Structural Dynamics
構造動力学
(11)

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6.9 Approximate Analysis by Response Spectrum Superposition (応答スペクトルを用いたモード重ね合わせ法による近似的な動的解析)

- The preceding section shows a general dynamic response analysis for linear MDOF structures.
- A simpler method, involving the application of response spectrum techniques, may be used to evaluate the peak seismic response (**最大地震応答**) of structures.
- This approach provides only an approximate indication of the peak response developed in linear MDOF structure.
- Advantage of this method (**応答スペクトル法**) is that this technique saves computer time.

- Response displacement is obtained by Eq. (6.68) as

$$\{u\} = \sum_{i=1}^{n} \{u_i\} \qquad (6.75)$$

where $\{u_i\}$ represents response displacement of the i-th mode and is given as

$$\{u_i\} = \{\phi_i\}\beta_i \tilde{q}_i \qquad (6.76)$$

where $\tilde{q}_r(t)$ is evaluated based on Eq. (4.11) as

$$\tilde{q}_r(t) = -\frac{1}{\omega_D}\int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \sin\omega_D(t-\tau)\,d\tau \qquad (6.76)$$

$$\{u(t)\} = \{\phi_1\}\beta_1\tilde{q}_1 + \{\phi_2\}\beta_2\tilde{q}_2 + \cdots\{\phi_r\}\beta_r\tilde{q}_r + \cdots\{\phi_n\}\beta_n\tilde{q}_n \qquad (6.68)$$

$$u(t) = -\frac{1}{\omega_D}\int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \sin\omega_D(t-\tau)\,d\tau \qquad (4.11)$$

● Because the peak value of $\tilde{q}_i$ is defined as response displacement spectrum $S_D(T,\xi)$ (refer to Eq. (6.73), and because each mass in a normal mode vibration reaches its maximum displacement at the same time for any one mode, the maximum values of response displacement of the i-th mode are given as

$$\{u_{i,\max}\} = \{\phi_i\}\beta_i \tilde{q}_{i,\max} = \{\phi_i\}\beta_i S_D \qquad (6.77)$$

$$S_D(T,\xi) \equiv |u(t)|_{\max} \qquad (6.73)$$

where,

$$u(t) = -\frac{1}{\omega_D}\int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \sin\omega_D(t-\tau)\,d\tau \qquad (4.11)$$

- From Eq. (6.75), the peak response displacement of the structure $\{u_{\max}\}$ can be obtained, if the peak value of response displacement of each mode $\{u_{i,\max}\}$ occurs at the same time

- However generally the peak response displacement of each mode $\{u_{i,\max}\}$ does not occur at the same time, the sum of the maximum absolute value of each mode $\{u_{i,\max}\}$ clearly gives the upper bound to the total system response $\{u_{\max}\}$, which, in general, would too much conservative.

$$\{u_{\max}\} < \sum_{i=1}^{n} \{u_{i,\max}\} \qquad (6.78)$$

$$\boxed{\{u\} = \sum_{i=1}^{n} \{u_i\} \qquad (6.75)}$$

● The error arising from an absolute superposition of the spectral maxima can be overcome by taking the total response as equal to the square root of the sum of the squares of the individual modal maxima.

●This root-mean-square (RMS) approximation (2乗和の平方根法) yields the maximum probable seismic response.

●When the maximum value of each response displacement has been obtained for each mode, the RMS approximation is given by

$$\{u_{\max}\} \approx \sqrt{\{u_{1,\max}{}^2\} + \{u_{2,\max}{}^2\} + \cdots \{u_{n,\max}{}^2\}}$$

$$= \sqrt{\sum_{i=1}^{n} \{u_{i,\max}{}^2\}} \qquad (6.79)$$

● Similarly, the peak velocities and accelerations (absolute accelerations) are approximately obtained by

$$\left\{\ddot{u}_{abs,\max}\right\} \approx \sqrt{\sum_{i=1}^{n}\left\{\ddot{u}_{abs,i,\max}{}^{2}\right\}} \qquad (6.80)$$

$$\left\{\dot{u}_{\max}\right\} \approx \sqrt{\sum_{i=1}^{n}\left\{\dot{u}_{i,\max}{}^{2}\right\}} \qquad (6.81)$$

- For seismic design, forces developed in a structure is important. The force $\{P\}$ can be evaluated from Eq. (6.8) by

$$\{P(t)\} = [K]\{u(t)\}$$

$$= \{P_1(t)\} + \{P_2(t)\} \cdots \{P_n(t)\} \qquad (6.82)$$

where

$$\{P_i(t)\} = [K]\{u_i(t)\} \qquad (6.83)$$

- In the analysis of a structure shown in Fig. 6.1, we have only shear force induced in each spring, however there are generally three forces; that is
  - ✓ Bending moment
  - ✓ Shear force
  - ✓ Axial force

$$\boxed{\{P\} = [K]\{u\} \qquad (6.8)}$$

●From Eq. (6.82), the peak force $\{P_{\max}\}$ may be evaluated by

$$\{P_{\max}\} = [K]\{u_{\max}\} \qquad (6.84)$$

● Note however that the peak response force does not necessarily occur at the same time depending on locations and M-Q-V (moment, shear and axial forces). Hence except very simple structure, the peak response forces are determined directly by Eq. (6.84).

$$\boxed{\{P(t)\} = [K]\{u(t)\} \qquad (6.82)}$$

Example 6.8 Evaluate the peak response displacements and accelerations of the 3 DOF structure of Example 6.5 based on the response superposition method. Use the computed response displacement and absolute accelerations at Example 6.7. Assume that damping ratio is 0.05.

● The maximum response displacements and absolute response accelerations are computed by LDRA-2 as

$$S_D(1.0s, 0.05) = 0.378m$$
$$S_D(0.357s, 0.05) = -0.0836m$$
$$S_D(0.247s, 0.05) = -0.00223m$$

$$S_V(1.0s, 0.05) = 2.45m/s$$
$$S_V(0.357s, 0.05) = 1.39m/s$$
$$S_V(0.247s, 0.05) = -0.453m/s$$

$$S_A(1.0s, 0.05) = -18.3m/s^2$$
$$S_A(0.357s, 0.05) = 22.6m/s^2$$
$$S_A(0.247s, 0.05) = 10.5m/s^2$$

- The peak response displacements are

$$\{u_{1,\max}\} = \beta_1 S_D(1.0, 0.05)\{\phi_1\}$$

$$= 1.221 \times 0.378 \times \begin{Bmatrix} 1.0 \\ 0.802 \\ 0.445 \end{Bmatrix} = \begin{Bmatrix} 0.462 \\ 0.370 \\ 0.205 \end{Bmatrix} \quad \text{(m)}$$

$$\{u_{2,\max}\} = \beta_2 S_D(0.357, 0.05)\{\phi_2\}$$

$$= 0.349 \times (-0.0836) \times \begin{Bmatrix} -0.802 \\ 0.445 \\ 1.00 \end{Bmatrix} = \begin{Bmatrix} 0.0394 \\ -0.0130 \\ -0.0292 \end{Bmatrix} \quad \text{(m)}$$

$$\{u_{3,\max}\} = \beta_3 S_D(0.247, 0.05)\{\phi_3\}$$

$$= -0.134 \times 0.0223 \times \begin{Bmatrix} -0.445 \\ 1.00 \\ -0.802 \end{Bmatrix} = \begin{Bmatrix} -0.00133 \\ 0.00299 \\ -0.00234 \end{Bmatrix} \quad \text{(m)}$$

As we can see from the above results, the peak displacements become smaller at higher modes.

● The RMS approximation of the peak response displacements are

$$\{u_{\max}\} \approx \sqrt{\begin{Bmatrix} 0.452^2 \\ 0.362^2 \\ 0.201^2 \end{Bmatrix} + \begin{Bmatrix} 0.0228^2 \\ (-0.0126)^2 \\ (-0.0284)^2 \end{Bmatrix} + \begin{Bmatrix} (-0.00130)^2 \\ 0.00292^2 \\ (-0.00234)^2 \end{Bmatrix}}$$

$$= \begin{Bmatrix} 0.453 \\ 0.362 \\ 0.203 \end{Bmatrix} \quad \longleftrightarrow \quad \begin{Bmatrix} 0.452 \\ 0.362 \\ 0.201 \end{Bmatrix}$$

Exact values (Refer to Example 9.6)

- Note in the example in the previous page, this is a good example to show that the spectral method provides good approximation to the exact response.
- However such a close approximation was obtained because the structure is very simple cantilevered shape, thus the contribution of the fundamental mode was significantly predominant. For more realistic structures, the accuracy of response spectral method does not provide such a good approximation.

- The peak absolute response accelerations are

$$\{\ddot{u}_{abs,1,\max}\} = \beta_1 S_A(1.0, 0.05)\{\phi_1\}$$

$$= 1.221 \times (-15.1) \times \begin{Bmatrix} 1.0 \\ 0.802 \\ 0.445 \end{Bmatrix} = \begin{Bmatrix} -18.4 \\ -14.8 \\ -8.20 \end{Bmatrix} (m/s^2)$$

$$\{\ddot{u}_{abs,2,\max}\} = \beta_2 S_A(0.357, 0.05)\{\phi_2\}$$

$$= 0.349 \times 25.5 \times \begin{Bmatrix} -0.802 \\ 0.445 \\ 1.00 \end{Bmatrix} = \begin{Bmatrix} -8.10 \\ 3.96 \\ 8.90 \end{Bmatrix} (m/s^2)$$

$$\{\ddot{u}_{abs,3,\max}\} = \beta_3 S_A(0.247, 0.05)\{\phi_3\}$$

$$= -0.134 \times 14.4 \times \begin{Bmatrix} -0.445 \\ 1.00 \\ -0.802 \end{Bmatrix} = \begin{Bmatrix} 0.859 \\ -1.93 \\ 1.55 \end{Bmatrix} (m/s^2)$$

- The RMS approximation of the peak relative response accelerations are

$$\{\ddot{u}_{abs,\max}\} \approx \sqrt{\begin{Bmatrix} (-18.4)^2 \\ (-14.8)^2 \\ (-8.20)^2 \end{Bmatrix} + \begin{Bmatrix} (-8.10)^2 \\ 3.96^2 \\ 8.90^2 \end{Bmatrix} + \begin{Bmatrix} 0.859^2 \\ (-1.93)^2 \\ 1.55^2 \end{Bmatrix}}$$

$$= \begin{Bmatrix} 20.1 \\ 15.4 \\ 12.2 \end{Bmatrix} (m/s^2)$$

(m/s^2)

● Note that because ground acceleration is included in the evaluation of the absolute acceleration, the approximation of absolute acceleration to the exact values is generally poor. Hence, if exact response values are required, it has to be computed by the direct time step analysis.