Structural Dynamics 構造動力学 (10)

Kazuhiko Kawashima Department of Civil Engineering Tokyo institute of Technology 東京工業大学大学院理工学研究科土木工学専攻 川島一彦 6.7 Equations of Motion of MDOF System Subjected to Earthquake Ground Motion (地震 動を受ける多自由度系の運動方程式)

• Based on Eq. (6.16), the equations of motion of a MDOF system subjected to a ground motion \ddot{u}_g are

$$[M]{\ddot{u}} + [K]{u} = -\ddot{u}_g[M]{I}$$
(6.16)

• Based on Eq. (6.42c), the generalized force (一般 化された荷重) for the r-th mode P_{μ}^{*} is

$$P_r^* = \{\phi_r\}^T \{P\} = -\ddot{u}_g \{\phi_r\}^T [M] \{I\}$$
(6.64)

$$[M]{\ddot{u}} + [K]{u} = {P} - \ddot{u}_g[M]{I}$$
(616)
$$P_r^* = {\phi_r}^T {P}$$
(6.42c)

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 Based on Eq. (6.60), the equation of motion for the r-th mode is written in the form of Eq. (9.62), and its solution is given by Eq. (9.63).

$$M_{r}^{*}\ddot{q}_{r} + C_{r}^{*}\dot{q}_{r} + K_{r}^{*}q_{r} = P_{r}^{*} \qquad (6.60)$$

$$\ddot{q}_{r} + 2h_{r}\omega_{r}\dot{q}_{r} + \omega_{r}^{2}q_{r} = \frac{P_{r}^{*}}{M_{r}^{*}} \qquad (6.62)$$

$$q_{r}(t) = e^{-h_{r}\omega_{nr}t} \left\{ q_{r}(0)\cos\omega_{d}t + (\frac{\dot{q}_{r}(0)}{\omega_{dr}} + \frac{h_{r}\omega_{nr}q_{r}(0)}{\omega_{dr}})\sin\omega_{dr}t \right\}$$

$$+ \frac{1}{M_{r}^{*}\omega_{dr}} \int_{0}^{t} P_{r}^{*}(\tau)e^{-h\omega_{nr}(t-\tau)}\sin\omega_{dr}(t-\tau)d\tau \qquad (6.63)$$

• Because it is likely for a MDOF system under a seismic ground motion that the initial velocity and displacements are zero, Eq. (6.63) becomes

$$q_{r} = \frac{1}{M_{r}^{*}\omega_{dr}} \int_{0}^{t} P_{r}^{*}(\tau) e^{-h\omega_{nr}(t-\tau)} \sin \omega_{dr}(t-\tau) d\tau \quad (6.65)$$

$$\left| q_{r}(t) = e^{-h_{r}\omega_{nr}t} \left\{ q_{r}(0)\cos\omega_{d}t + (\frac{\dot{q}_{r}(0)}{\omega_{dr}} + \frac{h_{r}\omega_{nr}q_{r}(0)}{\omega_{dr}})\sin\omega_{dr}t \right\} + \frac{1}{M_{r}^{*}\omega_{dr}} \int_{0}^{t} P_{r}^{*}(\tau)e^{-h\omega_{nr}(t-\tau)}\sin\omega_{dr}(t-\tau)d\tau \quad (6.63)$$

• Substitution of Eq. (6.64) into Eq. (6.65) leads to

$$q_r = -\frac{\beta_r}{\omega_{dr}} \int_0^t \ddot{u}_g(\tau) e^{-\xi_r \omega_{nr}(t-\tau)} \sin \omega_{dr}(t-\tau) d\tau$$
(6.66a)

n

(6.66b)

where

$$\beta_{r} = \frac{\{\phi_{r}\}^{T} [M]\{I\}}{M_{r}^{*}} = \frac{\{\phi_{r}\}^{T} [M]\{I\}}{\{\phi_{r}\}^{T} [M]\{\phi_{r}\}} = \frac{\sum m_{i} \phi_{ir}}{\sum i=1}^{n} \frac{m_{i} \phi_{ir}}{\sum m_{i} \phi_{ir}}$$

in which β_r is called mode participation factor of the r-th mode (r次の刺激係数)

$$P_{r}^{*} = \{\phi_{r}\}^{T}\{P\} = -\ddot{u}_{g}\{\phi_{r}\}^{T}[M]\{I\}$$
(6.64)
$$q_{r} = \frac{1}{M_{r}^{*}\omega_{dr}} \int_{0}^{t} P_{r}^{*}(\tau) e^{-\xi_{r}\omega_{nr}(t-\tau)} \sin \omega_{dr}(t-\tau) d\tau$$
(6.65)

• Denoting Eq. (6.66a) below

$$\widetilde{q}_r = -\frac{1}{\omega_{dr}} \int_0^t \ddot{u}_g(\tau) e^{-\xi_r \omega_{nr}(t-\tau)} \sin \omega_{dr}(t-\tau) d\tau$$

(6.66a)

represents solution of a SDOF system subjected to a ground motion $\ddot{u}_g(t)$, Eq. (6.66a) can be rewritten

$$q_r(t) = \beta_r \tilde{q}_r(t) \tag{6.67}$$

• It should be noted that $\tilde{q}_r(t)$ can be computed by the direct integration method (Newmark's β method)

• Eq. (6.67) shows that the mode participation factor of the r-th mode represents degree of contribution of the r-th mode to the total response. •When $|\beta_r|$ is large, the contribution of the r-th mode is predominant

• The generalize mass M_r^* (一般化された質量) given by Eq. (6.42a) is also important to show how the r-th mode is predominant to the total response.

$$q_{r}(t) = \beta_{r} \tilde{q}_{r}(t)$$
 (6.67)
$$M_{r}^{*} = \{\phi_{r}\}^{T} [M] \{\phi_{r}\}$$
 (6.42a)

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$\{u\} = [\Phi] \{q\} = \{\phi_1\} \beta_1 \tilde{q}_1 + \{\phi_2\} \beta_2 \tilde{q}_2 + \dots \{\phi_r\} \beta_r \tilde{q}_r + \dots \{\phi_n\} \beta_n \tilde{q}_n$ $\{\ddot{u}\} = [\Phi] \{\ddot{q}\} = \{\phi_1\} \beta_1 \tilde{q}_1 + \{\phi_2\} \beta_2 \tilde{q}_2 + \dots \{\phi_r\} \beta_r \tilde{q}_r + \dots \{\phi_n\} \beta_n \tilde{q}_n$ (6.68) $\{\dot{u}\} = [\Phi] \{\dot{q}\} = \{\phi_1\} \beta_1 \tilde{q}_1 + \{\phi_2\} \beta_2 \tilde{q}_2 + \dots \{\phi_r\} \beta_r \tilde{q}_r + \dots \{\phi_n\} \beta_n \tilde{q}_n$ (6.70)

• It is noted that \tilde{q}_r , \tilde{q}_r and $\tilde{\ddot{q}}_r$ are the response displacement, velocity and acceleration of SDOF system with natural frequency ω_r and damping ratio ξ_r . •Hence, \tilde{q}_r , \tilde{q}_r and $\tilde{\ddot{q}}_r$ can be easily computed by Excel Sheet (LDRA-2) based on Newmark β method • It should be noted that the acceleration response $\{\ddot{u}\}\$ by Eq. (6.69) is the relative acceleration. Because absolute acceleration $\{\ddot{u}_{abs}\}\$ is important for evaluation of the inertia force, $\{\ddot{u}_{abs}\}\$ has to be computed by

$$\{\ddot{u}_{abs}\} = \{\ddot{u}\} + \{I\}\ddot{u}_g \tag{6.71}$$

$$\{\ddot{u}\} = [\Phi]\{\ddot{q}\} = \{\phi_1\}\beta_1\tilde{q}_1 + \{\phi_2\}\beta_2\tilde{q}_2 + \cdots \{\phi_r\}\beta_r\tilde{q}_r + \cdots \{\phi_n\}\beta_n\tilde{q}_n$$
(6.69)



Example 6.5 Evaluate the mode participation factor for a 3 DOFS structure shown



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• The characteristic equation is

$$\begin{vmatrix} k_1 - \omega^2 m & -k_1 & 0 \\ -k_1 & k_1 + k_2 - \omega^2 m & -k_2 \\ 0 & -k_2 & k_2 + k_3 - \omega^2 m \end{vmatrix} = 0$$

• Hence

 $k - \omega^2 150/9.8$ -k \cap $-k \qquad 2k - \omega^2 150/9.8$ -k= 0 $-k \qquad 2k - \omega^2 150/9.8$ ()

Natural periods and natural mode shapes are

 $\omega_1 = 6.28 rad / s$ $T_1 = 1.0s$ $\omega_2 = 17.6 rad / s$ $T_2 = 0.357 s$ $\omega_3 = 25.4 rad / s$ $T_3 = 0.247 s$





• Modal matrix $\begin{bmatrix} \Phi \end{bmatrix} = \begin{bmatrix} 1.00 & -0.802 & -0.445 \\ 0.802 & 0.445 & 1.00 \\ 0.445 & 1.00 & -0.802 \end{bmatrix}$

Check of the orthogonal condition

$$[\Phi]^{T}[M][\Phi] = [\Phi]^{T} m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\Phi]$$

$$= 1.841m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Generalized mass

 $M_1^* = M_2^* = M_3^* = 1.841m = 1.841 \times 150/9.8 = 28.18$ kNs^2/m

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Check of the orthogonal condition

$$\begin{bmatrix} \Phi \end{bmatrix}^{T} \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} \Phi \end{bmatrix} = \begin{bmatrix} \Phi \end{bmatrix}^{T} k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \Phi \end{bmatrix}$$
$$= k \begin{bmatrix} 0.365 & 0 & 0 \\ 0 & 2.863 & 0 \\ 0 & 0 & 5.978 \end{bmatrix}$$

• Generalized stiffness $K_1^* = 0.365k = 1,113kN / m$ $K_2^* = 2.863k = 8,735kN / m$ $K_3^* = 5.978k = 18,238kN / m$

• Check of Eq. (6.44)

$$\omega_{1} = \sqrt{\frac{K_{1}^{*}}{M_{1}^{*}}} = \sqrt{\frac{1113}{28.79}} = 6.28rad/s \leftrightarrow 6.28rad/s$$
OK

$$\omega_{2} = \sqrt{\frac{K_{2}^{*}}{M_{2}^{*}}} = \sqrt{\frac{8735}{28.79}} = 17.6rad/s \leftrightarrow 17.4rad/s$$
OK

$$\omega_{3} = \sqrt{\frac{K_{3}^{*}}{M_{3}^{*}}} = \sqrt{\frac{18238}{28.79}} = 25.4/s \leftrightarrow 25.4rad/s$$
OK

$$\omega_{r} = \sqrt{\frac{K_{r}^{*}}{M_{r}^{*}}} = \sqrt{\frac{\{\phi_{r}\}^{T}[K]\{\phi_{r}\}}{\{\phi_{r}\}^{T}[M]\{\phi_{r}\}}} \quad (6.44)$$
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 Compute mode participation factor n $\beta_{r} = \frac{\{\phi_{r}\}^{T} [M]\{I\}}{M_{r}^{*}} = \frac{\{\phi_{r}\}^{T} [M]\{I\}}{\{\phi_{r}\}^{T} [M]\{\phi_{r}\}} = \frac{\sum_{i=1}^{n} m_{i}\phi_{ir}}{\sum_{i=1}^{n} m_{i}\phi_{ir}^{2}} \quad (6.66b)$ i=1 $\beta_1 = \{\phi_1\}^T [M] \{I\} / M_1^*$ $= \{1.0 \quad 0.802 \quad 0.445\} \times m \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} / M_1^*$ $=\frac{2.247m}{1.841m}=1.221$ $M_1^* = M_2^* = M_3^* = 1.841m = 1.841 \times 150/9.8 = 28.18$ kNs^{29}/m

$$\beta_{2} = \{\phi_{2}\}^{T} [M] \{I\} / M_{2}^{*}$$

$$= \{-0.802 \quad 0.445 \quad 1.00\} \times m \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} / M_{2}^{*}$$

$$= \frac{0.643m}{1.841m} = 0.349$$

$$\beta_{3} = \{\phi_{3}\}^{T} [M] \{I\} / M_{3}^{*}$$

$$= \{-0.445 \quad 1.00 \quad -0.802\} \times m \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} / M_{3}^{*}$$

$$= \frac{-0.247m}{1.841m} = -0.134$$

Mode participation factors are

 $\omega_1 = 6.28 rad / s$ $T_1 = 1.0s$ $\beta_1 = 1.221$ $\omega_2 = 17.6 rad / s$ $T_2 = 0.357 s$ $\beta_2 = 0.349$ $\omega_3 = 25.4 rad / s$ $T_3 = 0.247 s$ $\beta_3 = -0.134$





Example 6.6 Compute response displacement, velocity and acceleration (absolute acceleration) of the 3 DOFS structure of Example 6.5 under JMA Kobe ground motion. Assume that damping ratio is 0.05 for the three modes

 Compute response displacement, velocity and acceleration (absolute acceleration) of 3 DOF system based on Eqs. (6.68), (6.69), (6.70), and (6.71) using LDRA-2 as follows.

 The mode participation factors are already computed in Example 3.

• Compute \tilde{q}_r , \tilde{q}_r , \tilde{q}_r using LDRA-2. Note that mass M_r^* , natural period $T_r = 2\pi / \omega_r$ and damping ratio ξ_r have to be set in LDRA-2.



$$q_r(t) = \beta_r \tilde{q}_r(t)$$
$$\dot{q}_r(t) = \beta_r \tilde{\dot{q}}_r(t)$$
$$\ddot{q}_r(t) = \beta_r \tilde{\dot{q}}_r(t)$$

$$\{u(t)\} = \{u_1(t)\} + \{u_2(t)\} + \{u_3(t)\}$$

$$= \{\phi_1\}q_1(t) + \{\phi_2\}q_2(t) + \{\phi_3\}q_3(t)$$

$$\{\dot{u}(t)\} = \{\dot{u}_1(t)\} + \{\dot{u}_2(t)\} + \{\dot{u}_2(t)\}$$

$$= \{\phi_1\}\dot{q}_1(t) + \{\phi_2\}\dot{q}_2(t) + \{\phi_3\}\dot{q}_3(t)$$

$$\{\ddot{u}(t)\} = \{\ddot{u}_1(t)\} + \{\ddot{u}_2(t)\} + \{\ddot{u}_2(t)\}$$

$$= \{\phi_1\}\ddot{q}_1(t) + \{\phi_2\}\ddot{q}_2(t) + \{\phi_3\}\ddot{q}_3(t)$$

• Compute absolute acceleration by

$$\{\ddot{u}_{abs}\} = \{\ddot{u}\} + \{I\}\ddot{u}_g \tag{6.71}$$

Example 5 Time History Analysis of a 3 DOF structure

• Consider a 3 DOF structure shown below.

$$m$$

$$m$$

$$k_1$$

$$m$$

$$k_2$$

$$k_3$$

$$m = 150kN / g = 150/9.8 kNs^2 / m$$
$$k_1 = k_2 = k_3 = k = 3050.9kN / m$$







Peak Response Displacements and Time when they Occur

Mode	1st Mode	2nd Mode	3rd mode	Total
Mass 1	0.462m	0.0234m	-0.00133m	0.465m
	(5.18s)	(7.35s)	(8.54s)	(5.18s)
Mass 2	0.370m	-0.0130m	0.00298m	0.369m
	(5.18s)	(7.35s)	(8.54s)	(5.18s)
Mass 3	0.205m	-0.0292m	-0.00239m	0.200m
	(5.18s)	(7.35s)	(8.54s)	(5.18s)

Peak Response Velocity and Time when they Occur

Mode	1st Mode	2nd Mode	3rd mode	Total
Mass 1	2.99m/s	-0.388m/s	-0.0270m/s	3.00m/s
	(4.96s)	(7.44s)	(8.47s)	(4.96s)
Mass 2	2.40m/s	0.215m/s	0.0607m/s	2.38m/s
	(4.96s)	(7.44s)	(8.46s)	(4.96s)
Mass 3	1.33m/s	0.484m/s	-0.0487m/s	0.132m/s
	(4.96s)	(7.44s)	(8.47s)	(4.96s)

Peak relative acceleration response and time when they occur

Mode	1st Mode	2nd Mode	3rd mode	Total
Mass 1	-22.3m/s ²	-6.34m/s²	0.628m/s ²	-22.7m/s ²
	(5.18s)	(7.35s)	(8.53s)	(5.18s)
Mass 2	-17.9m/s ²	3.52m/s ²	-1.41m/s ²	-18.1m/s ²
	(5.18s)	(7.35s)	(8.53s)	(5.15s)
Mass 3	-9.94m/s²	7.90m/s ²	1.13m/s ²	-12.5m/s ²
	(5.18s)	(7.35s)	(8.53s)	(7.17s)

Peak absolute acceleration response and time when they occur

Mode	1st Mode	2nd Mode	3rd mode	Total
Mass 1	-22.3m/s ²	-6.34m/s ²	0.628m/s ²	-22.7m/s ²
	(5.18s)	(7.35s)	(8.53s)	(5.18s)
Mass 2	-17.9m/s ²	3.52m/s ²	-1.41m/s ²	-18.1m/s²
	(5.18s)	(7.35s)	(8.53s)	(5.15s)
Mass 3	-9.94m/s²	7.90m/s ²	1.13m/s ²	-12.5m/s ²
	(5.18s)	(7.35s)	(8.53s)	(7.17s)

Comparison of Peak Relative Acceleration and the Absolute acceleration

Mode	Relative acceleration	Absolute acceleration
Mode 1	-22.7m/s ²	-19.5m/s ²
	(5.18s)	(5.18s)
Mode 2	-18.1m/s ²	-14.8m/s ²
	(5.15s)	(5.16s)
Mode 3	-12.5m/s ²	-10.1m/s ²
	(7.17s)	(7.16s)



Since the first mode is dominant in the relative displacement, the variation of the peak relative displacement is close to the mode shape







6.8 Earthquake Response Spectra (地震応答スペク

Response displacement, velocity and absolute acceleration of a SDOF system subjected to a ground acceleration are given by Eqs. (4.11), (4.12), and (4.13)

$$u(t) = \frac{1}{\omega_D} \int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \sin \omega_D(t-\tau) d\tau \quad (4.11)$$

$$\dot{u}(t) = -\int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \cos \omega_D(t-\tau) d\tau + \frac{\xi}{\sqrt{1-\xi^2}} \int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \sin \omega_D(t-\tau) d\tau$$

$$\ddot{u}_{abs}(t) = \ddot{u}(t) + \ddot{u}_g(t) \quad (4.12)$$

$$= \omega_D \int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \left\{ \left(1 - \frac{\xi^2}{1-\xi^2} \sin \omega_D(t-\tau) \right) + \frac{2\xi}{\sqrt{1-\xi^2}} \cos \omega_D(t-\tau) d\tau \right\} \quad (4.13)$$

• Response acceleration spectrum $S_A(T,\xi)$ is defined as the peak absolute response acceleration of a SDOF oscillator, i.e., from Eq. (4.13)

$$S_A(T,\xi) \equiv \left| \ddot{u}_{abs}(t) \right|_{\max} \tag{6.72}$$

where

$$\ddot{u}_{abs}(t) = \omega_D \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \left\{ \left\{ 1 - \frac{\xi^2}{1 - \xi^2} \sin \omega_D (t-\tau) \right\} + \frac{2\xi}{\sqrt{1 - \xi^2}} \cos \omega_D (t-\tau) d\tau \right\}$$
(4.13)

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• Similarly, response displacement spectrum $S_D(T,\xi)$ and response velocity spectrum $S_V(T,\xi)$ are defined as the peak response displacement and velocity of a SDOF oscillator

$$S_D(T,\xi) \equiv |u(t)|_{\max} \tag{6.73}$$

where,

$$u(t) = \frac{1}{\omega_D} \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \sin \omega_D (t-\tau) d\tau \quad (4.11)$$

$$S_V(T,\xi) \equiv \left| \dot{u}(t) \right|_{\max}$$
(6.74)
where
$$\dot{u}(t) = -\int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \cos \omega_D (t-\tau) d\tau$$
$$+ \frac{\xi}{\sqrt{1-\xi^2}} \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \sin \omega_D (t-\tau) d\tau$$
(4.12)
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Ground acceleration recorded at Kobe Observatory of Japan Meteorological Agency One of the most significant ground motions ever recorded



Typical Near-Field ground Accelerations (代 表的な断層近傍地震動加速度)



Response Accelerations of Typical Near-Field Acceleration Records



 Design force is generally represented in terms of response acceleration spectra

Design Specifications of Highway Bridges Japan Road Association

 $\xi = 0.05$

Function Evaluation



