\{F(t)\} \qquad (7.2)$$

in which,

$\{F_I\}$: inertia force vector (**慣性力ベクトル**)

$\{F_D\}$: damping force vector (**減衰力ベクトル**)

$\{F_R\}$: restoring force vector (**復元力ベクトル**)

$\{F\}$: external force vector (**外力ベクトル**)

where,

$$\{F_I\}=[M]\{\ddot{u}\}$$

$$\{F_D\}=[C]\{\dot{u}\}$$

● If the system is linear, the restoring force vector is given by

$$\{F_R\} = [K]\{u\} \qquad (7.3)$$

Then, Eq. (7.2) becomes completely the same with Eq. (6.45).

●However, note in Eq. (7.2) that because the system is not linear elastic, the restoring force vector cannot be evaluated by Eq. (7.3). Therefore, the equations of motion given by Eq. (10.2) cannot be solved by the mode superposition method whic we studied in Chapter 9.

$$\{F_I(t)\} + \{F_D(t)\} + \{F_R(t)\} = \{F(t)\} \qquad (7.2)$$

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{P\} \qquad (6.45)$$

- In linear system

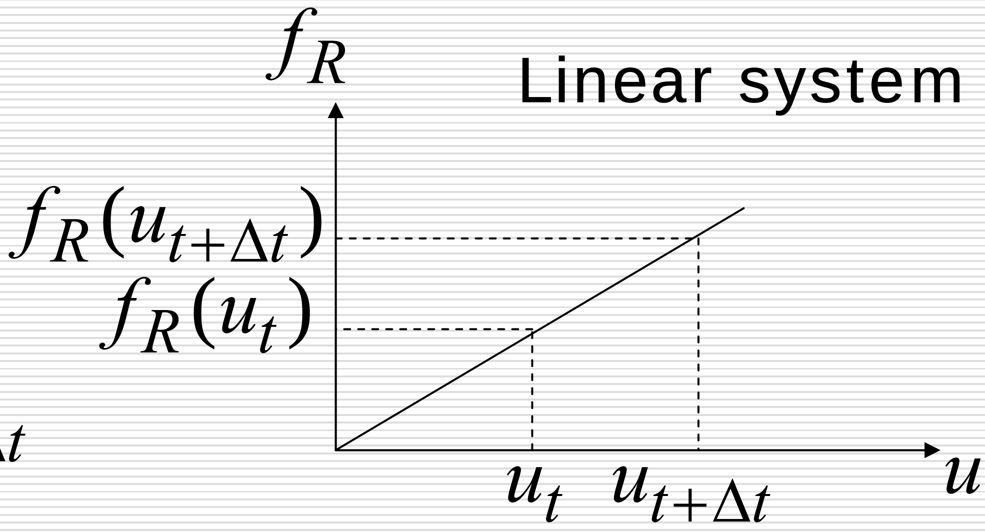
$$f_R(u_{t+\Delta t}) = k u_{t+\Delta t}$$

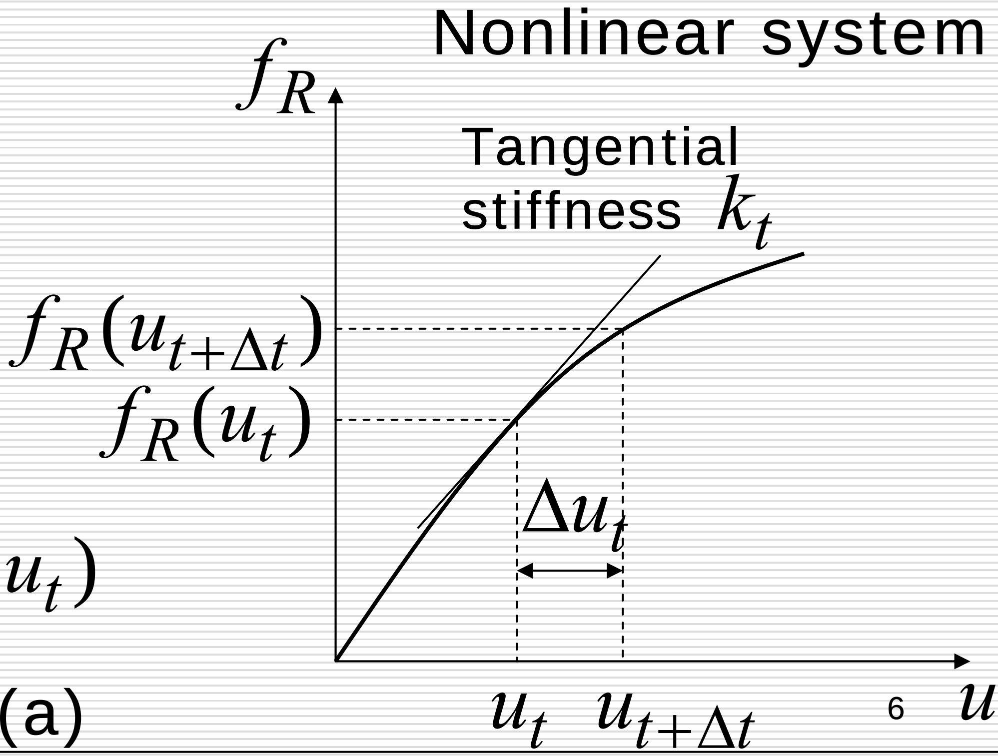
- However in nonlinear system

$$f_R(u_{t+\Delta t}) \neq k(u) u_{t+\Delta t}$$

- However, if $f_R(t)$ is the restoring force at u_t, the restoring force at $u_{t+\Delta t}$ may be approximately evaluated using the tangential stiffness (接線剛性) by

$$\Delta f_{Rt} \equiv f_R(u_{t+\Delta t}) - f_R(u_t)$$
$$\approx k_t(u_t) \Delta u_t$$



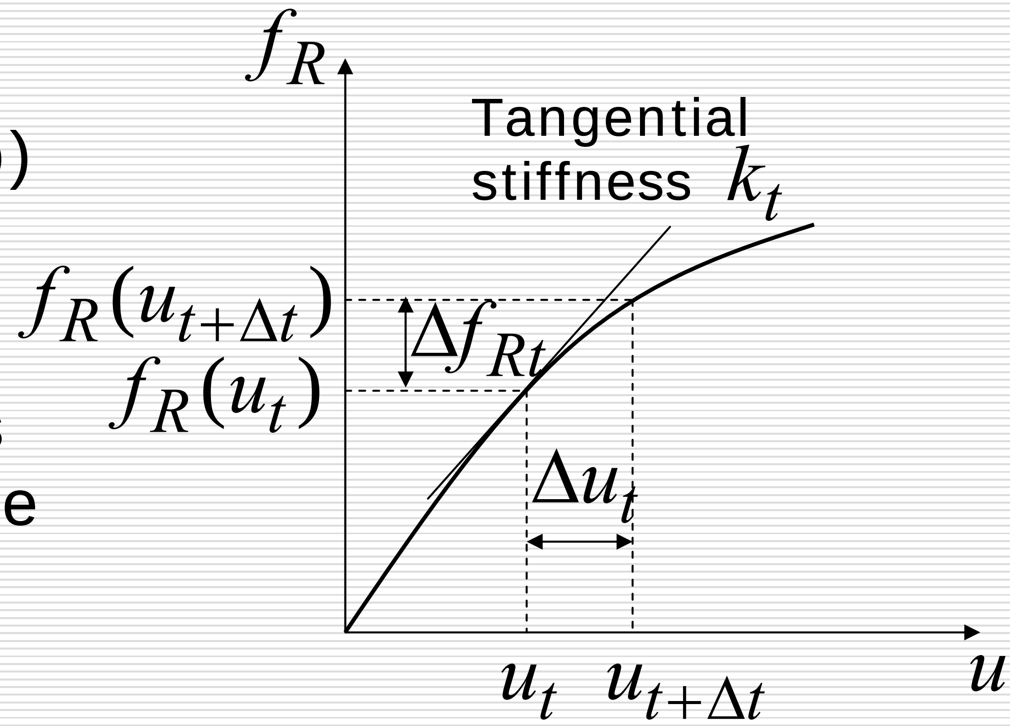


(a)

- In Eq. (a),

$$\Delta u_t = u_{t+\Delta t} - u_t \qquad \text{(b)}$$

is called incremental displacement (**増分変位**) at time t, and this represents an increase of displacement during a small time interval Δt.



- Δf_{Rt} is called incremental restoring force (**増分復元力**) at time t (during Δu_t).

- Tangential stiffness is defined by

$$k_t = \left(\frac{\partial f_R}{\partial u} \right)_{u=u_t} \qquad \text{(c)}$$

$$\boxed{\Delta f_{Rt} \equiv f_R(u_{t+\Delta t}) - f_R(u_t) \approx k_t(u_t)\Delta u_t \qquad \text{(a)}}$$

- If time interval Δt is small enough such as 1/100s-1/10,000s, the approximation of the incremental restoring force by Eq. (a) may be sufficient.

- Extending Eq. (a) to MDOF system, the incremental restoring force vector at time t is written by

$$\begin{aligned}\{\Delta F_{Rt}\} &\equiv \{F_R(u_{t+\Delta t})\} - \{F_R(u_t)\} \\ &= [K_t(u_t)]\{\Delta u_t\}\end{aligned} \qquad \text{(d)}$$

where, $\{\Delta u_t\}$ is called the incremental displacement vector (**増分変位ベクトル**) during time t+Δt and t, and is expressed by

$$\{\Delta u_t\} \equiv \{u_{t+\Delta t}\} - \{u_t\} \qquad \text{(e)}$$

$$\boxed{\Delta f_{Rt} \equiv f_R(u_{t+\Delta t}) - f_R(u_t) \approx k_t(u_t)\Delta u_t \qquad \text{(a)}}$$

- Based on Eq. (7.2), the dynamic equilibrium at time t+Δt can be written by

$$\{F_I(t+\Delta t)\}+\{F_D(t+\Delta t)\}+\{F_R(t+\Delta t)\}=\{F(t+\Delta t)\} \tag{7.4}$$

- Subtracting Eq. (7.2) from Eq. (7.4), one obtains the dynamic equilibrium in an incremental form

$$\{\Delta F_{It}\}+\{\Delta F_{Dt}\}+\{\Delta F_{Rt}\}=\{\Delta F_t\} \tag{7.5}$$

where,

$$\begin{aligned}\{\Delta F_{It}\} &\equiv \{F_I(t+\Delta t)\}-\{F_I(t)\}\\ &=[M]\{\ddot{u}_{t+\Delta t}\}-[M]\{\ddot{u}_t\}\\ &=[M]\{\Delta\ddot{u}_t\}\end{aligned} \tag{7.6}$$

in which $\{\Delta\ddot{u}_t\}\equiv\{\ddot{u}_{t+\Delta t}\}-\{\ddot{u}_t\}$ (7.7)

$$\{F_I(t)\}+\{F_D(t)\}+\{F_R(t)\}=\{F(t)\} \tag{7.2}$$

$$[M]\{\ddot{u}\}+[K]\{u\}=-\ddot{u}_g[M]\{I\} \tag{6.16}$$

- Similarly,

$$\{\Delta F_{Dt}\} \equiv \{F_D(t+\Delta t)\} - \{F_D(t)\}$$

$$= [C]\{\Delta \dot{u}_t\} \qquad (7.8)$$

$$\{\Delta F_{Rt}\} \equiv \{F_R(t+\Delta t)\} - \{F_R(t)\}$$

$$\approx [K_t]\{\Delta u_t\} \qquad (7.9)$$

$$\{\Delta F_t\} = \{F(t+\Delta t)\} - \{F(t)\} \qquad (7.10)$$

in which

$$\{\Delta \dot{u}_t\} \equiv \{\dot{u}_{t+\Delta t}\} - \{\dot{u}_t\} \qquad (7.11)$$

$$\{\Delta u_t\} \equiv \{u_{t+\Delta t}\} - \{u_t\} \qquad (7.12)$$

$$\boxed{\begin{aligned} &\{F_I(t+\Delta t)\} + \{F_D(t+\Delta t)\} + \{F_R(t+\Delta t)\} = \{F(t+\Delta t)\} \qquad (7.4) \\ &\{F_I(t)\} + \{F_D(t)\} + \{F_R(t)\} = \{F(t)\} \qquad (7.2) \end{aligned}}$$

● Substitution of Eqs. (7.6), (7.8), (7.9) and (7.10) into Eq. (7.5) leads to

$$[M]\{\Delta \ddot{u}_t\} + [C]\{\Delta \dot{u}_t\} + [K_t]\{\Delta u_t\} = \{\Delta F_t\} \qquad (7.12)$$

where,

$$\{\Delta \ddot{u}_t\} \equiv \{\ddot{u}_{t+\Delta t}\} - \{\ddot{u}_t\} \qquad (7.7)$$

$$\{\Delta \dot{u}_t\} \equiv \{\dot{u}_{t+\Delta t}\} - \{\dot{u}_t\} \qquad (7.11)$$

$$\{\Delta u_t\} \equiv \{u_{t+\Delta t}\} - \{u_t\} \qquad (7.12)$$

$$\{\Delta F_{It}\} + \{\Delta F_{Dt}\} + \{\Delta F_{Rt}\} = \{\Delta F_t\} \qquad (7.5)$$

$$\{\Delta F_{It}\} \equiv \{F_I(t+\Delta t)\} - \{F_I(t)\} = [M]\{\Delta \ddot{u}_t\} \qquad (7.6)$$

$$\{\Delta F_{Dt}\} \equiv \{F_D(t+\Delta t)\} - \{F_D(t)\} = [C]\{\Delta \dot{u}_t\} \qquad (7.8)$$

$$\{\Delta F_{Rt}\} \equiv \{F_R(t+\Delta t)\} - \{F_R(t)\} = [K_t]\{\Delta u_t\} \qquad (7.9)$$

$$\{\Delta F_t\} = \{F(t+\Delta t)\} - \{F(t)\} \qquad (7.10)$$

10.2 Direct Integration Method (直接積分法)

●Based on the Newmark's β method for SDOF system given by Eq. (5.27), the basic integration equations can be extended to MDOF system as

$$\{\Delta \ddot{u}_t\} = c_1\{\Delta u_t\} - c_3\{\dot{u}_t\} - c_4\{\ddot{u}_t\} \qquad (7.13a)$$

$$\{\Delta \dot{u}_t\} = c_2\{\Delta u_t\} - c_4\{\dot{u}_t\} - c_5\{\ddot{u}_t\} \qquad (7.13b)$$

$$\Delta \ddot{u}_t = c_1 \Delta u_t - c_3 \dot{u}_t - c_4 \ddot{u}_t \qquad (5.27a)$$

$$\Delta \dot{u}_t = c_2 \Delta u_t - c_4 \dot{u}_t - c_5 \ddot{u}_t \qquad (5.27b)$$

in which

$$c_1 = \frac{1}{\sigma \Delta t^2} \qquad c_2 = \frac{\delta}{\sigma \Delta t} \qquad c_3 = \frac{1}{\sigma \Delta t}$$

$$c_4 = \frac{1}{2\sigma} \qquad c_5 = \left(\frac{\delta}{2\sigma} - 1\right)\Delta t \qquad (5.28)$$

- In Eq. (7.13), combination of $\delta = 1/2$ and $\sigma = 1/4$ is the integration scheme corresponding to the constant acceleration method (**一定加速度法**), and combination of $\sigma = 1/6$ and $\delta = 1/2$ the linear acceleration method (**線形加速度法**)

- Similar to Eq. (5.29), substitution of Eq. (7.13) into Eq. (7.12) leads to

$$\{\Delta\ddot{u}_t\} = c_1\{\Delta u_t\} - c_2\{\dot{u}_t\} - c_3\{\ddot{u}_t\} \qquad (7.13a)$$

$$\{\Delta\dot{u}_t\} = c_2\{\Delta u_t\} - c_4\{\dot{u}_t\} - c_5\{\ddot{u}_t\} \qquad (7.13b)$$

$$[M]\{\Delta\ddot{u}_t\} + [C]\{\Delta\dot{u}_t\} + [K_t]\{\Delta u_t\} = \{\Delta F_t\} \qquad (7.12)$$

$$\left[\tilde{K}_t\right]\{\Delta u_t\} = \{\Delta \tilde{F}_t\} \qquad (7.14)$$

in which,

$$\left[\tilde{K}_t\right] = [K_t] + c_1[M] + c_2[C] \qquad (7.15)$$

$$\{\Delta \tilde{F}_t\} = \{\Delta F_t\} + [c_3[M] + c_4[C]]\{\dot{u}_t\}$$
$$+ [c_4[M] + c_5[C]]\{\ddot{u}_t\} \qquad (7.16)$$

$$\tilde{k}_t \Delta u_t = \Delta \tilde{p}_t \qquad (5.29)$$

in which,

$$\tilde{k}_t = k_t + c_1 m + c_2 c_t \qquad (5.30)$$

$$\Delta \tilde{p}_t = \Delta p_t + (c_3 m + c_4 c_t)\dot{u}_t$$
$$+ (c_4 m + c_5 c_t)\ddot{u}_t \qquad (5.31)$$

● Once the incremental displacements $\{\Delta u_t\}$ are computed by solving Eq. (7.14), the incremental accelerations $\{\Delta \ddot{u}_t\}$ and incremental velocities $\{\Delta \dot{u}_t\}$ during Δt can be obtained from Eq. (7.13).

●Then the response quantities at time t+Δt can be computed based on Eqs. (7.7), (7.11) and (7.12) by

$$\{\ddot{u}_{t+\Delta t}\} = \{\ddot{u}_t\} + \{\Delta \ddot{u}_t\} \qquad (7.17)$$

$$\{\dot{u}_{t+\Delta t}\} = \{\dot{u}_t\} + \{\Delta \dot{u}_t\} \qquad (7.18)$$

$$\{u_{t+\Delta t}\} = \{u_t\} + \{\Delta u_t\} \qquad (7.19)$$

$$\{\Delta \ddot{u}_t\} = c_1\{\Delta u_t\} - c_2\{\dot{u}_t\} - c_3\{\ddot{u}_t\} \qquad (7.13a)$$

$$\{\Delta \dot{u}_t\} = c_2\{\Delta u_t\} - c_4\{\dot{u}_t\} - c_5\{\ddot{u}_t\} \qquad (7.13b)$$

$$\{\Delta \ddot{u}_t\} \equiv \{\ddot{u}_{t+\Delta t}\} - \{\ddot{u}_t\} \qquad (7.7)$$

$$\{\Delta \dot{u}_t\} \equiv \{\dot{u}_{t+\Delta t}\} - \{\dot{u}_t\} \qquad (7.11)$$

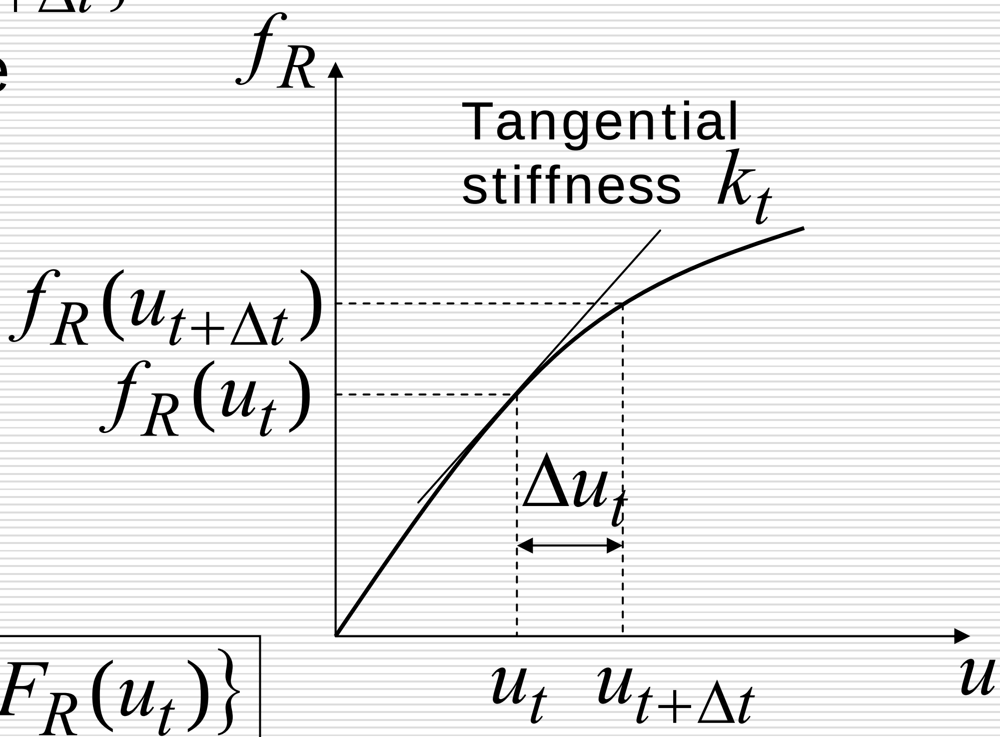
$$\{\Delta u_t\} \equiv \{u_{t+\Delta t}\} - \{u_t\} \qquad (7.12)$$

● As shown for SDOF system, the restoring force at time t+Δt cannot be directly computed by simply multiplying the displacement at time t+Δt by the tangential stiffness at time t, that is,

$$\{F_R(t+\Delta t)\} \neq [K_t]\{u_{t+\Delta t}\}$$

● $\{F_R(u_{t+\Delta t})\}$ has to be computed based on Eq. (d) by

$$\{F_R(u_{t+\Delta t})\} = \{F_R(u_t)\} + [K_t]\{\Delta u_t\} \qquad (7.20)$$



$$\boxed{\begin{aligned}\{\Delta F_{Rt}\} &\equiv \{F_R(u_{t+\Delta t})\} - \{F_R(u_t)\} \\ &= [K_t(u_t)]\{\Delta u_t\} \qquad \text{(d)}\end{aligned}}$$

10.3 Idealization of Damping Matrix

- In the analysis of linear MDOF system, damping ratio is assigned for each mode by Eqs. (6.60) and (6.61) after decoupling the equations of motion into n-sets of equation of motion of SDOF system. Hence, it is not needed to formulate damping matrix.
- However in the analysis of nonlinear MDOF system, damping matrix has to be formulated because damping ratio cannot be assigned for each mode by Eqs. (6.60) and (6.61).

$$M_r^* \ddot{q}_r + C_r^* \dot{q}_r + K_r^* q_r = P_r^* \qquad (6.60)$$

$$\frac{C_r^*}{M_r^*} = 2\xi_r \omega_r \qquad (6.61)$$

● Rayleigh damping given by Eq. (6.56) is generally assumed in nonlinear MDOS system.

● Pre-multiplying $[\Phi]^T$ and post-multiplying $[\Phi]$ to Eq. (6.56), one obtains

$$[\Phi]^T[C][\Phi] = \alpha[\Phi]^T[M][\Phi] + \beta[\Phi]^T[K][\Phi] \qquad (10.21)$$

● Based on the orthogonal condition by Eqs. (6.52) and (6.53), one obtains

$$C_i^* = \alpha M_i^* + \beta K_i^* \qquad (7.22)$$

● Dividing Eq. (7.22) by M_i^*,

$$\frac{C_i^*}{M_i^*} = \alpha + \beta \frac{K_i^*}{M_i^*} \qquad (7.23)$$

$$\boxed{[C] = \alpha[M] + \beta[K] \qquad (6.56)}$$

- Since

$$\frac{C_i^*}{M_i^*} = 2\xi_i \omega_i$$

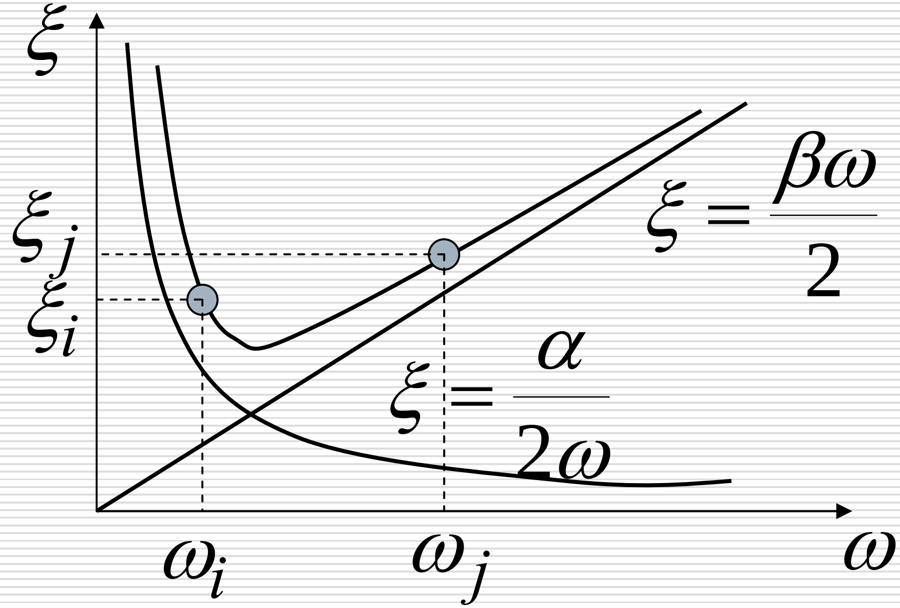
$$\frac{K_i^*}{M_i^*} = \omega_i^2$$

Eq. (7.23) can be written as

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right) \qquad (7.24)$$

$$\boxed{\frac{C_i^*}{M_i^*} = \alpha + \beta\frac{K_i^*}{M_i^*} \qquad (7.23)}$$

- Eq. (7.24) shows that ξ is consisted of two contributions.



- Two parameters α and β can be determined by assigning two pairs of (ω_i, ξ_i) and (ω_j, ξ_j)
- Two modes i and j have to be determined so that the damping ratio at the predominant modes can be captured by Eq. (7.24)

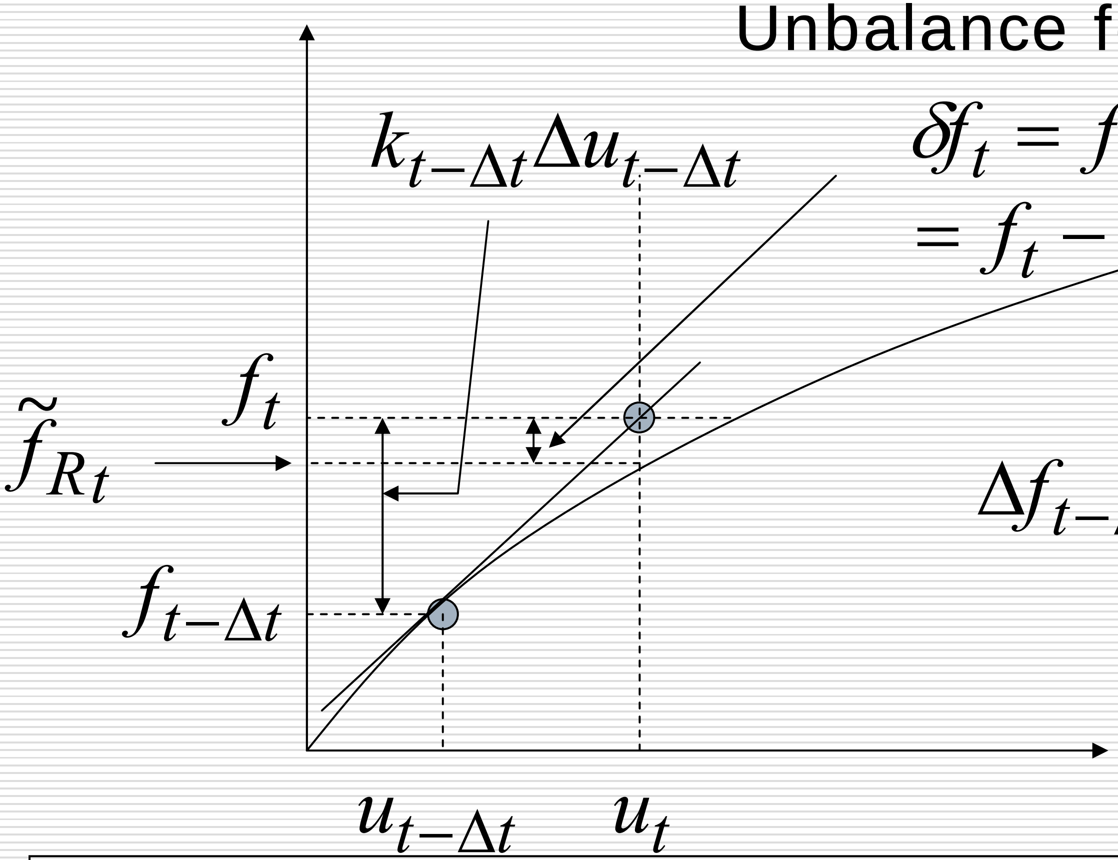
$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right) \qquad (7.24)$$

- α and β are determined as

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \frac{2\omega_i \omega_j}{\omega_j^2 - \omega_i^2} \begin{bmatrix} \omega_j & -\omega_i \\ -1/\omega_j & 1/\omega_i \end{bmatrix} \begin{Bmatrix} \xi_i \\ \xi_j \end{Bmatrix} \qquad (7.25)$$

7.4 Accuracy of Computed Responses (解析結果の精度)

●Approximation of the restoring force by Eq. (10.20) is sometimes insufficient to compute reliable response of a structure.



Unbalance force at time t

$$\delta f_t = f_t - \tilde{f}_{Rt} = f_t - \left(m\ddot{u}_t + c\dot{u}_t + f_{Rt}\right) \qquad (7.26)$$

$$\Delta f_{t-\Delta t} = f_t - f_{t-\Delta t} = k_{t-\Delta t}\Delta u_{t-\Delta t} = k_{t-\Delta t}\left(u_t - u_{t-\Delta t}\right)$$

$$\boxed{\{F_R(u_t)\} = \{F_R(u_{t-\Delta t})\} + [K_{t-\Delta t}]\{\Delta u_{t-\Delta t}\} \quad (7.20)}$$

- By extending the unbalance force of SDOF system to that of MDOF system, accuracy of the computed response at time t is generally represented in terms of unbalance force (**不つり合い力**)

$$\{\delta F_t\} = \{F_t\} - ([M]\{\ddot{u}_t\} + [C]\{\dot{u}_t\} + \{F_{Rt}\})$$

(7.27)

- Amount of the unbalance force can be represented in various ways. One of the expressions may be to define a ratio between the norm of the unbalance force and the norm of the external force as

$$\Delta_P = \frac{\|\delta F_t\|}{\|F_t\|} < \Delta_{PS} \qquad (7.28)$$

where Δ_{PS} is the threshold value (acceptable error)

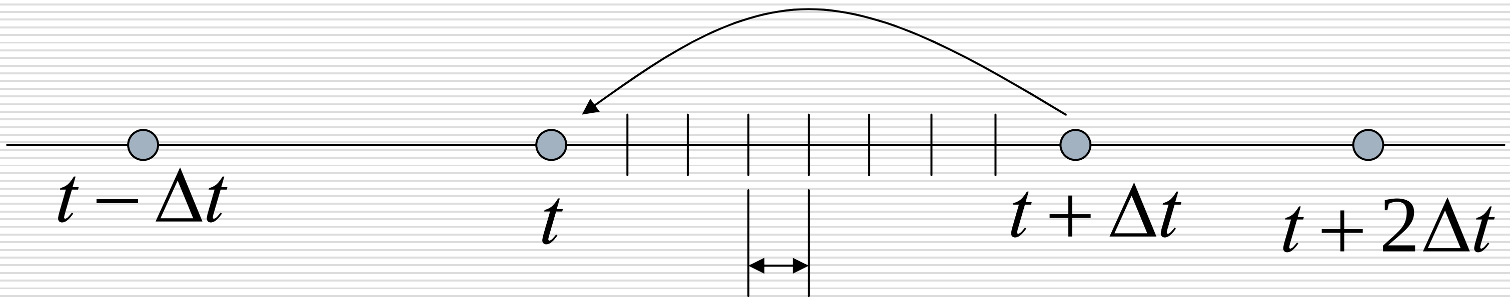
$$\boxed{\delta f_t = f_t - \tilde{f}_{Rt} = f_t - (m\ddot{u}_t + c\dot{u}_t + f_{Rt}) \quad (7.26)}$$

7.5 Improvement of Accuracy of Solutions

1) Use of smaller time interval for numerical integration

- It is always effective to use smaller time interval for numerical integration although computer time required increases.

- If strong nonlinearity exists only at several time intervals, it is useful to subdivide time interval only where the accuracy by Eq. (7.27) is insufficient.



$$\Delta\tilde{t} = \frac{\Delta t}{n}$$

2) Add unbalance force to incremental external force at the next time step

- It is often used to add the unbalance force by Eq. (7.27) to the incremental load in the next time step. Rewriting Eq. (7.27) at time t, the unbalance force at time t is

$$\{\delta F_t\} = \{F_t\} - ([M]\{\ddot{u}_t\} + [C]\{\dot{u}_t\} + \{F_{Rt}\}) \qquad (7.29)$$

and adding this unbalanced force to the incremental load in Eq. (10.12), one obtains

$$[M]\{\Delta\ddot{u}_t\} + [C]\{\Delta\dot{u}_t\} + [K_t]\{\Delta u_t\} = \{\Delta F_t\} + \{\partial F_t\} \qquad (7.30)$$

$$\{\delta F_{t+\Delta t}\} = \{F_{t+\Delta t}\} - ([M]\{\ddot{u}_{t+\Delta t}\} + [C]\{\dot{u}_{t+\Delta t}\} + \{F_{Rt+\Delta t}\}) \qquad (7.27)$$

$$[M]\{\Delta\ddot{u}_t\} + [C]\{\Delta\dot{u}_t\} + [K_t]\{\Delta u_t\} = \{\Delta F_t\} \qquad (7.12)$$

- Based on Eq. (7.10), the right hand side of Eq. (7.30) becomes

$$\{\Delta F_t\}+\{\partial F_t\}=\{F_{t+\Delta t}\}-\{F_t\}+\{\partial F_t\}$$

$$=\{F_{t+\Delta t}\}-\{F_t\}-\{\{F_t\}-([M]\{\ddot{u}_t\}+[C]\{\dot{u}_t\}+\{F_{Rt}\})\}$$

$$=\{F_{t+\Delta t}\}+([M]\{\ddot{u}_t\}+[C]\{\dot{u}_t\}+\{F_{Rt}\})$$

- Thus, Eq. (7.30) becomes

$$[M]\{\Delta\ddot{u}_t\}+[C]\{\Delta\dot{u}_t\}+[K_t]\{\Delta u_t\}$$
$$=\{F_{t+\Delta t}\}+([M]\{\ddot{u}_t\}+[C]\{\dot{u}_t\}+\{F_{Rt}\}) \quad (7.31)$$

$$\{\Delta F_t\}=\{F(t+\Delta t)\}-\{F(t)\} \quad (7.10)$$

$$[M]\{\Delta\ddot{u}_t\}+[C]\{\Delta\dot{u}_t\}+[K_t]\{\Delta u_t\}=\{\Delta F_t\}+\{\partial F_t\} \quad (7.30)$$

- By adding unbalanced force to the incremental force, accuracy of the solution of Eq. (7.12) is generally improved.