# Structural Dynamics
# 構造動力学
# (8)

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# 6.5 Equations of Motion for Undamped MDOF Systems in General Coordinate (基準座標で表した非減衰系の運動方程式)

●Assume to represent the response displacement vector (**相対変位ベクトル**) and the response acceleration vector (**相対加速度ベクトル**) as a linear combination of the mode shape vector (**振動モードベクトル**) and the generalized coordinate (**一般化された座標**)

Space function (**空間座標**)

Time function (**時間関数**)

$$\{u(x,t)\} = [\Phi(x)]\{q(t)\}$$

$$\{\ddot{u}(x,t)\} = [\Phi(x)]\{\ddot{q}(t)\} \quad (6.34)$$

where,

$\{q(t)\}$: generalized coordinate (**一般化された座標**)

●Generalized coordinate (一般化された座標) $\{q(t)\}$ is defined as

$$\{q(t)\} = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ . \\ q_i(t) \\ . \\ q_n(t) \end{Bmatrix} \qquad \{\ddot{q}(t)\} = \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \\ . \\ \ddot{q}_i(t) \\ . \\ \ddot{q}_n(t) \end{Bmatrix} \qquad (6.35)$$

in which $q_i(t)$ and $\ddot{q}_i(t)$ represents displacement quantity and acceleration quantity associated with the i-th mode normalized by the mode shape

- Assuming undamped MDOF system with $\ddot{u}_g = 0$ the equations of motion are written from Eq. (6.16) as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{P\} \qquad (6.36)$$

- Substituting Eq. (6.34), Eq. (6.36) becomes

$$[M][\Phi]\{\ddot{q}\} + [K][\Phi]\{q\} = \{P\} \qquad (6.37)$$

$$[M]\{\ddot{u}\} + [K]\{u\} = \{P\} - \ddot{u}_g [M]\{I\} \qquad (6.16)$$

$$\{u\} = [\Phi]\{q\}$$
$$\{\ddot{u}\} = [\Phi]\{\ddot{q}\} \qquad (6.34)$$

- Recalling Eq. (6.25),

$$[\Phi] = [\phi_1 \quad \phi_2 \quad \cdot \quad \cdot \quad \phi_n]$$

$$= \begin{bmatrix} \phi_{11} & \phi_{12} & \cdot & \cdot & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdot & \cdot & \phi_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{n1} & \cdot & \cdot & \cdot & \phi_{nn} \end{bmatrix} \quad (6.25)$$

$$\phi_1 \qquad \phi_2$$

$[\Phi]\{q\}$ can be written as

$$[\Phi]\{q\} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdot & \cdot & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdot & \cdot & \phi_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{n1} & \cdot & \cdot & \cdot & \phi_{nn} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ q_n \end{Bmatrix}$$

$$\begin{aligned} &= \phi_{11}q_1 + \phi_{12}q_2 + \cdots\cdots + \phi_{1r}q_r + \cdots\cdots + \phi_{1n}q_n \\ &+ \phi_{21}q_1 + \phi_{22}q_2 + \cdots\cdots + \phi_{2r}q_r + \cdots\cdots + \phi_{2n}q_n \\ &\qquad\vdots \\ &+ \phi_{r1}q_1 + \phi_{r2}q_2 + \cdots\cdots + \phi_{rr}q_r + \cdots\cdots + \phi_{rn}q_n \\ &\qquad\cdot \\ &+ \phi_{n1}q_1 + \phi_{n2}q_2 + \cdots\cdots + \phi_{nr}q_r + \cdots\cdots + \phi_{nn}q_n \end{aligned}$$

● $[\Phi]\{q\}$ can be arranged as

$$
\begin{aligned}
[\Phi]\{q\} = {} & \phi_{11}q_1 + \phi_{12}q_2 + \cdots\cdots + \phi_{1r}q_r + \cdots\cdots + \phi_{1n}q_n \\
& + \phi_{21}q_1 + \phi_{22}q_2 + \cdots\cdots + \phi_{2r}q_r + \cdots\cdots + \phi_{2n}q_n \\
& \vdots \\
& + \phi_{r1}q_1 + \phi_{r2}q_2 + \cdots\cdots + \phi_{rr}q_r + \cdots\cdots + \phi_{rn}q_n \\
& \vdots \\
& + \phi_{n1}q_1 + \phi_{n2}q_2 + \cdots\cdots + \phi_{nr}q_r + \cdots\cdots + \phi_{nn}q_n
\end{aligned}
$$

$\{\phi_1\}$

$$
= \begin{Bmatrix} \phi_{11} \\ \cdot \\ \phi_{i1} \\ \cdot \\ \phi_{i1} \end{Bmatrix} q_1(t) + \begin{Bmatrix} \phi_{12} \\ \cdot \\ \phi_{i2} \\ \cdot \\ \phi_{n2} \end{Bmatrix} q_2(t) + \cdots\cdots \begin{Bmatrix} \phi_{1n} \\ \cdot \\ \phi_{in} \\ \cdot \\ \phi_{nn} \end{Bmatrix} q_n(t)
$$

- Hence,

$$[\Phi]\{q\} = \{\phi_1\}q_1 + \{\phi_2\}q_2 + \cdots\cdots\{\phi_r\}q_r + \cdots\cdots\{\phi_n\}q_n$$

(6.38)

- Similarly,

$$[\Phi]\{\ddot{q}\} = \{\phi_1\}\ddot{q}_1 + \{\phi_2\}\ddot{q}_2 + \cdots\cdots\{\phi_r\}\ddot{q}_r + \cdots\cdots\{\phi_n\}\ddot{q}_n$$

(6.39)

- Premultiplying $\{\phi_r\}^T$ to Eq. (6.37),

$$\{\phi_r\}^T[M][\Phi]\{\ddot{q}\}+\{\phi_r\}^T[K][\Phi]\{q\}=\{\phi_r\}^T\{P\}$$

- Then substitution of Eq. (6.25) leads to

$$\{\phi_r\}^T[M][\phi_1 \quad \phi_2 \quad \cdot \quad \cdot \quad \phi_r \quad \cdot \quad \cdot \quad \phi_n]\{\ddot{q}\}$$
$$+\{\phi_r\}^T[K][\phi_1 \quad \phi_2 \quad \cdot \quad \cdot \quad \phi_r \quad \cdot \quad \cdot \quad \phi_n]\{q\}=\{\phi_r\}^T\{P\} \tag{6.40}$$

- From the orthogonal condition by Eq. (6.33),

$$[\Phi]=[\phi_1 \quad \phi_2 \quad \cdot \quad \cdot \quad \phi_n] \tag{6.25}$$

$$\{\phi_r\}^T[M]\{\phi_s\}=0 \tag{6.33a}$$

$$\{\phi_r\}^T[K]\{\phi_s\}=0 \tag{6.33b}$$

$$[M][\Phi]\{\ddot{q}\}+[K][\Phi]\{q\}=\{P\} \tag{6.37}$$

Eq. (6.40) becomes

$$\{\phi_r\}^T[M]\{\phi_r\}\{\ddot{q}_r\}+\{\phi_r\}^T[K]\{\phi_r\}\{q_r\}=\{\phi_r\}^T\{P\} \tag{6.41}$$

because all the term $\{\phi_r\}^T[M]\{\phi_s\}\quad (r \neq s)$ vanish.

- Representing

$$M_r^* = \{\phi_r\}^T[M]\{\phi_r\} \tag{6.42a}$$

$$K_r^* = \{\phi_r\}^T[K]\{\phi_r\} \tag{6.42b}$$

$$P_r^* = \{\phi_r\}^T\{P\} \tag{6.42c}$$

$$\{\phi_r\}^T[M][\phi_1 \quad \phi_2 \quad \cdot \quad \cdot \quad \phi_r \quad \cdot \quad \cdot \quad \phi_n]\{\ddot{q}\} + \{\phi_r\}^T[K][\phi_1 \quad \phi_2 \quad \cdot \quad \cdot \quad \phi_r \quad \cdot \quad \cdot \quad \phi_n]\{q\}=\{\phi_r\}^T\{P\} \tag{6.40}$$

Eq. (6.41) can be expressed

$$M_r^* \ddot{q}_r + K_r^* q_r = P_r^* \qquad (6.43)$$

in which $M_r^*$ , $K_r^*$ and $P_r^*$ are called generalized mass, generalized stiffness and generalized force of the r-th mode (**一般化されたr次の質量、一般化されたr次のばね定数、一般化されたr次の荷重**).

● Eq. (6.43) is an uncoupled equation of the SDOF system representing the dynamic equilibrium for the r-th mode of the system.

●A similar independent equation may be written for each of the n-modes of oscillation.

$$\{\phi_r\}^T [M]\{\phi_r\}\{\ddot{q}\} + \{\phi_r\}^T [K]\{\phi_r\}\{q\} = \{\phi_r\}^T \{P\} \qquad (6.41)$$

● It is emphasized that n-th coupled dynamic equations of motion of MDOF system with n degree-of-freedom system (Eq. (6.41))(**n自由度を持つ連成した運動方程式**) was decoupled into n-sets of dynamic equation of motion of SDOF system (**n個の連成していない(非連成)の**1**自由度系の運動方程式に分解できた**) using the orthogonal condition (**直交条件**).

●This is the great advantage of using the orthogonal condition.

$$\{\phi_r\}^T[M]\{\phi_r\}\{\ddot{q}_r\} + \{\phi_r\}^T[K]\{\phi_r\}\{q_r\} = \{\phi_r\}^T\{P\} \quad (6.41)$$

$$M_r^*\ddot{q}_r + K_r^* q_r = P_r^* \quad (6.43)$$

● From Eqs. (6.42) and (6.43), one obtains the r-th angular natural frequency as

$$\omega_r = \sqrt{\frac{K_r^*}{M_r^*}} = \sqrt{\frac{\{\phi_r\}^T[K]\{\phi_r\}}{\{\phi_r\}^T[M]\{\phi_r\}}} \qquad (6.44)$$

$$M_r^* = \{\phi_r\}^T[M]\{\phi_r\} \qquad (6.42a)$$

$$K_r^* = \{\phi_r\}^T[K]\{\phi_r\} \qquad (6.42b)$$

$$M_r^*\ddot{q}_r + K_r^* q_r = P_r^* \qquad (6.43)$$