Structural Dynamics
構造動力学
(5)

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Chapter 4 Response to General Dynamic Loading: Superposition Methods一般的な荷重に対する応答（重ね合わせ法に基づく）

This chapter is described based on “Dynamics of Structures” by Shelton Cherry.

4.1 Impulsive Excitation: The Impulse Function
インパルス応答 (インパルス関数)

● Consider an idealized SDOF viscous damped system (under critically damped system) at rest at t=0 at which time a force F_0 is applied to the mass for a very short interval Δt.

● Therefore at time t=0, the system is subjected to a single impulse $I=F_0\Delta t$ (**力積**) as shown in Fig. 4.1

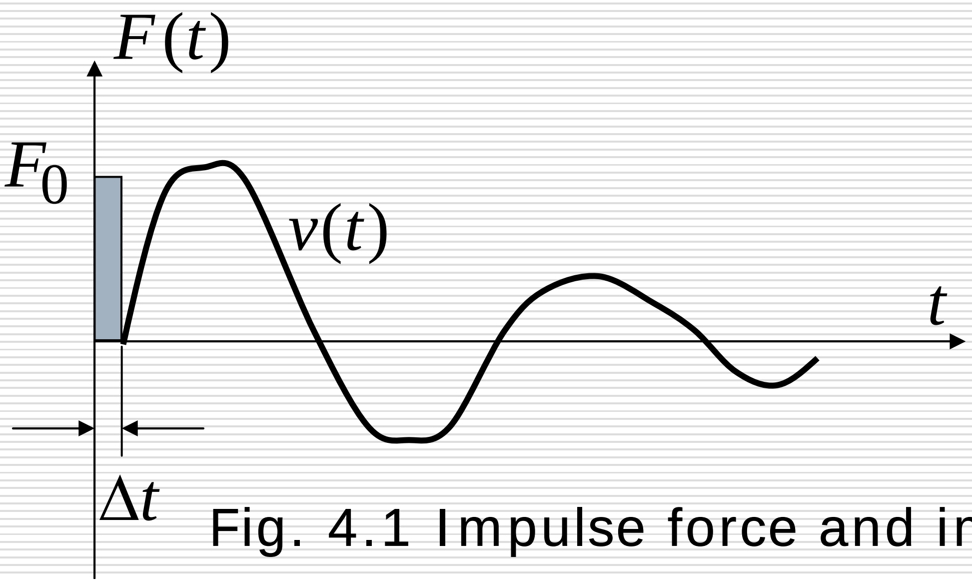

Fig. 4.1 Impulse force and impulse response

● From impulse-momentum principle, we have

$$F_0 \Delta t = m\dot{v}_0 \qquad (4.1)$$

● Eq. (4.1) implies that as a result of the impulse *I*, the mass of the system acquires an instantaneous velocity

$$\dot{v}_0 = \frac{F_0 \Delta t}{m} = \frac{I}{m} \qquad (4.2)$$

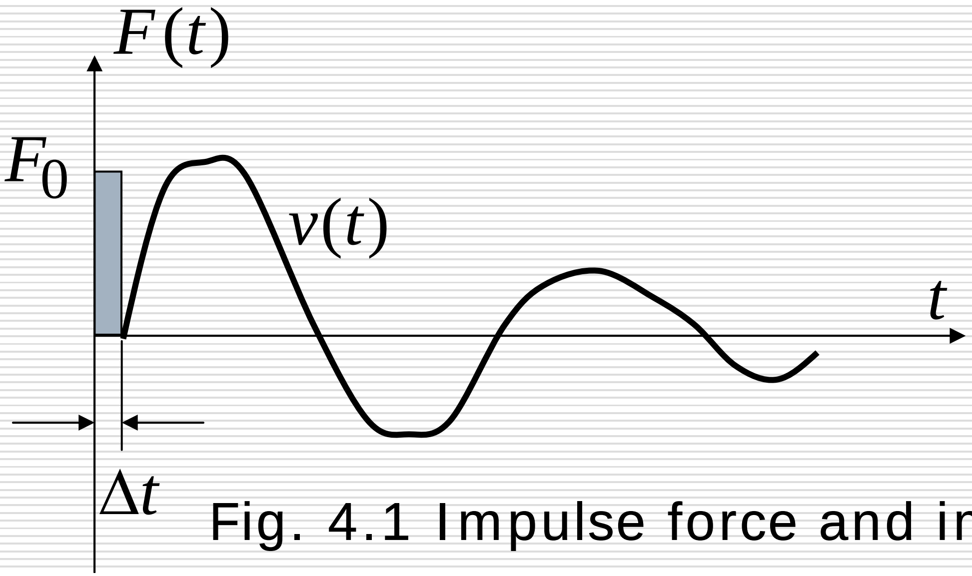

Fig. 4.1 Impulse force and impulse response

● Since F(t)=0 for $t>\tau$, there is no further excitation and hence no particular integral in the solution of the basic equation of motion.

● The system therefore executes free vibration

● If we measure time from the end of the impulse and assume no appreciable change in displacement during the short time Δt, the initial conditions are $v(0)=v_0=0$ and $\dot{v}(0)=\dot{v}_0$.

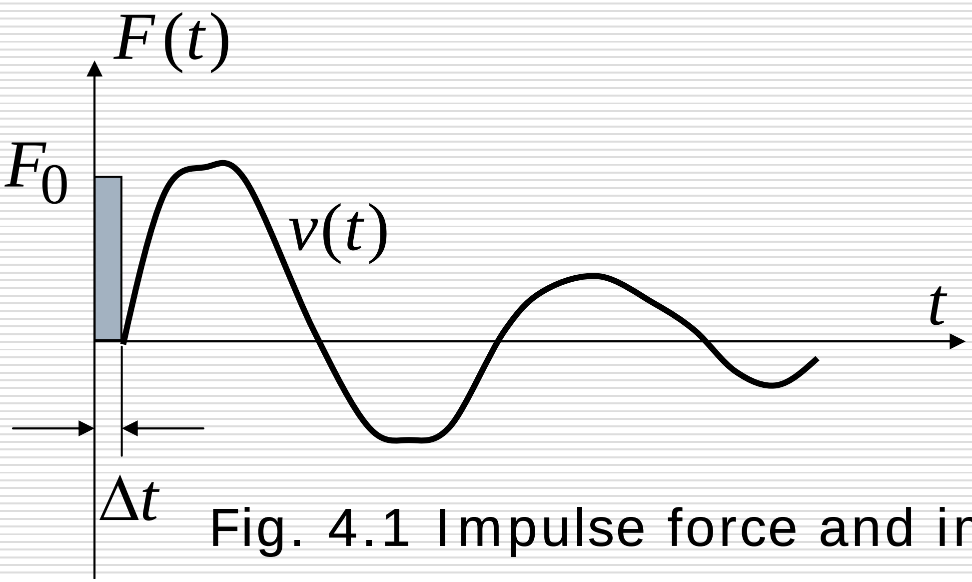

Fig. 4.1 Impulse force and impulse response

● Eq. (2.50) represents the damped free vibration of the system with $v(0)=0$ and $\dot{v}(0)$.

$$v(t)=\rho\cos(\omega_D t+\theta)e^{-\xi\omega t} \qquad (2.50)$$

$$\rho=\sqrt{v(0)^2+\left(\frac{\dot{v}(0)+v(0)\xi\omega}{\omega_D}\right)^2} \qquad (2.51)$$

$$\theta=-\tan^{-1}\left(\frac{\dot{v}(0)+v(0)\xi\omega}{\omega_D v(0)}\right) \qquad (2.52)$$

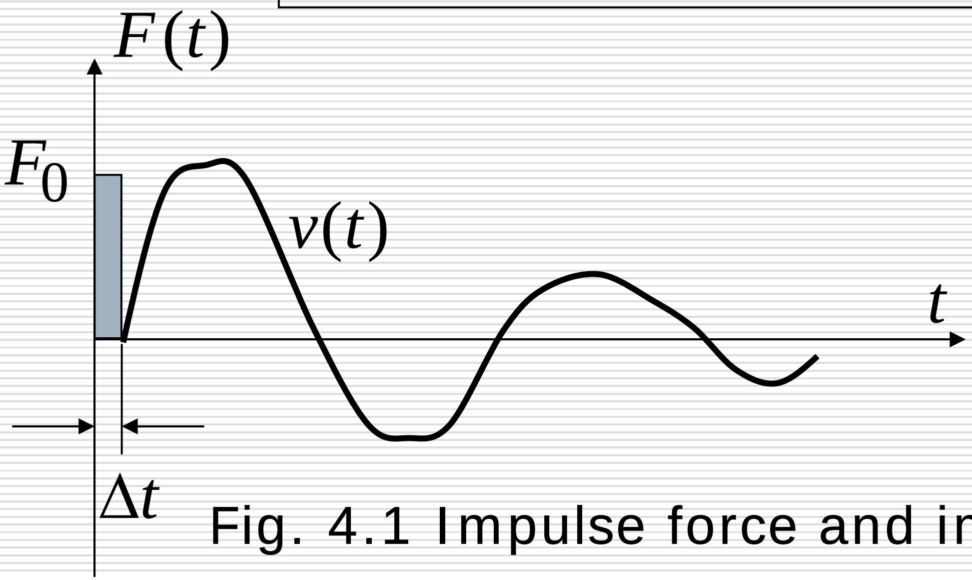

Fig. 4.1 Impulse force and impulse response

- Denoting from Eq.(2.52), $\theta=-\pi/2$, Eq. (2.50) becomes

$$v(t)=\frac{\dot{v}_0}{\omega_D}\cos(\omega_D t-\frac{\pi}{2})e^{-\omega\xi t}$$

or

$$v(t)=\frac{I}{m\omega_D}e^{-\omega\xi t}\sin\omega_D t \qquad (4.3)$$

$$v(t)=\rho\cos(\omega_D t+\theta)e^{-\xi\omega t} \qquad (2.50)$$

$$\rho=\sqrt{v(0)^2+\left(\frac{\dot{v}(0)+v(0)\xi\omega}{\omega_D}\right)^2} \qquad (2.51)$$

$$\theta=-\tan^{-1}\left(\frac{\dot{v}(0)+v(0)\xi\omega}{\omega_D v(0)}\right) \qquad (2.52)$$

- Eq.(4.3) shows a damped sine-wave of circular frequency ω_D and decay of amplitude by exp(-ξωt)

$$v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin \omega_D t \qquad (4.3)$$

- If we define the notation *h(t)* to represent the response to a unit impulse, then Eq. (4.3) becomes

$$v(t) = I \cdot h(t) \qquad (4.4)$$

where, *h(t)* is referred to as the impulse-response function **(インパルス応答関数)**.

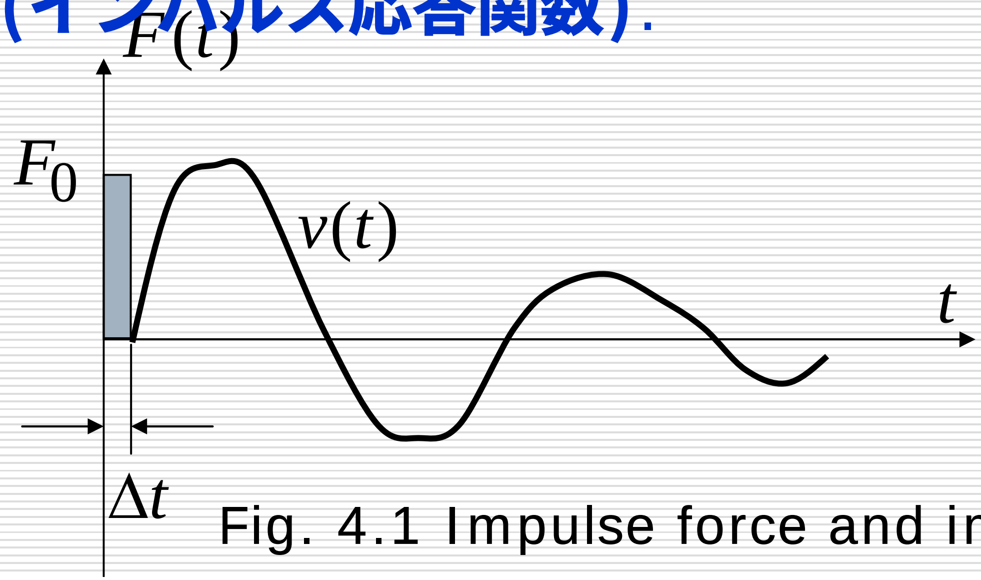

Fig. 4.1 Impulse force and impulse response

4.2 Response to the General Transient (過渡応答)

- Transient response of a SDOF system subjected to an arbitrary external force F(t) may be considered that it is developed as a sequence of infinitesimal impulses each of which contributes to the total motion according to Eq. (4.3).

$$v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin\omega_D t \qquad (4.3)$$

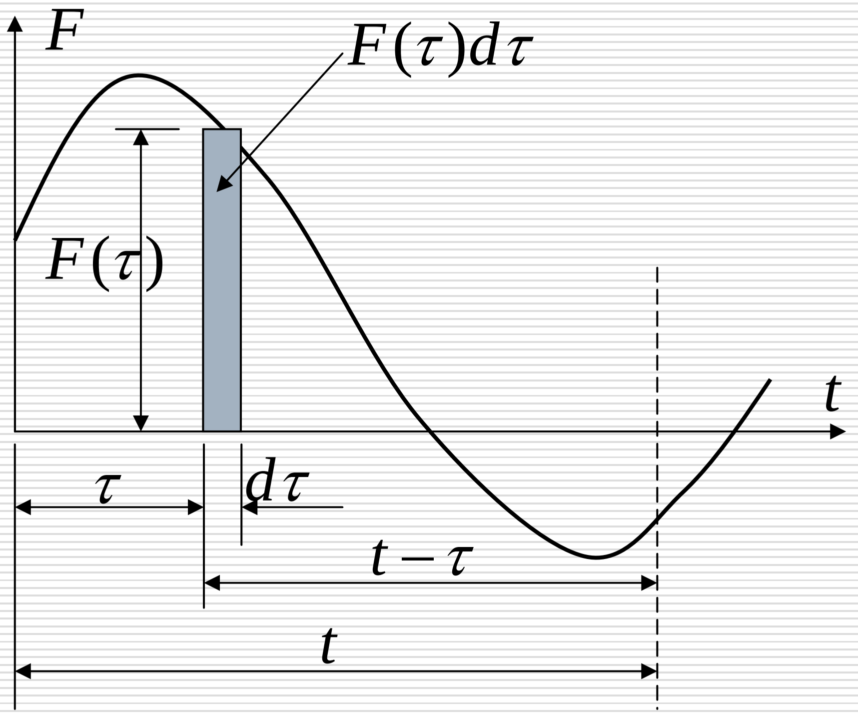

Fig. 4.2 Transient response as a sequence of impulses

● Assuming v(0) = $\dot{v}(0)$ = 0 at t = 0 (this restriction shall later be removed), the response due to any one impulse in the absence of all others may be written from Eq. (4.3) as

$$dv(t) = \frac{F(\tau)d\tau}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin\omega_D(t-\tau) \qquad (4.5)$$

● In Eq. (4.5) we have to carefully distinguish between t (the time at which the response is desired) and τ (the time at which the impulse applies)

$$\boxed{v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin\omega_D t \qquad (4.3)}$$

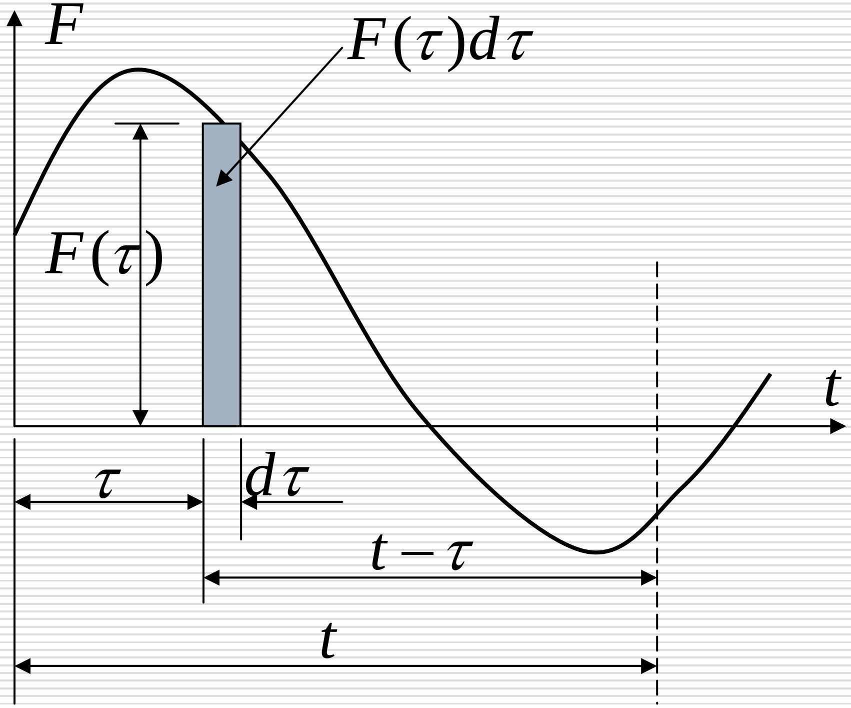

Fig. 4.2 Transient response as a sequence of impulses

● The effect of all impulses in the interval t=0 and t=t contribute to the total response at time t.

●Because we now consider the linear system, the total response due to all impulses may be evaluated by superposition (**重ね合わせ**) of Eq. (4.5) as

$$v(t)=\int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin\omega_D(t-\tau)d\tau \qquad (4.6)$$

where τ is a dummy variable which disappears on integration.

● Eq. (4.6) is known as Duhamel integral (**デュアメル積分**), superposition integral (**重ね合わせ積分**) or convolution integral (**畳み込み積分**).

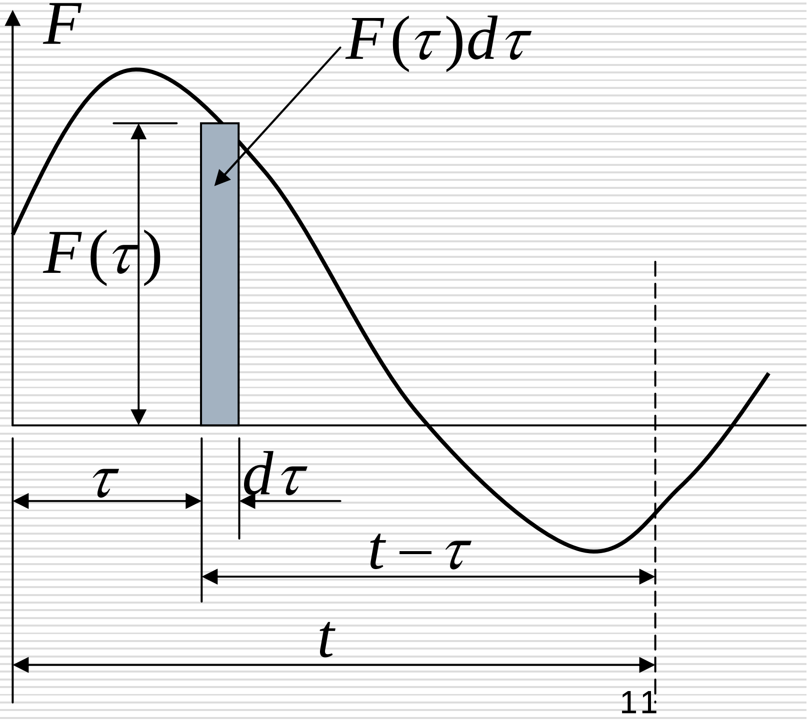

Fig. 4.2

- If the system has initial condition $v(0)=v_0$, and $\dot{v}(0)=\dot{v}_0$, the complete solution of the response becomes from Eq. (2.49) and Eq. (4.6) as

$$v(t)=\left\{v(0)\cos\omega_D t+\left(\frac{\dot{v}(0)+v(0)\xi\omega}{\omega_D}\right)\sin\omega_D\right\}e^{-\xi\omega t}$$
$$+\int_0^t \frac{F(\tau)}{m\omega_D}e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)d\tau \qquad (4.7)$$

in which, the first term represents the free vibration of the system.

$$v(t)=\int_0^t \frac{F(\tau)}{m\omega_D}e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)d\tau \qquad (4.6)$$

$$v(t)=\left\{v(0)\cos\omega_D t+\left(\frac{\dot{v}(0)+v(0)\xi\omega}{\omega_D}\right)\sin\omega_D\right\}e^{-\xi\omega t} \qquad (2.49)$$

● Eq. (4.6) can be rewritten using the impulse response function (インパルス応答関数) given by Eq. (4.4) as

$$v(t) = \int_0^t F(\tau) h(t-\tau) d\tau \qquad (4.8)$$

where

$$h(t-\tau) = \frac{1}{m\omega_D} \left\{ e^{-\xi\omega(t-\tau)} \sin \omega_D (t-\tau) \right\} \qquad (4.9)$$

●Depending on the complexity of the system, it may or may not be easy to find h(t-τ) analytically. It is sometimes possible to obtain good approximations to h(t-τ) experimentally.

$$v(t) = I \cdot h(t) \qquad (4.4)$$

$$v(t) = \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin \omega_D (t-\tau) d\tau \qquad (4.6)$$

4.3 Response of SDOF System to Earthquake Ground Motions (地震動を受ける1自由度系の応答)

- Based on Eq. (2.17), the external force $F(t)$ which applies to a SDOF system is

$$F(t) = -m\ddot{u}_g(t) \qquad (4.10)$$

- Hence substitution of the above equation to Eq. (4.6) leads to

$$v(t) = \frac{1}{\omega_D}\int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \sin\omega_D(t-\tau)\,d\tau \qquad (4.11)$$

$$v(t) = -\int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega_n(t-\tau)} \sin\omega_D(t-\tau)\,d\tau \qquad (4.6)$$

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = -m\ddot{v}_g(t) \equiv p_{eff}(t) \qquad (4.17)$$

●Denoting

$$\frac{d}{dt}\left\{\int_0^t f(\tau,t)d\tau\right\} = \int_0^t \frac{\partial f(\tau,t)}{\partial t}d\tau + f(\tau,t)_{\tau=t}$$

differentiation of Eq. (4.11) leads to

$$\dot{v}(t) = -\int_0^t \ddot{u}_g(\tau)e^{-\xi\omega_n(t-\tau)}\cos\omega_D(t-\tau)d\tau$$
$$+\frac{\xi}{\sqrt{1-\xi^2}}\int_0^t \ddot{u}_g(\tau)e^{-\xi\omega_n(t-\tau)}\sin\omega_D(t-\tau)d\tau$$

(4.12)

$$v(t) = -\frac{1}{\omega_D}\int_0^t \ddot{u}_g(\tau)e^{-\xi\omega_n(t-\tau)}\sin\omega_D(t-\tau)d\tau \quad (4.11)$$

●A further differentiation of Eq. (4.11) yields $\ddot{v}(t)$.

●Absolute acceleration $\ddot{v}_a$ (note that $\ddot{v}_a$ was represented as $\ddot{v}^t$ in Eq. (2.15), but it is represented as $\ddot{v}_a$ here for convenience of the notation), which is important for evaluation of the inertia force, is

$$
\begin{aligned}
\ddot{v}_a(t) &= \ddot{v}(t) + \ddot{u}_g(t) \\
&= \frac{\omega_n(1-2\xi^2)}{\sqrt{1-\xi^2}} \int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \sin\omega_D(t-\tau)\,d\tau \\
&\qquad + 2\omega_n\xi \int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \cos\omega_D(t-\tau)\,d\tau \\
&= \omega_D \int_0^t \ddot{u}_g(\tau) e^{-\xi\omega_n(t-\tau)} \left\{ \left(1 - \frac{\xi^2}{1-\xi^2}\sin\omega_D(t-\tau)\right) \right. \\
&\qquad \left. + \frac{2\xi}{\sqrt{1-\xi^2}} \cos\omega_D(t-\tau)\,d\tau \right\} \qquad (4.13)
\end{aligned}
$$