Structural Dynamics
構造動力学
(6)

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CHAPTER 5 RESPONSE TO GENERAL DYNAMIC LOADING:STEP-BY-STEP METHOD (一般的な動的荷重に対する応答 (逐次積分法)

5.3 and 5.4 are described based on Kawashima’s note

5.1 General Concept

● Because superposition which is used in the Duhamel integration is applied only to linear elastic structure, it cannot be used to structures which induce inelastic deformation (**非線形域の変形を生じる構造物**).

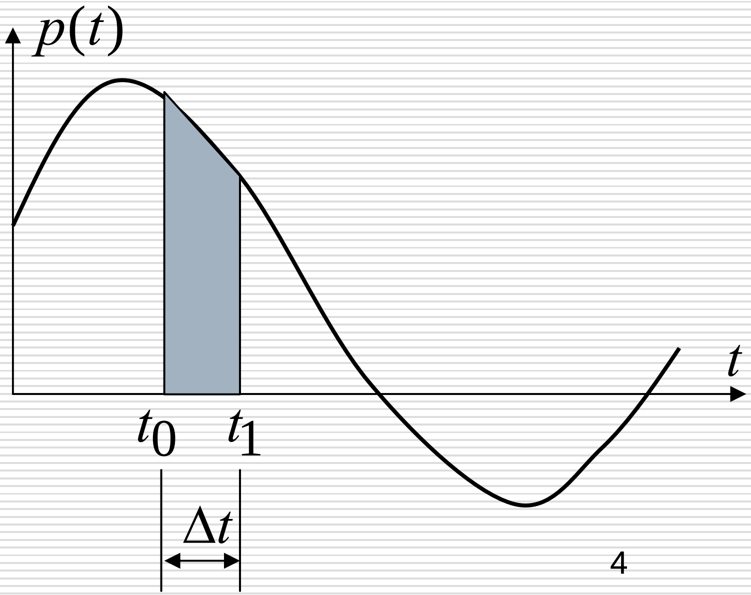
●The step-by-step integration procedure (**時刻歴応答解析法**) is a general approach to dynamic response analysis, and it is well suited to analysis of nonlinear response because it avoids any use of superposition.

●The step-by-step method provides the only completely general approach to analysis of nonlinear response; however, the method is equally valuable in the analysis of linear response because the same algorithms can be applied regardless of whether the structure is behaving linearly or not.

5.2 Piecewise Exact Method (逐次厳密解)

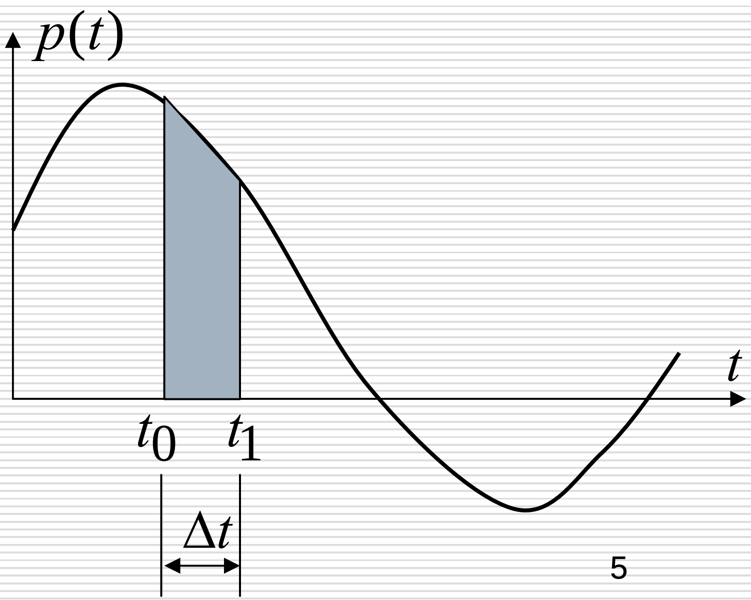
- The simplest step-by-step method for analysis is the so-called "piecewise exact" method. In this method, the load time history is divided into time intervals, usually defined by significant changes of slope in the loading history.

- It is assumed that the slope of the load curve remains constant between these points.
- It must be recognized that the actual loading history is only approximated by the constant slope steps.
- Thus the calculated response is not an exact representation of the true response



- However, the error can be reduced to any acceptable value merely by reducing the length of the time steps.
- If desired, the length of the time steps can be varied from one interval to the next in order to achieve the best possible fit of the loading time history by the sequence of straight line segments. (**直線区間の連続で実際の外力を当てはめる(近似する)**)

- This method is called step-by-step response analysis method. (**時刻歴応答解析法**)



● The duration of the step is denoted by Δt, and it spans from t_0 to t_1.

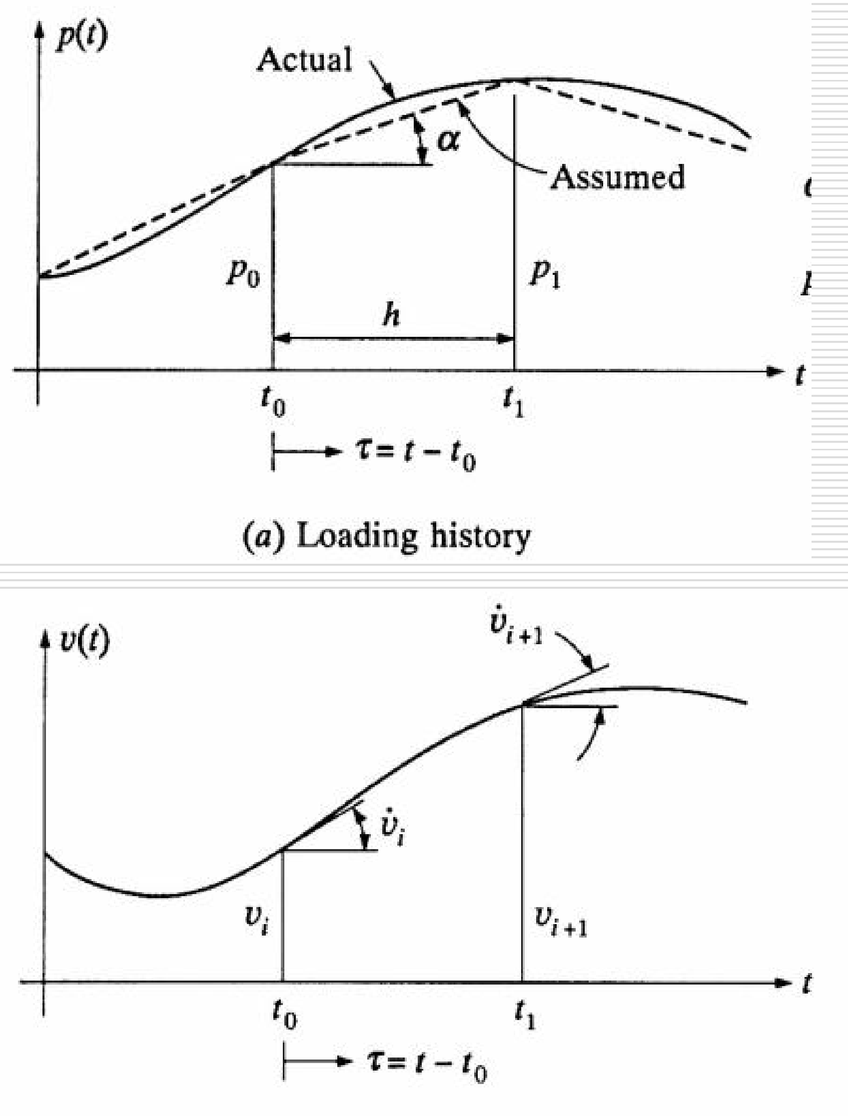
●The assumed linearly varying loading during the time step is given as

$$p(\tau) = p_0 + \alpha\tau \qquad (5.1)$$

where, a is the constant, τ is the time valuable during the step, and p_0 is the initial loading.

● The equation of motion for a SDOF system with viscous damping becomes

$$m\ddot{v} + c\dot{v} + kv = p_0 + \alpha\tau \qquad (5.2)$$



(a) Loading history

(b) Response history

Fig. 5.1

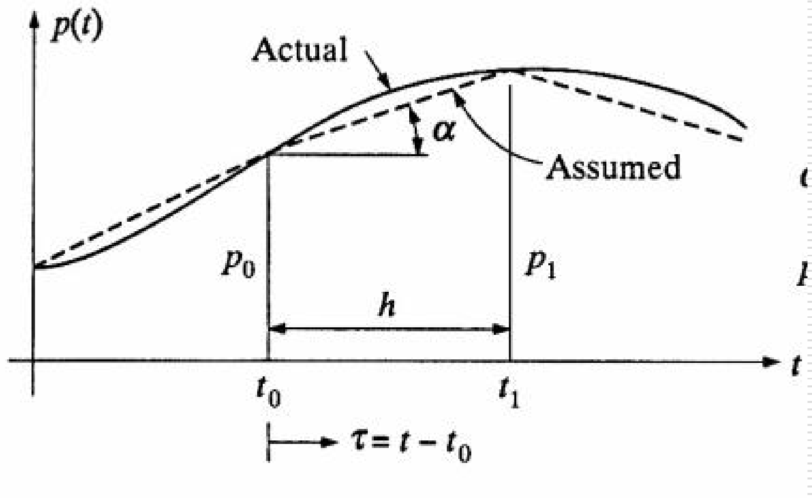
$$m\ddot{v} + c\dot{v} + kv = p_0 + \alpha\tau \qquad (5.2)$$

- The response $v(\tau)$ during any time step consists of a free vibration term $v_h(\tau)$ plus the particular solution to the specific linear load variation $v_p(\tau)$, thus

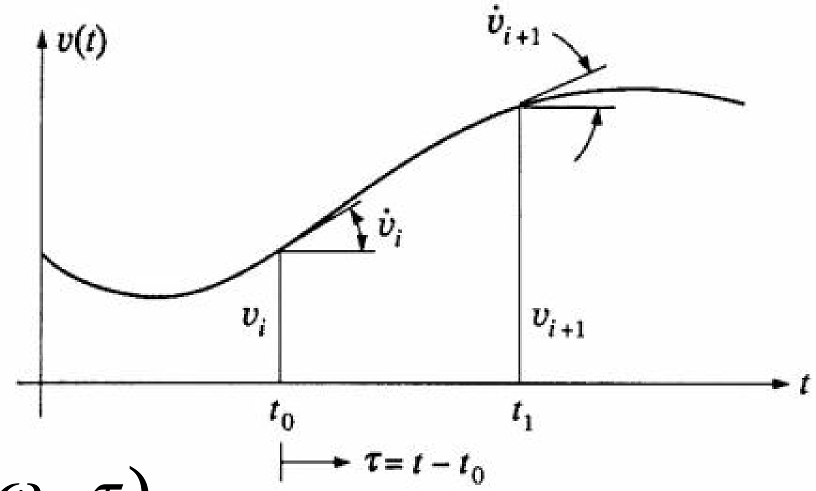
$$v(\tau) = v_h(\tau) + v_p(\tau) \qquad (5.3)$$

where, free vibration is given by Eq. (2.48) as

$$v_h(\tau) = e^{-\xi\omega\tau}(A\cos\omega_D\tau + B\sin\omega_D\tau)$$



(a) Loading history



(b) Response history

Fig. 5.1

$$v(t) = \{A\cos\omega_D t + B\sin\omega_D t\}e^{-\xi\omega t} \qquad (2.48)$$

● It is easy to verify that the linearly varying particular solution of Eq. (5.2) is

$$v_p(\tau) = \frac{1}{k}(p_0 + \alpha\tau) - \frac{\alpha c}{k^2} \qquad (5.4)$$

$$\boxed{m\ddot{v} + c\dot{v} + kv = p_0 + \alpha\tau \qquad (5.2)}$$

● Combining these expressions and evaluating A and B considering the initial conditions at time τ=0 ($v(0)=v_0$ and , $\dot{v}(0) = \dot{v}_0$), the displacement during the time step is given

$$v(\tau) = A_0 + A_1\tau + A_2 e^{-\xi\omega\tau}\cos\omega_D\tau + A_3 e^{-\xi\omega\tau}\sin\omega_D\tau \qquad (5.5)$$

$$\dot{v}(\tau) = A_1 + (\omega_D A_3 - \xi\omega A_2)e^{-\xi\omega\tau}\cos\omega_D\tau - (\omega_D A_2 + \xi\omega A_3)e^{-\xi\omega\tau}\sin\omega_D\tau \qquad (5.6)$$

$$v(\tau) = A_0 + A_1\tau + A_2 e^{-\xi\omega\tau}\cos\omega_D\tau + A_3 e^{-\xi\omega\tau}\sin\omega_D\tau \tag{5.5}$$

$$\dot{v}(\tau) = A_1 + (\omega_D A_3 - \xi\omega A_2)e^{-\xi\omega\tau}\cos\omega_D\tau - (\omega_D A_2 + \xi\omega A_3)e^{-\xi\omega\tau}\sin\omega_D\tau \tag{5.6}$$

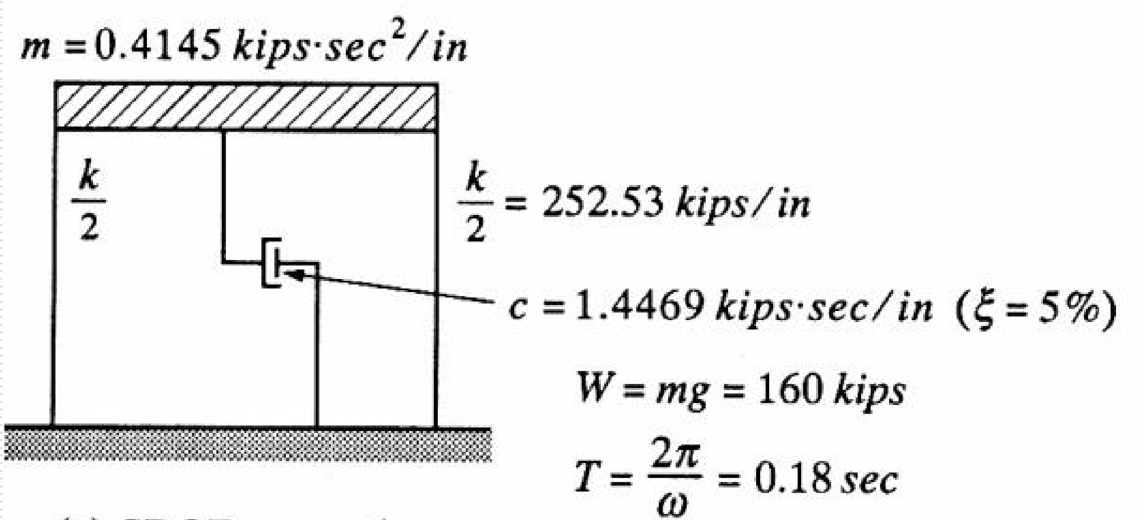
in which $A_0 = \frac{v_0}{\omega^2} - \frac{2\xi\alpha}{\omega^3}$ $\quad A_1 = \frac{\alpha}{\omega^2}$

$A_2 = v_0 - A_0$ $\quad A_3 = \frac{1}{\omega_D}(\dot{v}_0 + \xi\omega A_2 - \frac{\alpha}{\omega^2})$

- Of course, the velocity and displacement at the end of this time step become the initial condition for the next time step, and the equivalent equations can be used to step forward to the end of that steps.

Example 5.1

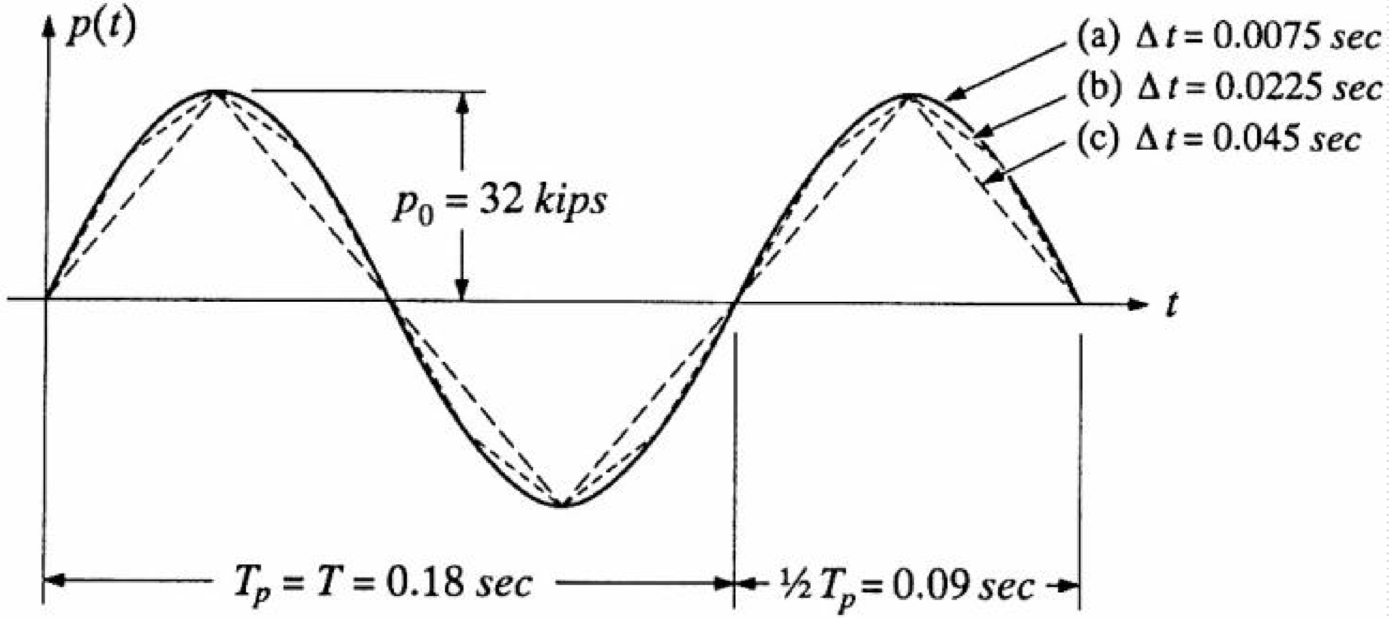
- The response of a SDOF structure to various approximations of a single sine-wave loading was calculated by the piecewise exact method. The properties of the structure are shown in Fig. E5.1(a).



(a) SDOF properties

Fig. E5.1(a)

● Three straight line approximations (**直線近似**) of the one and one-half cycle loading are defined by discrete values spaced at time intervals of (a) 0.0075s, (b) 0.0225 s and (c) 0.045 s, respectively, (1/12, ¼ and ½ of the 0.09s half cycle period).



(*b*) Straight line approximations of sine-wave loading

Fig. 5.1(b)

Piecewise exact example – SDOF structure and loading.

● The calculated responses to those three loadings are shown in Fig. E5.1(b).

●It may be concluded that the results for case (a), using 0.0075s load segments, are quite close to the exact solution.

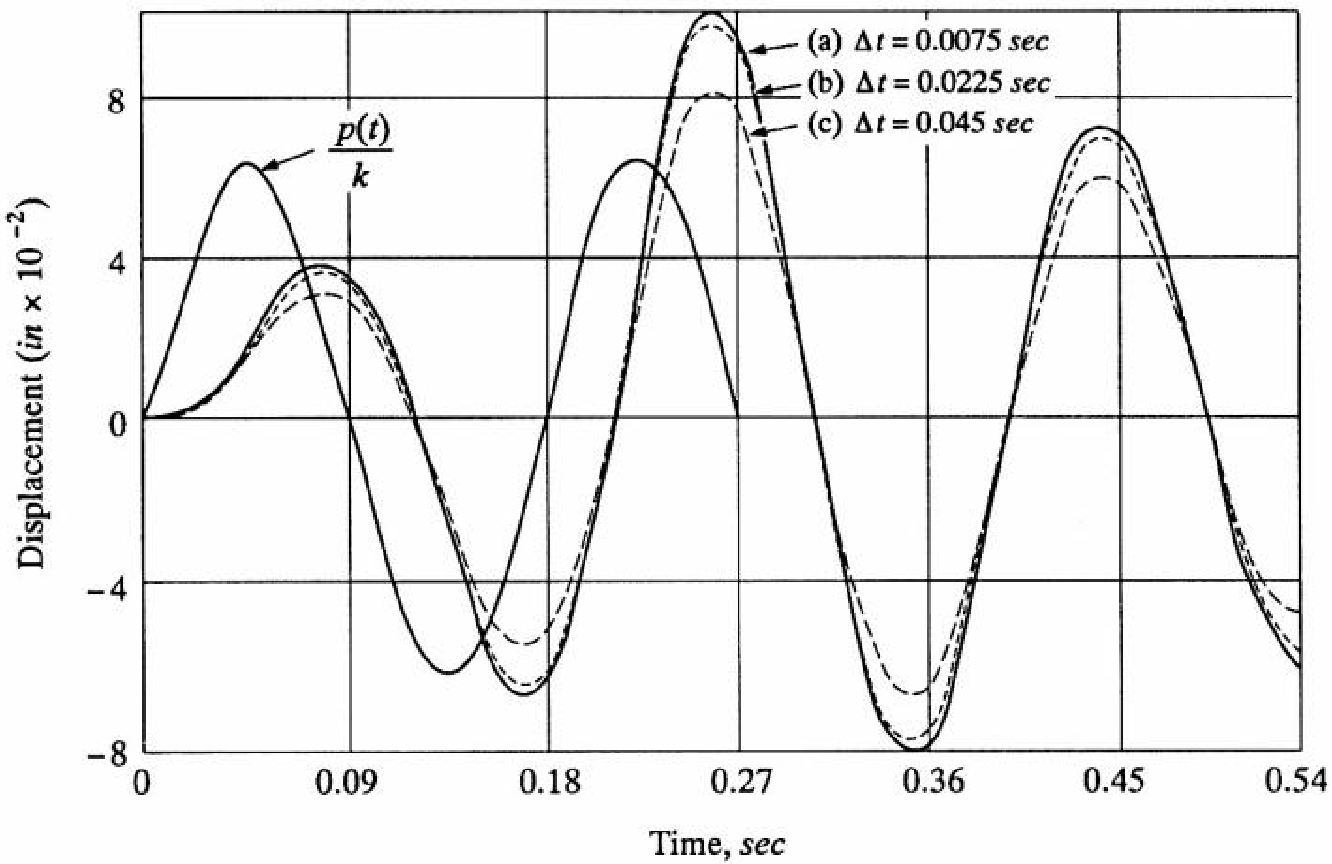
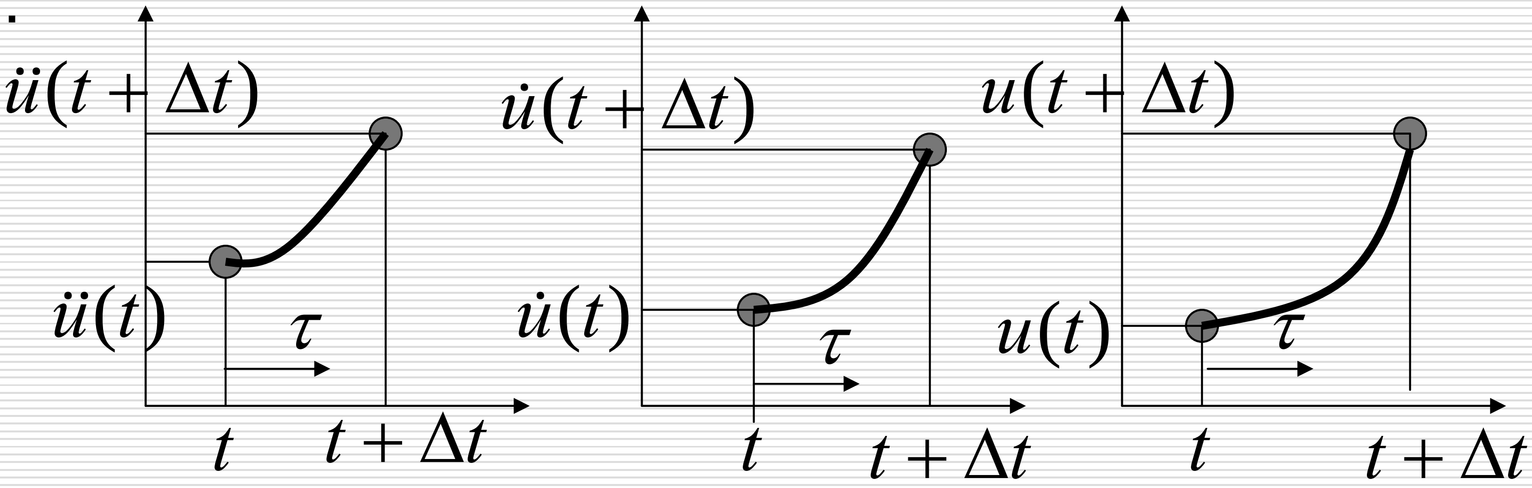


Fig. 5.3

Piecewise exact calculated response.

5.3 Step-by-Step Dynamic Response Analysis

- Step-by-step dynamic response analysis (**時刻歴動的解析法**) is to use integration to step forward from the initial to the final conditions for each time step.
- The essential concept is represented by the following equations:



$$\dot{u}(\tau)=\int_0^{\tau}\ddot{u}(\tau)d\tau+C_1 \quad (5.7a)$$

$$u(\tau)=\int_0^{\tau}\dot{u}(\tau)d\tau+C_2=\int\int_0^{\tau}\ddot{u}(\tau)d\tau+C_1\tau+C_2 \quad (5.7b)$$

- In order to carry out the analysis, it is necessary to assume how the acceleration varies during the time step.

$$\dot{u}(\tau) = \int_0^\tau \ddot{u}(\tau)d\tau + C_1 \qquad (5.7a)$$

$$u(\tau) = \int\int_0^\tau \ddot{u}(\tau)d\tau + C_1\tau + C_2 \qquad (5.7b)$$

- From the initial conditions, we obtain the following expressions

$$\dot{u}(t) = C_1 \qquad (5.8a)$$

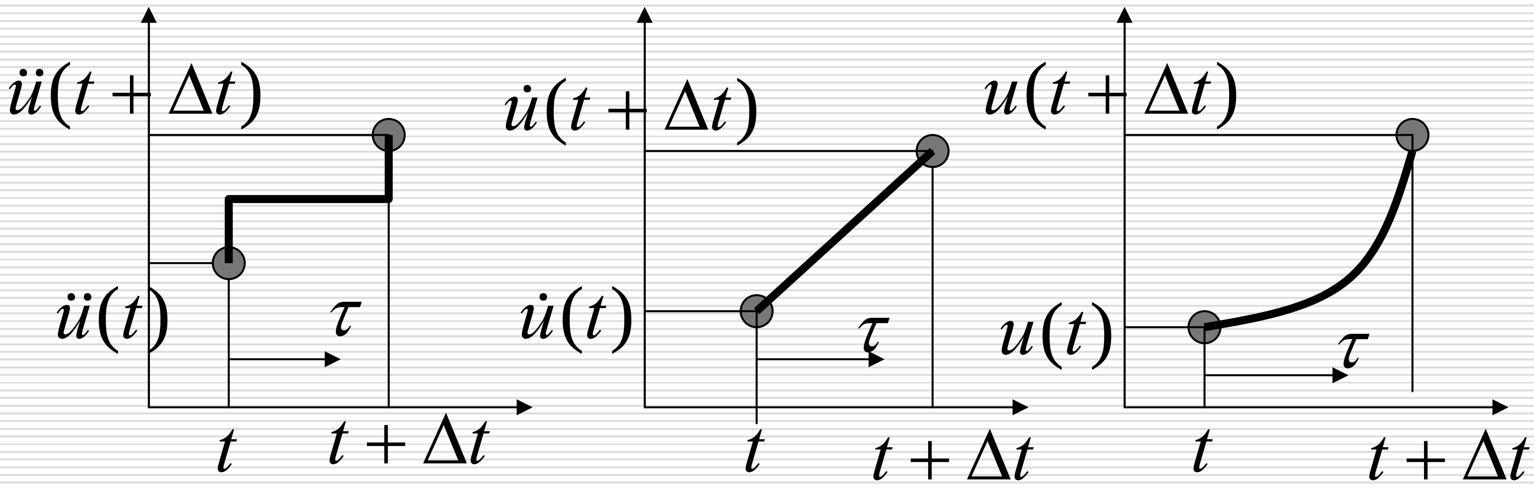
$$u(t) = C_2 \qquad (5.8b)$$

- Substitution of these initial conditions by Eq. (5.8) into Eqs. (5.7) leads to

$$\dot{u}(\tau) = \dot{u}(t) + \int_0^\tau \ddot{u}(\tau)d\tau \qquad (5.9a)$$

$$u(\tau) = u(t) + \dot{u}(t)\tau + \int\int_0^\tau \ddot{u}(\tau)d\tau \qquad (5.9b)$$

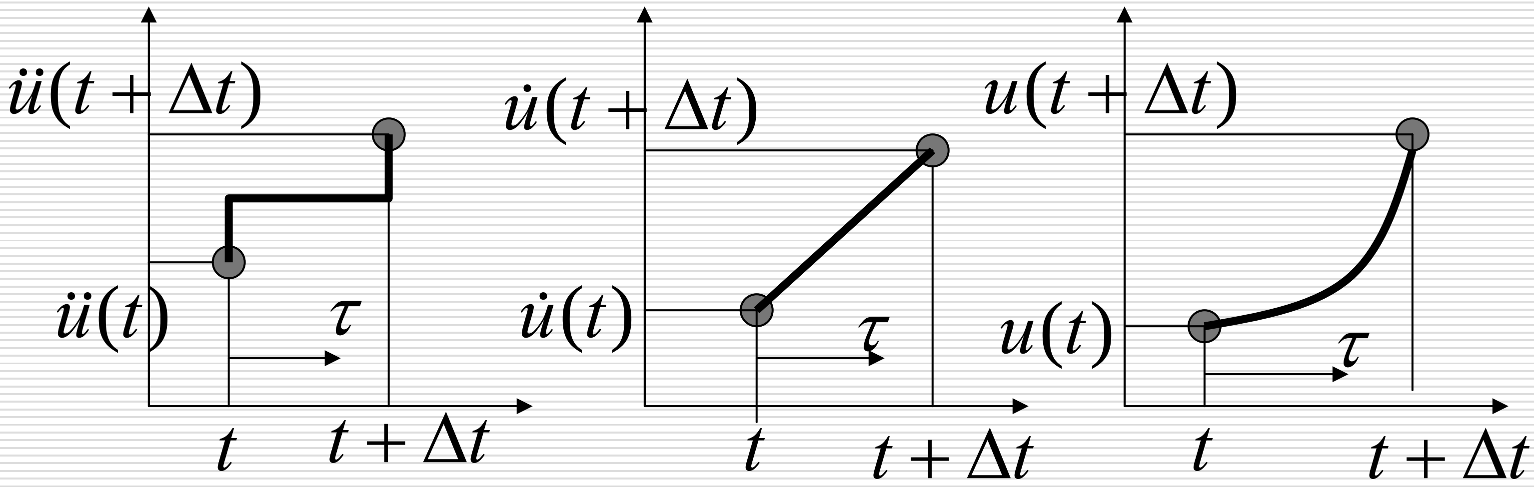
(2) Constant Acceleration Method (一定加速度法)



● Assuming that the acceleration between the time t and t+Δt is constant, and is the averaged value of the acceleration at the time t an d t+Δt as

$$\ddot{u}(\tau) = \frac{\ddot{u}(t) + \ddot{u}(t+\Delta t)}{2} \qquad (5.10)$$

Substitution of Eq. (5.10) into Eq. (5.9) leads to



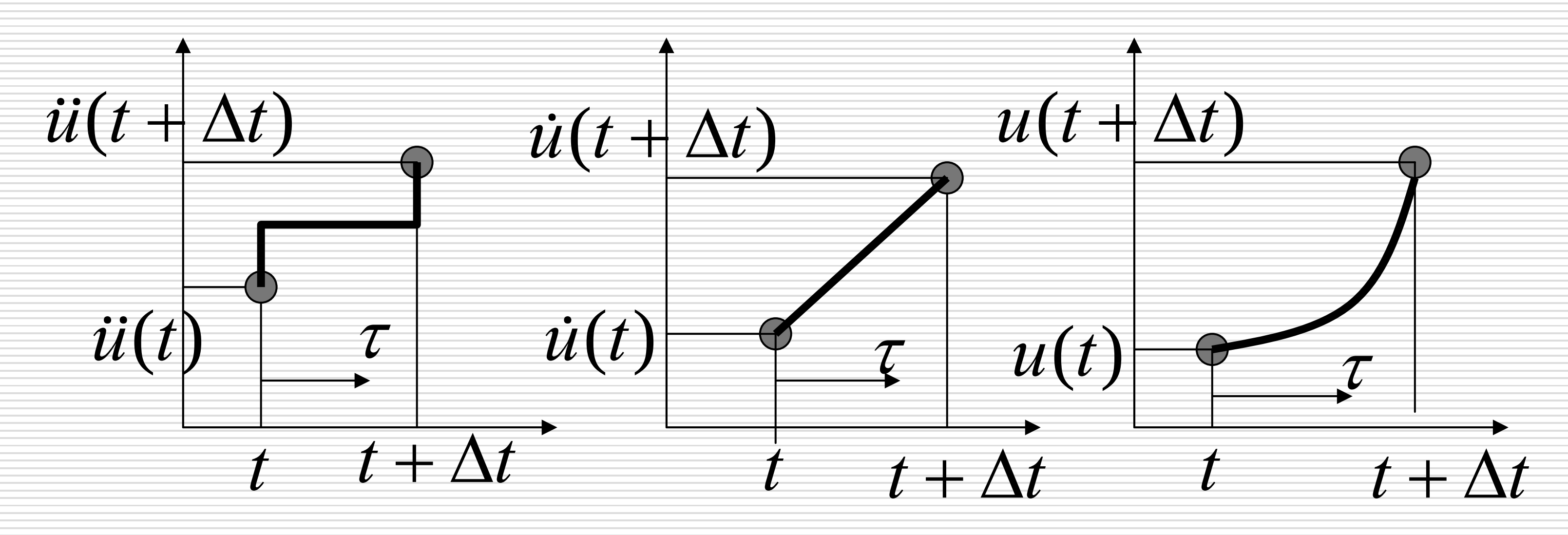
$$\dot{u}(\tau) = \dot{u}_t + \frac{\tau}{2}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad (5.11a)$$

$$u(\tau) = u_t + \dot{u}_t\tau + \frac{\tau^2}{4}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad (5.11b)$$

$$\dot{u}(\tau) = \dot{u}(t) + \int_0^\tau \ddot{u}(\tau)d\tau \qquad (5.9a)$$

$$u(\tau) = u(t) + \dot{u}(t)\tau + \int\int_0^\tau \ddot{u}(\tau)d\tau \qquad (5.9b)$$

$$\ddot{u}(\tau) = \frac{\ddot{u}(t) + \ddot{u}(t+\Delta t)}{2} \qquad (5.10)$$



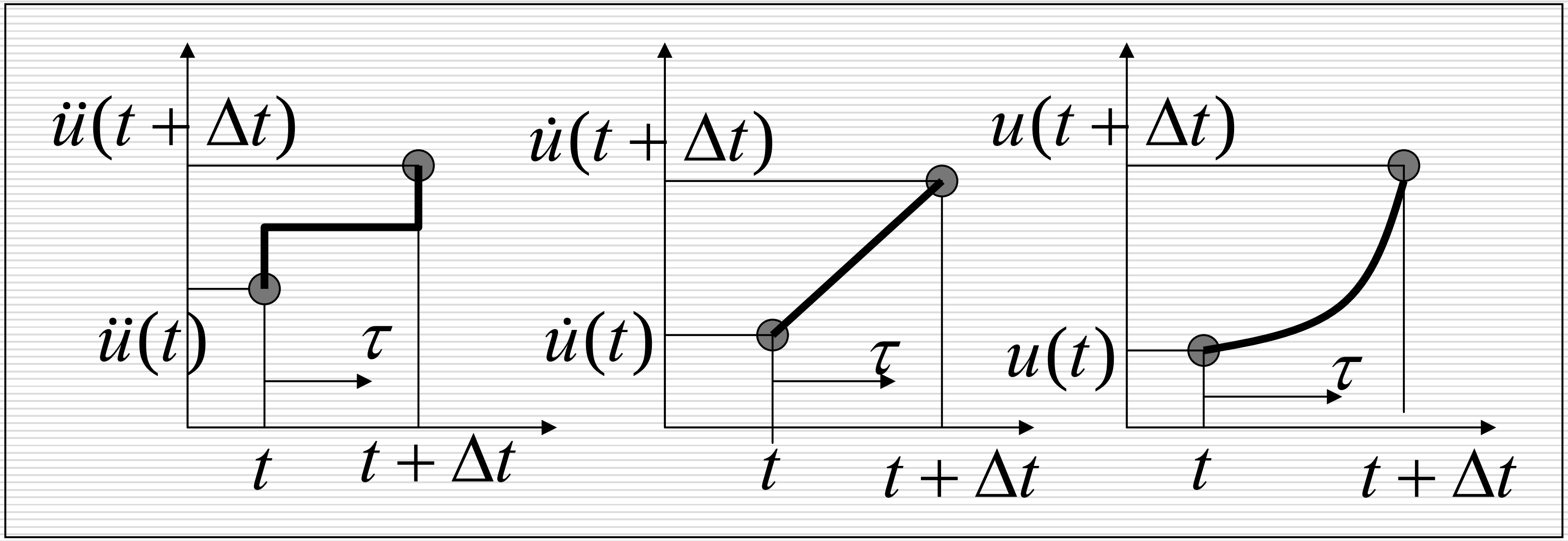
- Hence, substitution of $\tau=\Delta t$ gives

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \frac{\Delta t}{2}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad (5.12a)$$

$$u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \frac{\Delta t^2}{4}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad (5.12b)$$

$$\dot{u}(\tau) = \dot{u}_t + \frac{\tau}{2}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad (5.11a)$$

$$u(\tau) = u_t + \dot{u}_t \tau + \frac{\tau^2}{4}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad (5.11b)$$

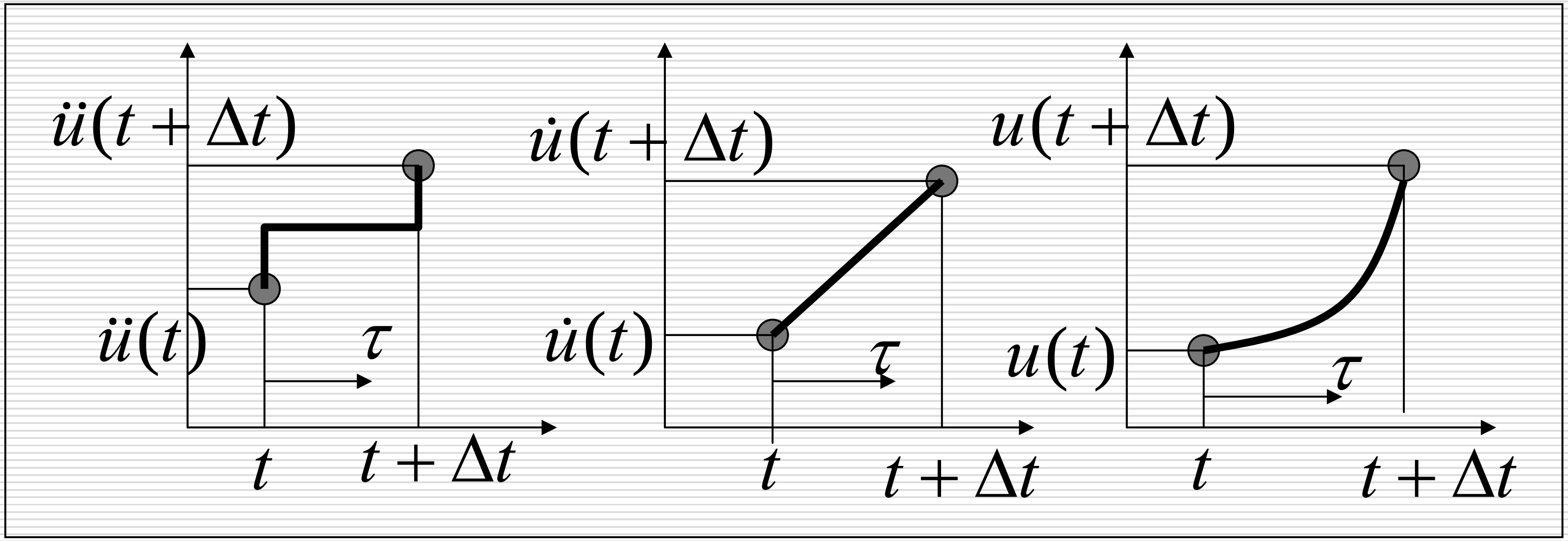


- Eq. (5.12) shows that since we know u_t , $\dot{u}_t$ and $\ddot{u}_t$, once we know the acceleration at the time t+Δt, $\ddot{u}_{t+\Delta t}$, we can know $\dot{u}_{t+\Delta t}$ and $u_{t+\Delta t}$.

- This method is called the constant acceleration method.

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \frac{\Delta t}{2}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad (5.12a)$$

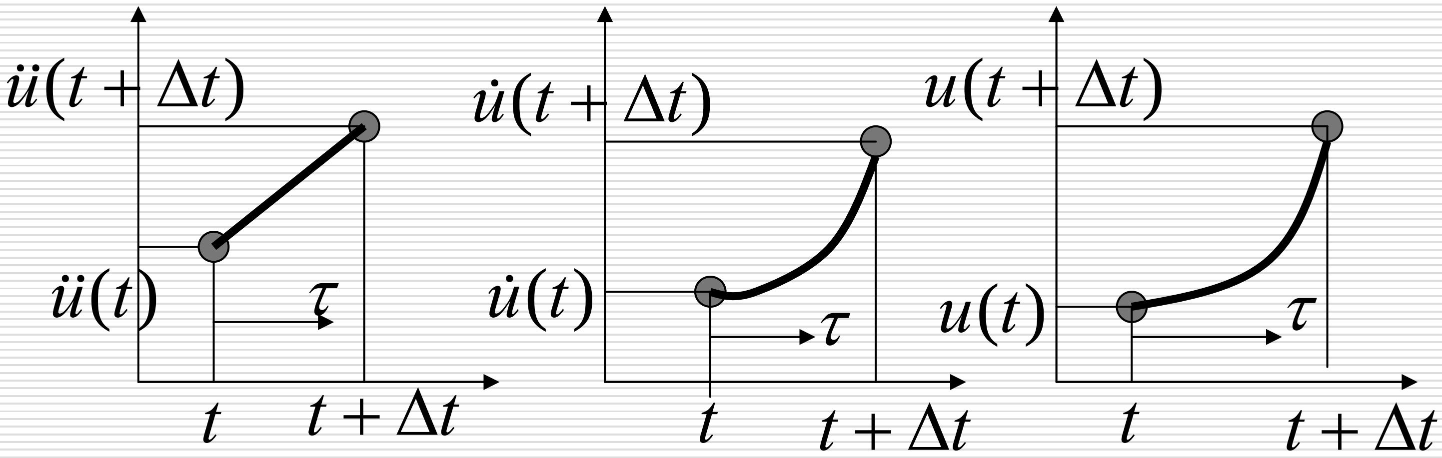
$$u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \frac{\Delta t^2}{4}(\ddot{u}_t + \ddot{u}_{t+\Delta t}) \qquad (5.12b)$$



● The great advantage of the constant acceleration method is that it is unconditionally stable (**無条件に安定**); that is, the error are not amplified from one step to the next no matter how a long time step is chosen.

●Consequently, the time step may be selected considering only the need for properly defining the dynamic excitation and vibration characteristics of the structure.

(3) Linear Acceleration Method (線形加速度法)



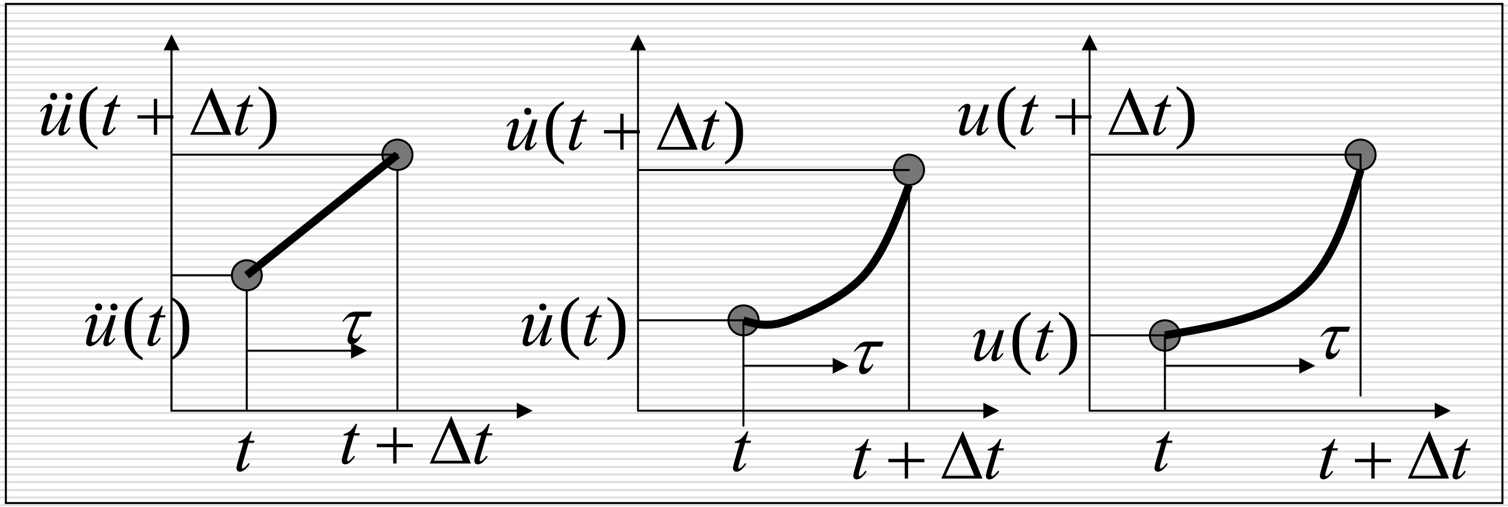
- Another assumption for the acceleration is that it varies linearly with the time between the time t and t+Δt as

$$\ddot{u}(\tau)=\ddot{u}_t+\frac{\tau}{\Delta t}(\ddot{u}_{t+\Delta t}-\ddot{u}_t) \qquad (5.13)$$

- Substitution of this into Eq. (5.9) yields

$$\dot{u}(\tau)=\dot{u}(t)+\int_0^{\tau}\ddot{u}(\tau)d\tau \qquad (5.9a)$$

$$u(\tau)=u(t)+\dot{u}(t)\tau+\int\int_0^{\tau}\ddot{u}(\tau)d\tau \qquad (5.9b)$$



$$\dot{u}(\tau) = \dot{u}_t + \ddot{u}_t\tau + \frac{\tau^2}{2\Delta t}(\ddot{u}_{t+\Delta t} - \ddot{u}_t) \qquad (5.14a)$$

$$u(\tau) = u_t + \dot{u}_t\tau + \frac{\ddot{u}_t\tau^2}{2} + \frac{\tau^3}{6}(\ddot{u}_{t+\Delta t} - \ddot{u}_t) \qquad (5.14b)$$

- Substitution of $\tau=\Delta t$, one obtains

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \ddot{u}_t\Delta t + \frac{\Delta t^2}{2}(\ddot{u}_{t+\Delta t} - \ddot{u}_t) \qquad (5.15a)$$

$$u_{t+\Delta t} = u_t + \dot{u}_t\Delta t + \frac{\ddot{u}_t\Delta t^2}{2} + \frac{\Delta t^3}{6}(\ddot{u}_{t+\Delta t} - \ddot{u}_t) \qquad (5.15b)$$

- Like the constant acceleration method, the linear acceleration method is widely used in practice.

- However, in contrast to the constant acceleration method, the linear acceleration method is only conditionally stable (条件付き安定); it will be unstable unless

$$\frac{\Delta t}{T} \leq \frac{\sqrt{3}}{\pi} = 0.55$$

- However this restriction has little significance in the analysis of SDOF systems, because a shorter time step than this must be used to obtain a satisfactory representation of the dynamic input and response.

(4) Newmark β method (ニューマークのβ法)

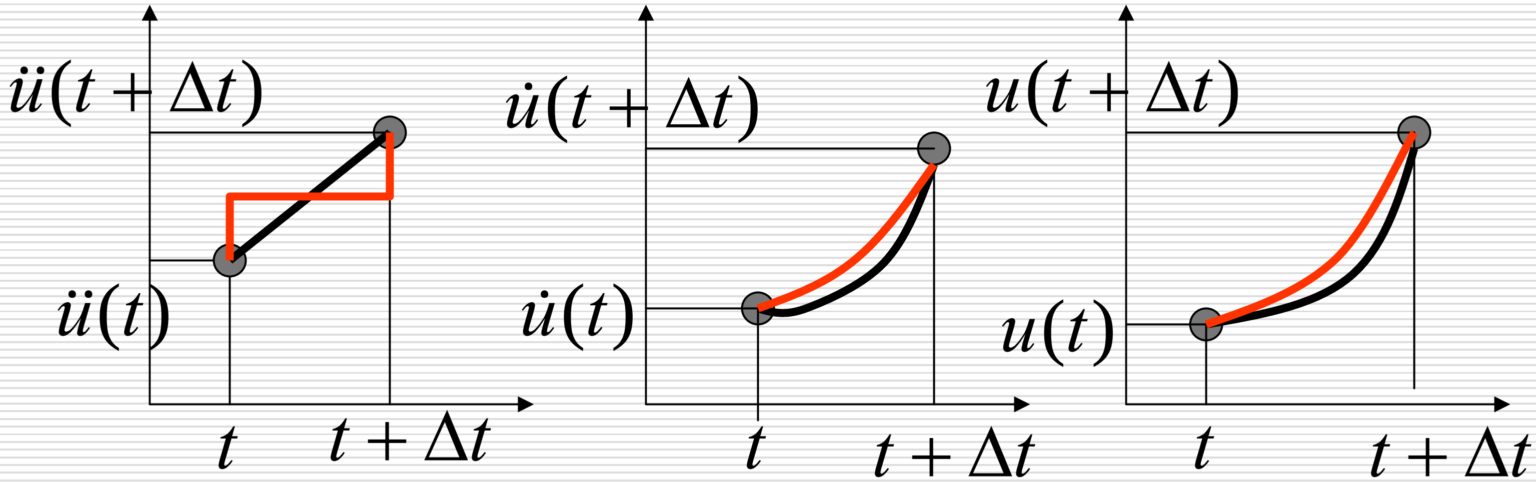
● A more generalized step-by-step formulation was proposed by Newmark, which includes the preceding methods as special cases, but may be applied in several other version.

● In the Newmark formulation , the basic integration equations are expressed as

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \{(1-\delta)\ddot{u}_t + \delta\ddot{u}_{t+\Delta t}\}\Delta t \qquad (5.16a)$$

$$u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \left\{(\frac{1}{2}-\sigma)\ddot{u}_t + \sigma\ddot{u}_{t+\Delta t}\right\}\Delta t^2 \qquad (5.16b)$$

● It is evident in Eq. (2.10) that the factor δ provides a linearly varying weighting between the initial and the final accelerations on the change of velocity; the factor σ similarly provides the weighting the contributions of these initial and the final accelerations to the change of displacement.



$$\dot{u}_{t+\Delta t} = \dot{u}_t + \{(1-\delta)\ddot{u}_t + \delta\ddot{u}_{t+\Delta t}\}\Delta t \qquad (5.16a)$$

$$u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \left\{(\frac{1}{2}-\sigma)\ddot{u}_t + \sigma\ddot{u}_{t+\Delta t}\right\}\Delta t^2 \qquad (5.16b)$$

$\delta = \frac{1}{2}$ and $\sigma = \frac{1}{4}$; constant acceleration method (**一定加速度法**)

$\delta = \frac{1}{2}$ and $\sigma = \frac{1}{6}$; Linear acceleration method (**線形加速度法**)

Conversion to Explicit Formulation

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \{(1-\delta)\ddot{u}_t + \delta\ddot{u}_{t+\Delta t}\}\Delta t \qquad (5.16a)$$

$$u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \left\{(\frac{1}{2} - \sigma)\ddot{u}_t + \sigma\ddot{u}_{t+\Delta t}\right\}\Delta t^2 \qquad (5.16b)$$

- From Eq. (5.16b), one can obtain

$$\ddot{u}_{t+\Delta t} = \frac{1}{\sigma}\left\{\frac{u_{t+\Delta t} - u_t}{\Delta t^2} - \frac{\dot{u}_t}{\Delta t} - \left(\frac{1}{2} - \sigma\right)\ddot{u}_t\right\} \qquad (5.17)$$

- Substituting Eq. (5.17) into Eq. (5.16a) leads to

$$\dot{u}_{t+\Delta t} = \dot{u}_t + (1-\delta)\Delta t\ddot{u}_t + \frac{\delta}{\sigma}\left\{\frac{u_{t+\Delta t} - u_t}{\Delta t} - \dot{u}_t - (\frac{1}{2} - \sigma)\Delta t\ddot{u}_t\right\} \qquad (5.18)$$

- Equation of motion at time t+Δt is

$$m u_{t+\Delta t} + c\dot{u}_{t+\Delta t} + k u_{t+\Delta t} = p_{t+\Delta t} \qquad (5.19)$$

- Substitution of Eqs. (5.17) and (5.18) into Eq. (5.19) yields

$$\ddot{u}_{t+\Delta t} = \frac{1}{\sigma}\left\{\frac{u_{t+\Delta t} - u_t}{\Delta t^2} - \frac{\dot{u}_t}{\Delta t} - \left(\frac{1}{2} - \sigma\right)\ddot{u}_t\right\} \qquad (5.17)$$

$$\dot{u}_{t+\Delta t} = \dot{u}_t + (1-\delta)\Delta t\ddot{u}_t + \frac{\delta}{\sigma}\left\{\frac{u_{t+\Delta t} - u_t}{\Delta t} - \dot{u}_t - (\frac{1}{2} - \sigma)\Delta t\ddot{u}_t\right\} \qquad (5.18)$$

$$\tilde{k}u_{t+t} = \tilde{p}_{t+\Delta t} \qquad (5.20)$$

where,

$$\tilde{k} = k + \frac{m}{\sigma \Delta t^2} + \frac{c\delta}{\sigma \Delta t} \qquad (5.21)$$

$$\tilde{p}_{t+\Delta t} = p_{t+\Delta t} + m\left\{ \frac{u_t}{\sigma \Delta t^2} + \frac{\dot{u}_t}{\sigma \Delta t} + \left(\frac{1}{2} - \sigma \right) \frac{\ddot{u}_t}{\sigma} \right\}$$

$$+ c\left\{ \frac{\delta}{\sigma \Delta t} u_t + \left(\frac{\delta}{\sigma} - 1 \right) \dot{u}_t + \left(\frac{\delta}{2\sigma} - 1 \right) \Delta t \ddot{u}_t \right\} \qquad (5.22)$$

- In Eqs. (5.21) and (5.22), the left hand sides are known quantities at the time t, therefore, Eq. (5.20) can be solved for $u_{t+\Delta t}$.

- Once $u_{t+\Delta t}$ is obtained, substitution of $u_{t+\Delta t}$ into Eqs. (5.17) and (5.18) yields $\ddot{u}_{t+\Delta t}$ and $\dot{u}_{t+\Delta t}$
- By repeating this process, we can calculate solution of Eq. (5.19).

$$\ddot{u}_{t+\Delta t} = \frac{1}{\sigma}\left\{\frac{u_{t+\Delta t} - u_t}{\Delta t^2} - \frac{\dot{u}_t}{\Delta t} - \left(\frac{1}{2} - \sigma\right)\ddot{u}_t\right\} \quad (5.17)$$

$$\dot{u}_{t+\Delta t} = \dot{u}_t + (1-\delta)\Delta t\ddot{u}_t + \frac{\delta}{\sigma}\left\{\frac{u_{t+\Delta t} - u_t}{\Delta t} - \dot{u}_t - (\frac{1}{2} - \sigma)\Delta t\ddot{u}_t\right\} \quad (5.18)$$

$$mu_{t+\Delta t} + c\dot{u}_{t+\Delta t} + ku_{t+\Delta t} = p_{t+\Delta t} \quad (5.19)$$

Constant acceleration method

- From Eqs. (5.20)-(5.22), substituting δ=1/2 and σ=1/4, one obtains

$$\tilde{k} u_{t+t} = \tilde{p}_{t+\Delta t} \qquad (5.20)$$

where,

$$\tilde{k} = k + \frac{4m}{\Delta t^2} + \frac{2c}{\Delta t} \qquad (5.21a)$$

$$\tilde{p}_{t+\Delta t} = p_{t+\Delta t} + m\left\{\frac{4u_t}{\Delta t^2} + \frac{4\dot{u}_t}{\Delta t} + \ddot{u}_t\right\} + c\left\{\frac{2}{\Delta t}u_t + \dot{u}_t\right\} \qquad (5.22a)$$

Linear acceleration method

● From Eqs. (5.20)-(5.22), substituting δ=1/2 and σ=1/6, one obtains

$$\tilde{k} u_{t+t} = \tilde{p}_{t+\Delta t} \qquad (5.20)$$

where,

$$\tilde{k} = k + \frac{6m}{\Delta t^2} + \frac{3c}{\Delta t} \qquad (5.21b)$$

$$\tilde{p}_{t+\Delta t} = p_{t+\Delta t} + m\left\{ \frac{6u_t}{\Delta t^2} + \frac{6\dot{u}_t}{\Delta t} + 2\ddot{u}_t \right\}$$
$$+ c\left\{ \frac{3}{\Delta t} u_t + 2\dot{u}_t + \frac{\Delta t}{2} \ddot{u}_t \right\} \qquad (5.22b)$$

5.4 Incremental Formulation for Nonlinear Analysis (非線形解析のための増分形定式化)

- The step-by-step procedure described above are suitable for analysis of linear systems. However for nonlinear systems, the procedure describe above cannot be directly applied.

- For nonlinear system, it is assumed that the physical properties remain constant only for short increments of time or deformation (微少時間の間、すなわち微少変形の間には物理的特性が変化しないと仮定する).

u_t

$\dot{u}_t$

$\ddot{u}_t$

$u_{t+\Delta t}$

$\dot{u}_{t+\Delta t}$

$\ddot{u}_{t+\Delta t}$

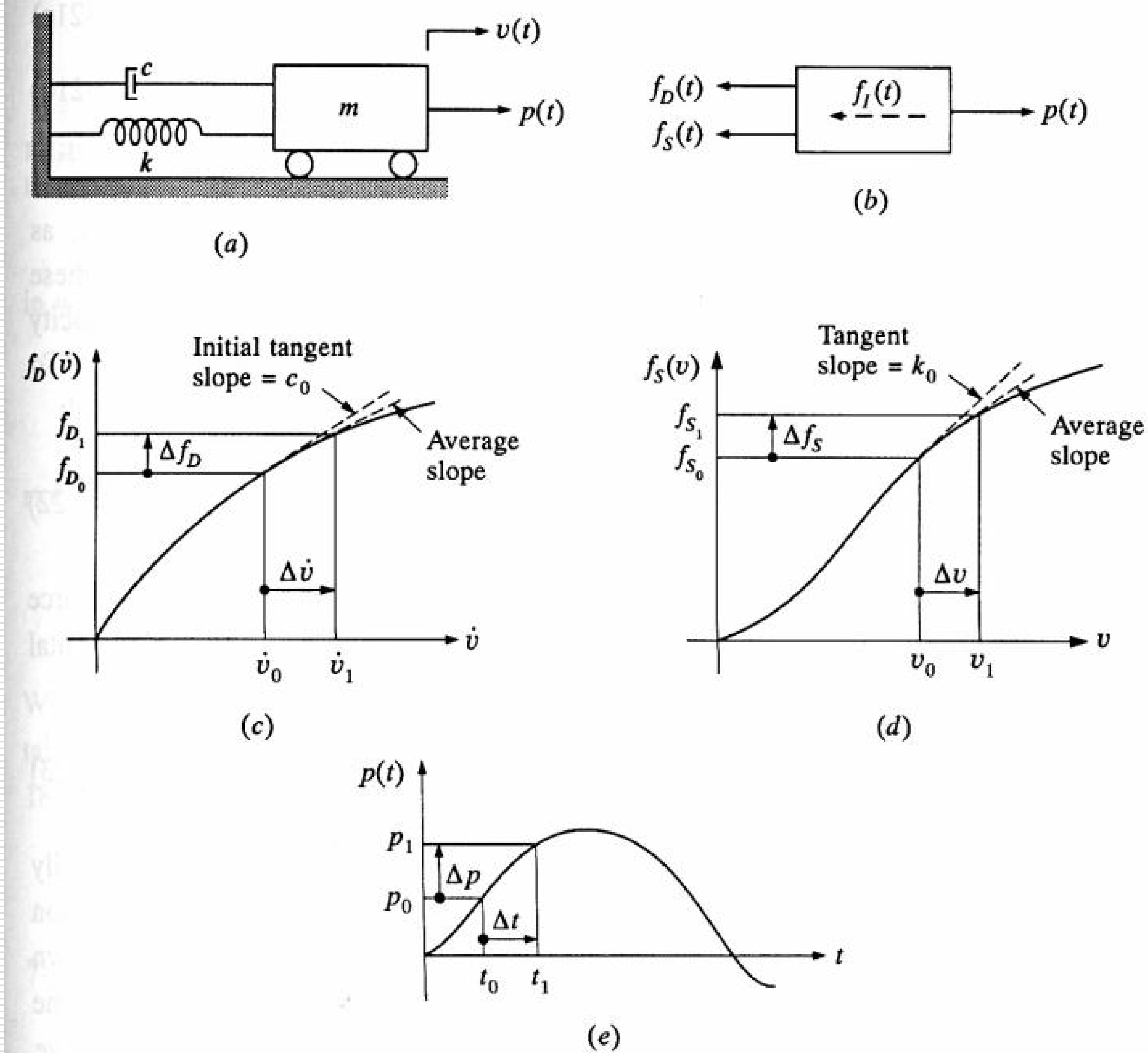


Fig. 5.2

Definition of a nonlinear dynamic system: (a) basic SDOF structure; (b) force equilibrium; (c) nonlinear damping; (d) nonlinear stiffness; (e) applied load.

- The equilibrium of forces acting on the mass at the time t may be written as

$$f_{It} + f_{Dt} + f_{Rt} = p_t \qquad (5.23)$$

- At the time t+Δt, eq.(7.23) can be written as

$$f_{It+\Delta t} + f_{Dt+\Delta t} + f_{Rt+\Delta t} = p_{t+\Delta t} \qquad (5.24)$$

- Subtracting Eq. (5.23) from Eq. (5.22) yields the incremental equation of motion (**増分形の運動方程式**).

$$\Delta f_{It} + \Delta f_{Dt} + \Delta f_{Rt} = \Delta p_t \qquad (5.25)$$

in which,

$$\Delta f_{It} = f_{It+\Delta t} - f_{It} = m\Delta \ddot{u}_t \qquad (5.26a)$$

$$\Delta f_{Dt} = f_{Dt+\Delta t} - f_{Dt} = c_t \Delta \dot{u}_t \qquad (5.26b)$$

$$\Delta f_{Rt} = f_{Rt+\Delta t} - f_{Rt} = k_t \Delta u_t \qquad (5.26c)$$

$$\Delta p_t = p_{t+\Delta t} - p_t \qquad (5.26d)$$

$\Delta \ddot{u}_t \equiv \ddot{u}_{t+\Delta t} - \ddot{u}_t$: Incremental acceleration (**増分加速度**)

$\Delta \dot{u}_t \equiv \dot{u}_{t+\Delta t} - \dot{u}_t$: Incremental velocity (**増分速度**)

$\Delta u_t \equiv u_{t+\Delta t} - u_t$: Incremental displacement (**増分変位**)

Substitution of Eq. (5.26) into Eq. (5.25) leads to the incremental equation of motion (**増分形の運動方程式**)

$$m\Delta \ddot{u}_t + c_t \Delta \dot{u}_t + k_t \Delta u_t = \Delta p_t \qquad (5.23)$$

$$\Delta f_{It} + \Delta f_{Dt} + \Delta f_{Rt} = \Delta p_t \qquad (5.25)$$

$$\Delta f_{It} = f_{It+\Delta t} - f_{It} = m\Delta \ddot{u}_t \qquad (5.26a)$$

$$\Delta f_{Dt} = f_{Dt+\Delta t} - f_{Dt} = c_t \Delta \dot{u}_t \qquad (5.26b)$$

$$\Delta f_{Rt} = f_{Rt+\Delta t} - f_{Rt} = k_t \Delta u_t \qquad (5.26c)$$

$$\Delta p_t = p_{t+\Delta t} - p_t \qquad (5.26d)$$

● Incremental acceleration, velocity and displacement are evaluated from Eq. (5.16) as

✓ From Eq. (5.16b), one obtains

$$\ddot{u}_{t+\Delta t} = \frac{\Delta u_t - \dot{u}_t \Delta t}{\sigma \Delta t^2} - \left(\frac{1}{2\sigma} - 1 \right) \ddot{u}_t \qquad (5.24)$$

✓Rearranging Eq. (5.24),

$$\Delta \ddot{u}_t \equiv \ddot{u}_{t+\Delta t} - \ddot{u}_t = \frac{\Delta u_t - \dot{u}_t \Delta t}{\sigma \Delta t^2} - \frac{1}{2\sigma} \ddot{u}_t \quad (5.25)$$

✓Substituting this into Eq. (5.16a), one obtains

$$\Delta \dot{u} \equiv \dot{u}_{t+\Delta t} - \dot{u}_t = \delta \left(\frac{\Delta u_t - \dot{u}_t \Delta t}{\sigma \Delta t} \right) + \left(1 - \frac{\delta}{2\sigma} \right) \Delta t \ddot{u}_{t+\Delta t} \qquad (5.26)$$

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \{(1-\delta)\ddot{u}_t + \delta \ddot{u}_{t+\Delta t}\} \Delta t \qquad (5.16a)$$

$$u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \left\{ \left(\frac{1}{2} - \sigma\right) \ddot{u}_t + \sigma \ddot{u}_{t+\Delta t} \right\} \Delta t^2 \qquad (5.16b)$$

- Eqs. (5.25) and (5.26) can be rearranged as

$$\Delta\ddot{u}_t = C_1\Delta u_t - C_3\dot{u}_t - C_4\ddot{u}_t \qquad (5.27a)$$

$$\Delta\dot{u}_t = C_2\Delta u_t - C_4\dot{u}_t - C_5\ddot{u}_t \qquad (5.27b)$$

in which

$$C_1 = \frac{1}{\sigma\Delta t^2} \qquad C_2 = \frac{\delta}{\sigma\Delta t} \qquad C_3 = \frac{1}{\sigma\Delta t}$$

$$C_4 = \frac{1}{2\sigma} \qquad C_5 = \left(\frac{\delta}{2\sigma} - 1\right)\Delta t \qquad (5.28)$$

$$\Delta\ddot{u}_t \equiv \ddot{u}_{t+\Delta t} - \ddot{u}_t = \frac{\Delta u_t - \dot{u}_t\Delta t}{\sigma\Delta t^2} - \frac{1}{2\sigma}\ddot{u}_t \quad (5.25)$$

$$\Delta\dot{u} \equiv \dot{u}_{t+\Delta t} - \dot{u}_t = \delta\left(\frac{\Delta u_t - \dot{u}_t\Delta t}{\sigma\Delta t}\right) + \left(1 - \frac{\delta}{2\sigma}\right)\Delta t\ddot{u}_{t+\Delta t} \quad (5.26)$$

- Substituting Eq. (5.27) into Eq. (5,23), one obtains

$$\tilde{k}_t \Delta u_t = \Delta \tilde{p}_t \qquad (5.29)$$

in which,

$$\tilde{k}_t = k_t + C_1 m + C_2 c_t \qquad (5.30)$$

$$\Delta \tilde{p}_t = \Delta p_t + \left(C_3 m + C_4 c_t\right)\dot{u}_t + \left(C_4 m + C_5 c_t\right)\ddot{u}_t \qquad (5.31)$$

- Solving Eq. (5.29) for Δu_t, and then substituting this to Eq. (5.27), we can obtain $u_{t+\Delta t}$, $\dot{u}_{t+\Delta t}$ and $\ddot{u}_{t+\Delta t}$

$$m\Delta \ddot{u}_t + c_t \Delta \dot{u}_t + k_t \Delta u_t = \Delta p_t \qquad (5.23)$$

$$\Delta \ddot{u}_t = C_1 \Delta u_t - C_3 \dot{u}_t - C_4 \ddot{u}_t \qquad (5.27a)$$

$$\Delta \dot{u}_t = C_2 \Delta u_t - C_4 \dot{u}_t - C_5 \ddot{u}_t \qquad (5.27b)$$

- Note that C_1-C_5 in Eq. (5.28) can be written as

	Constant acceleration	Linear acceleration
C_1	$4/\Delta t^2$	$6/\Delta t^2$
C_2	$2/\Delta t$	$3/\Delta t$
C_3	$4/\Delta t$	$6/\Delta t$
C_4	2	3
C_5	0	$\Delta t/2$

$$C_1 = \frac{1}{\sigma \Delta t^2} \qquad C_2 = \frac{\delta}{\sigma \Delta t} \qquad C_3 = \frac{1}{\sigma \Delta t}$$

$$C_4 = \frac{1}{2\sigma} \qquad C_5 = \left(\frac{\delta}{2\sigma} - 1\right)\Delta t \tag{5.28}$$

Example E5.2

- To demonstrate a hand-solution for applying the linear acceleration step-by-step method, the response of the elastoplastic SDOF frame (**弾塑性1自由度系フレーム**) shown in Fig. E5.4 to the load history indicated is calculated.

- A time step of 0.1 sec is used for this analysis, which is much longer than desirable for good accuracy but will be adequate for the present purpose.

- Damping coefficient is assumed to remain constant; hence the only nonlinearity in the system results from the change of stiffness as yield takes place.

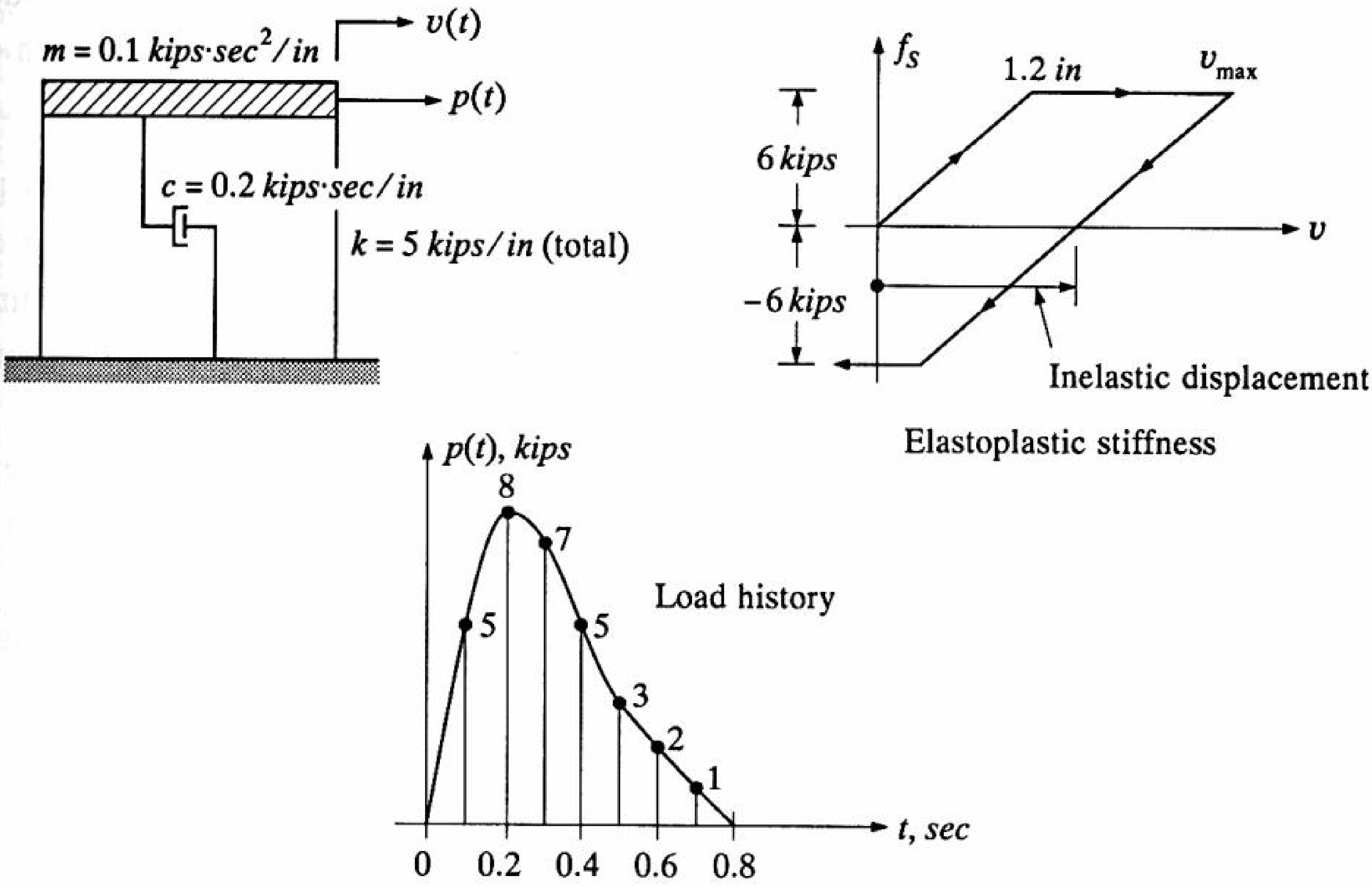


Fig. 5.4

Elastoplastic frame and dynamic loading.

●The effective stiffness thus may be expressed from Eq. (5.29) as

$$\tilde{k} = k_t + \frac{6}{\Delta t^2}m + \frac{3}{\Delta t}c = k_t + \frac{6}{0.1^2}m + \frac{3}{0.1}c$$

$$= k_t + 66$$

where, k_t is either 5 kips/in or zero depending on whether the frame is elastic or yield.

● Also the effective incremental loading is given by Eq. (5.31) as

$$\Delta\tilde{p}_t = \Delta p_t + (\frac{6}{\Delta t}m + 3c)\dot{u}_t + (3m + \frac{\Delta t}{2}c)\ddot{u}_t$$

$$= \Delta p_t + (\frac{6m}{0.1} + 3c)\dot{u}_t + (3m + \frac{0.1}{2}c)\ddot{u}_t$$

$$= \Delta p_t + 6.6\dot{u}_t + 0.31\ddot{u}_t$$

$$\tilde{k}_t = k_t + C_1 m + C_2 c_t \quad (5.29)$$

$$\Delta\tilde{p}_t = \Delta p_t + (C_3 m + C_4 c_t)\dot{u}_t + (C_4 m + C_5 c_t)\ddot{u}_t \quad (5.31)$$

Table 5.1

Nonlinear response analysis: linear acceleration step-by-step method

Structure and loading in Fig. E7-3

t	p	v	$\dot{v}$	f_S $5\,\bar{v}$ *	f_D $0.2\,\dot{v}$	f_I (2)−(5)−(6)	$\ddot{v}$ 10×(7)	Δp	$6.6\,\dot{v}$	$0.31\,\ddot{v}$	$\hat{p}$ (9)+(10)+(11)	k	$\tilde{k}$ 66+(13)	Δv (12)÷(14)	$30\,\Delta v$	$3\,\dot{v}$	$0.05\,\ddot{v}$	$\Delta\dot{v}$ (16)−(17)−(18)
sec	*kips*	*in*	*in/sec*															
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
0.0	0	0	0	0	0	0	0	5	0	0	0	5	71	0.070	2.11	0	0	2.11
0.1	5	0.070	2.11	0.35	0.42	4.23	42.3	3	13.92	13.12	30.04	5	71	0.423	12.68	6.33	2.11	4.24
0.2	8	0.493	6.35	2.46	1.27	4.27	42.7	−1	41.90	13.25	54.15	5	71	0.763	22.88	19.06	2.14	1.68
0.3	7	1.256	8.03	6	1.61	−0.61	−6.1	−2	53.02	−1.89	49.13	0**	66	0.744	22.33	24.08	−0.30	−1.45
0.4	5	2.000	6.58	6	1.32	−2.32	−23.2	−2	43.43	−7.19	34.24	0	66	0.519	15.57	19.74	−1.16	−3.01
0.5	3	2.519	3.57	6	0.71	−3.71	−37.1	−1	23.56	−11.50	11.06	0	66	0.168	5.02	10.72	−1.85	−3.85
0.6	2	2.687	−0.28	6	−0.06	−3.94	−39.4	−1	−1.85	−12.22	−15.07	5	71	−0.212	−6.36	−0.84	−1.97	−3.55
0.7	1	2.475	−3.83	4.94	−0.77	−3.17	−31.7	−1	−25.28	−9.82	−36.10	5	71	−0.508	−15.24	−11.49	−1.58	−2.17
0.8	0	1.967	−6.00	2.40	−1.20	−1.20	−12.0	0	−39.60	−3.72	−43.32	5	71	−0.610	−18.30	−18.00	−0.60	0.30
0.9	0	1.357	−5.70	−0.65	−1.14	1.79	17.9	0	−37.62	5.55	−32.07	5	71	−0.452	−13.56	−17.10	0.90	2.64
1.0	0	0.905	−3.06															

* $\bar{v} = v - v_i$, where v_i = inelastic displacement = $v_{max} - 1.2$ *in*;

** $k = 0$ while frame is yielding.

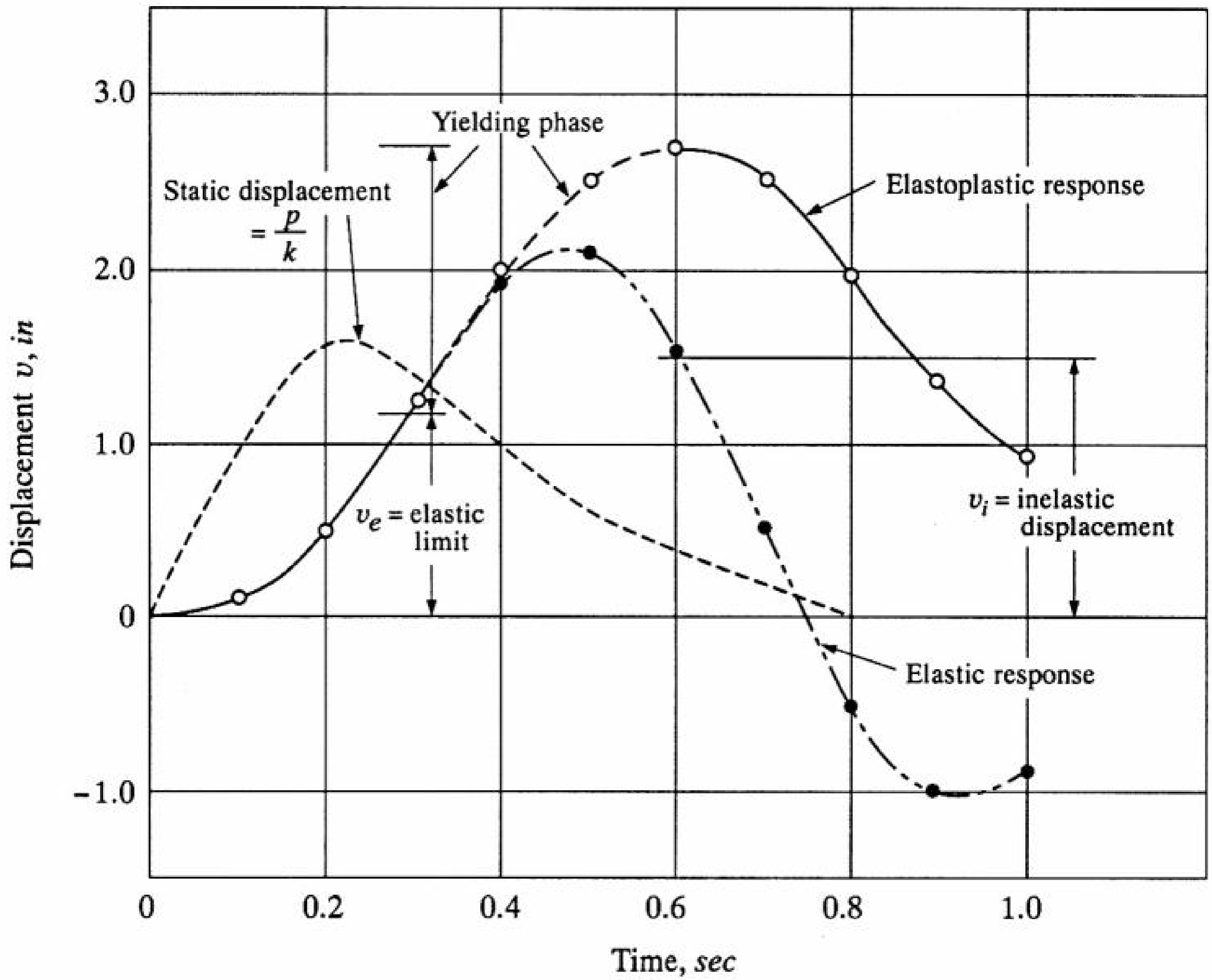


Fig. 5.4

Comparison of elastoplastic with elastic response (frame of Fig. E8-1).

Compute responses of a SDOF system by Newmark **β method**

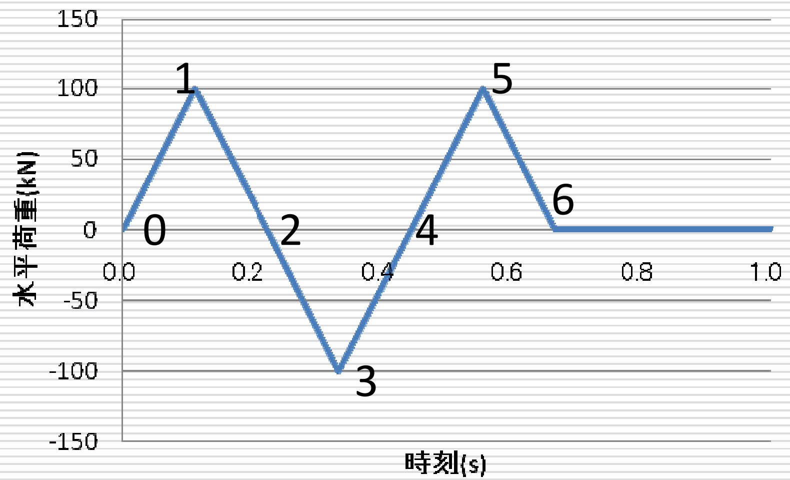
σ=0.25, δ=0.5, ξ =0.05, m=150/9.8ton, k_0=3050.9kN/m, P_0=100kN, T_0=0.445s

$$P = P_0 \sin\left\{\frac{2\pi}{T_p} t\right\}$$

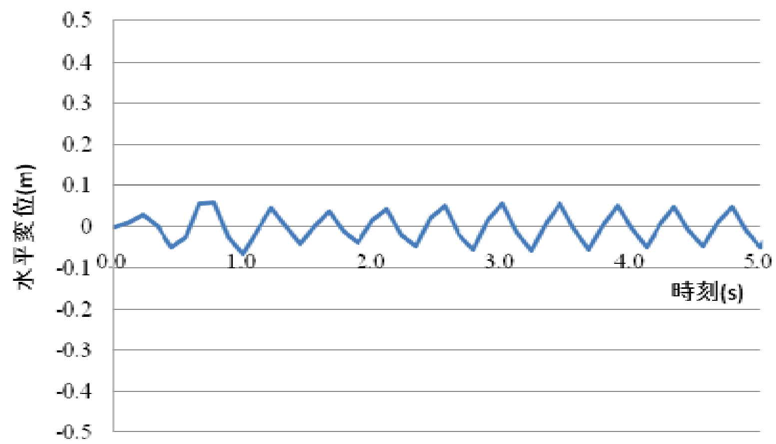
$$\tilde{k} = k + \frac{m}{\sigma \Delta t^2} + \frac{c\delta}{\sigma \Delta t} \qquad (5.21)$$

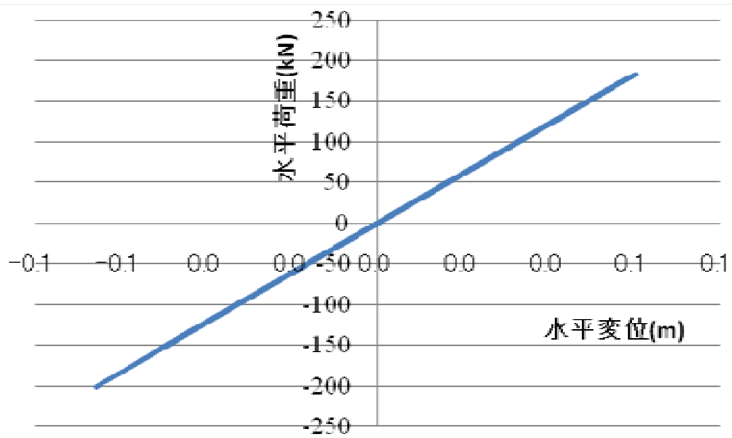
$$\tilde{p}_{t+\Delta t} = p_{t+\Delta t} + m\left\{\frac{u_t}{\sigma \Delta t^2} + \frac{\dot{u}_t}{\sigma \Delta t} + \left(\frac{1}{2} - \sigma\right)\frac{\dot{u}_t}{\sigma}\right\} + c\left\{\frac{\delta}{\sigma \Delta t} u_t + \left(\frac{\delta}{\sigma} - 1\right)\dot{u}_t + \left(\frac{\delta}{2\sigma} - 1\right)\Delta t \ddot{u}_t\right\} \qquad (5.22)$$

	時刻	水平荷重	変位	速度	加速度	剛性	減衰係数			変位増分	速度増分	加速度増分	復元力
	t(s)	p(kN)	u	u_dot	u_2dot	k	c	k_tilde	Δp_tilde	Δu	Δu_dot	Δu_2dot	Rf
1	0.00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	3050.9	21.60959	8385.2854	1.00E+02	1.19E-02	2.14E-01	3.85E+00	0.00E+00
2	0.11	1.00E+02	1.19E-02	2.14E-01	3.85E+00	3050.9	21.60959	8385.2854	1.45E+02	1.73E-02	-1.17E-01	-9.82E+00	3.64E+01
3	0.22	1.23E-14	2.92E-02	9.69E-02	-5.97E+00	3050.9	21.60959	8385.2854	-2.25E+02	-2.68E-02	-6.76E-01	-2.27E-01	8.92E+01
4	0.33	-1.00E+02	2.40E-03	-5.79E-01	-6.19E+00	3050.9	21.60959	8385.2854	-4.33E+02	-5.17E-02	2.30E-01	1.65E+01	7.31E+00
5	0.45	-2.45E-14	-4.93E-02	-3.50E-01	1.03E+01	3050.9	21.60959	8385.2854	2.08E+02	2.48E-02	1.15E+00	-3.68E-02	-1.50E+02
6	0.56	1.00E+02	-2.45E-02	7.96E-01	1.03E+01	3050.9	21.60959	8385.2854	6.87E+02	8.20E-02	-1.19E-01	-2.27E+01	-7.46E+01
7	0.67	3.68E-14	5.75E-02	6.77E-01	-1.24E+01	3050.9	21.60959	8385.2854	2.17E+01	2.59E-03	-1.31E+00	1.33E+00	1.76E+02



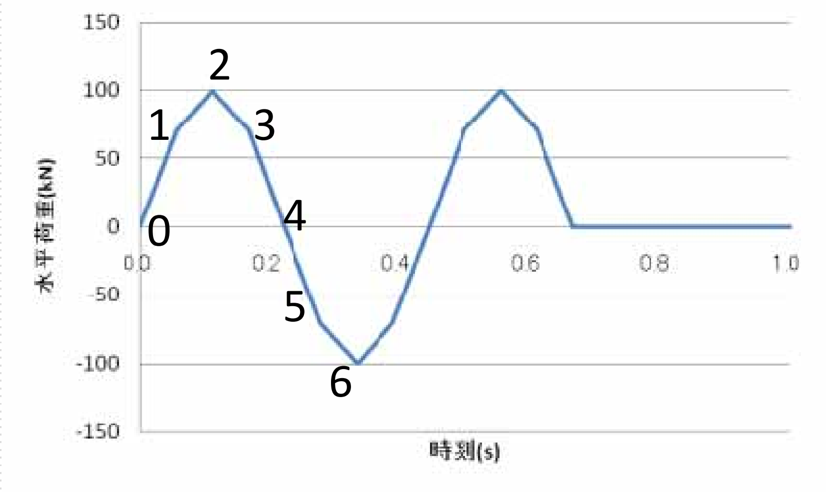
Δt=T_p/4の時

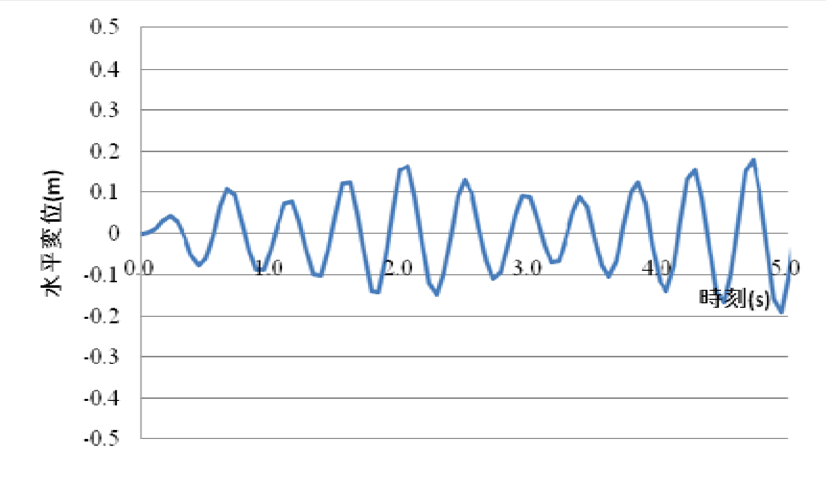


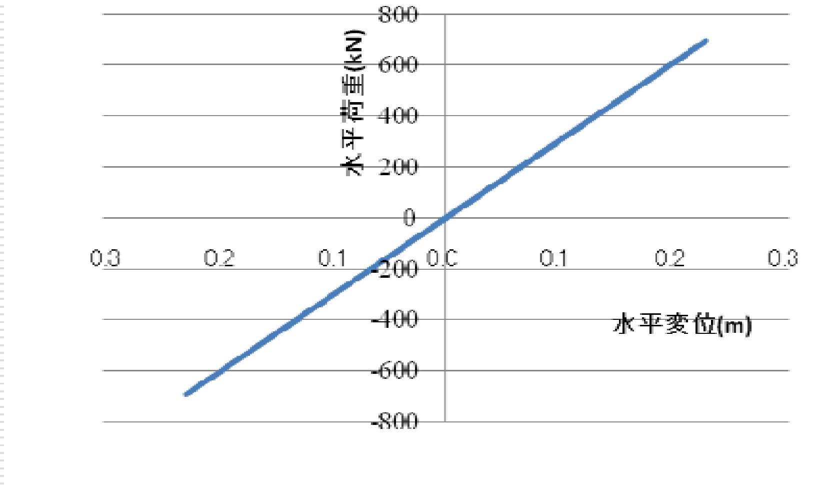


	時刻	水平荷重	変位	速度	加速度	剛性	減衰係数			変位増分	速度増分	加速度増分	復元力
	t(s)	p(kN)	u	u_dot	u_2dot	k	c	k_tilde	Δp_tilde	Δu	Δu_dot	Δu_2dot	Rf
0	0.00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	3050.9	21.60959	23611.536	7.07E+01	2.99E-03	1.08E-01	3.87E+00	0.00E+00
1	0.06	7.07E+01	2.99E-03	1.08E-01	3.87E+00	3050.9	21.60959	23611.536	2.71E+02	1.15E-02	1.97E-01	-6.52E-01	9.14E+00
2	0.11	1.00E+02	1.45E-02	3.05E-01	3.22E+00	3050.9	21.60959	23611.536	4.18E+02	1.77E-02	2.67E-02	-5.48E+00	4.41E+01
3	0.17	7.07E+01	3.22E-02	3.32E-01	-2.26E+00	3050.9	21.60959	23611.536	2.39E+02	1.01E-02	-2.99E-01	-6.22E+00	9.81E+01
4	0.22	1.23E-14	4.23E-02	3.28E-02	-8.48E+00	3050.9	21.60959	23611.536	-2.93E+02	-1.24E-02	-5.11E-01	-1.43E+00	1.29E+02
5	0.28	-7.07E+01	2.99E-02	-4.79E-01	-9.91E+00	3050.9	21.60959	23611.536	-8.80E+02	-3.73E-02	-3.83E-01	6.05E+00	9.12E+01
6	0.33	-1.00E+02	-7.36E-03	-8.61E-01	-3.85E+00	3050.9	21.60959	23611.536	-1.00E+03	-4.25E-02	1.95E-01	1.47E+01	-2.24E+01

Δt=T_p/8の時







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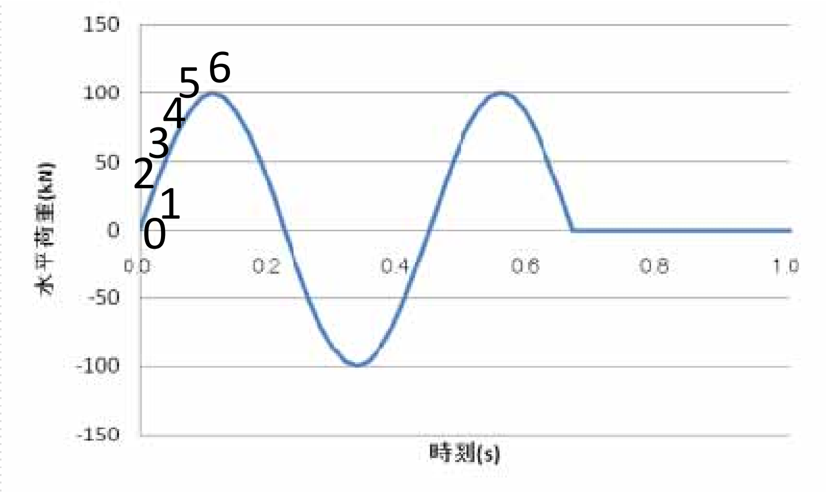
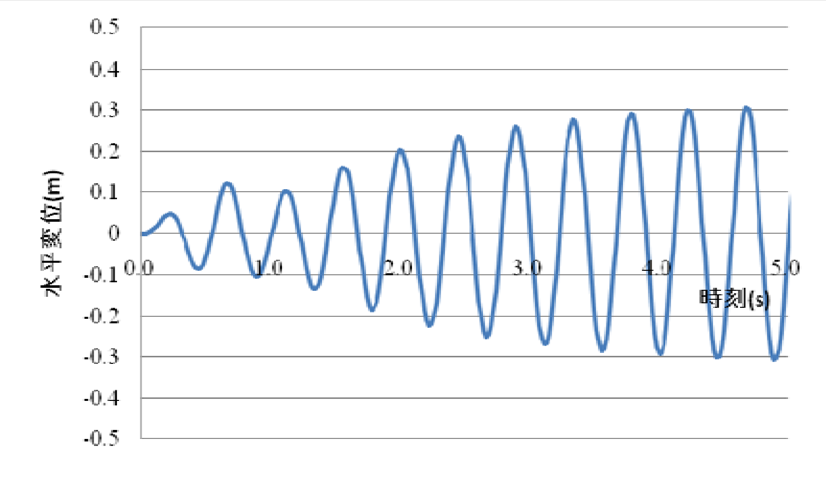
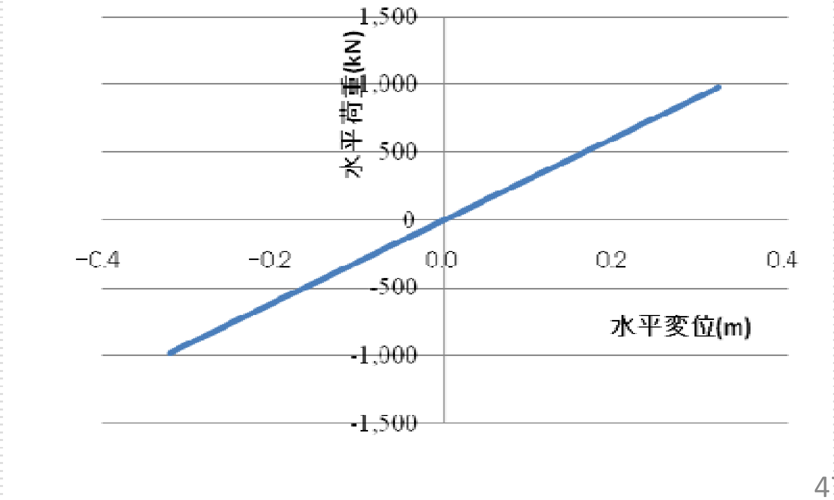
	時刻	水平荷重	変位	速度	加速度	剛性	減衰係数			変位増分	速度増分	加速度増分	復元力
	t(s)	p(kN)	u	u_dot	u_2dot	k	c	k_tilde	Δp_tilde	Δu	Δu_dot	Δu_2dot	Rf
0	0.00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	3050.9	21.60959	183435.2	2.59E+01	1.41E-04	1.52E-02	1.64E+00	0.00E+00
1	0.02	2.59E+01	1.41E-04	1.52E-02	1.64E+00	3050.9	21.60959	183435.2	1.25E+02	6.83E-04	4.32E-02	1.38E+00	4.30E-01
2	0.04	5.00E+01	8.24E-04	5.84E-02	3.02E+00	3050.9	21.60959	183435.2	3.09E+02	1.68E-03	6.46E-02	9.27E-01	2.51E+00
3	0.06	7.07E+01	2.51E-03	1.23E-01	3.95E+00	3050.9	21.60959	183435.2	5.48E+02	2.99E-03	7.63E-02	3.35E-01	7.65E+00
4	0.07	8.66E+01	5.50E-03	1.99E-01	4.28E+00	3050.9	21.60959	183435.2	8.08E+02	4.40E-03	7.63E-02	-3.33E-01	1.68E+01
5	0.09	9.66E+01	9.90E-03	2.76E-01	3.95E+00	3050.9	21.60959	183435.2	1.05E+03	5.70E-03	6.39E-02	-1.00E+00	3.02E+01
6	0.11	1.00E+02	1.56E-02	3.40E-01	2.94E+00	3050.9	21.60959	183435.2	1.13E+03	6.14E-03	-1.71E-02	-7.73E+00	4.76E+01

Δt=T_p/24の時







・Δt=T_p/4,T_p/8,T_p/24を重ね合わせたグラフ