Nash equilibrium

Definition 8.D.1: (Nash equilibrium)

$s = (s_1, \dots, s_I)$ is a Nash equilibrium

in $\Gamma_N = [N=\{0,1,\dots,I\}, \{S_i\}, \{u_i\}]$

if $\forall\ i = 1, \dots, I$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})\ \forall\ s'_i \in S_i$.

Note: Nash eq. $\rightarrow$ each player's strategy is a best response to the strategies actually played by her rivals

Rationalizable strategies

$\rightarrow$ best response to some justified strategies of the rivals

Nash equilibrium

Example 8.D.1:

_ denotes a best response

(M, m) is the unique Nash eq.

	l	m	r
U	$\underline{5}$, 3	0, 4	3, $\underline{5}$
M	4, 0	$\underline{5}$, $\underline{5}$	4, 0
D	3, $\underline{5}$	0, 4	$\underline{5}$, 3

Example 8.D.2:

(a_2, b_2) is the unique Nash eq.

rationalizable strategies $\rightarrow$
$\{a_1, a_2, a_3\}$ for 1,
$\{b_1, b_2, b_3\}$ for 2

	b_1	b_2	b_3	b_4
a_1	0, $\underline{7}$	2, 5	$\underline{7}$, 0	0, 1
a_2	5, 2	$\underline{3}$, $\underline{3}$	5, 2	0, 1
a_3	$\underline{7}$, 0	2, 5	0, $\underline{7}$	0, 1
a_4	0, $\underline{0}$	0, -2	0, $\underline{0}$	$\underline{10}$, -1

Note: Every strategy in Nash eq. $\rightarrow$ rationalizable

Nash equilibrium

Example 8.D.3:

_ denotes a best response

(E, E), (C, C) are Nash eq.

	E	C
E	<u>100</u>, <u>100</u>	0, 0
C	0, 0	<u>100</u>, <u>100</u>

Nash eq. theory says nothing which eq. we should expect.

Best-response correspondence

$b_i : S_{-i} \rightarrow S_i$

$$b_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \ \ \forall s'_i \in S_i\}$$

$s = (s_1, \ldots, s_I)$ is a Nash equilibrium

in $\Gamma_N = [N=\{0,1,\ldots,I\},\{S_i\}, \{u_i\}]$

iff $\quad s_i \in b_i(s_{-i}) \quad \forall\, i = 1, \ldots, I$

Nash equilibrium ー Discussion

Why should we concern ourselves with the concept of Nash eq. ?

How do players reach a Nash eq. ?

1. Nash eq. as a consequence of rational inference
2. Nash eq. as a necessary condition

   if there is a unique predicted outcome
3. Focal points
4. Nash eq. as a self-enforcing agreement
5. Nash eq. as a stable social convention

Mixed Strategy Nash equilibrium

Definition 8.D.1:

$\sigma = (\sigma_1, \dots, \sigma_I)$ is a Nash equilibrium

in $\Gamma_N = [N=\{0,1,\dots,I\}, \{\Delta(S_i)\}, \{u_i\}]$

if $\forall\, i = 1, \dots, I$, $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})\ \forall\, \sigma'_i \in \Delta(S_i)$.

Example 8.D.4 :

((1/2, 1/2), (1/2, 1/2)) is a unique Nash eq.

1's payoff: H $-1\times1/2 + 1\times1/2 = 0$

T $1\times1/2 + (-1)\times1/2 = 0$

same for 2

	H	T
H	-1, +1	+1, -1
T	+1, -1	-1, +1

Mixed Strategy Nash equilibrium

Proposition 8.D.1: $S^+_i \subseteq S_i$ set of pure str. played with positive prob. in $\sigma = (\sigma_1, \dots, \sigma_I)$. σ is a Nash eq. in $\Gamma_N = [N=\{0,1,\dots,I\}, \{\Delta(S_i)\}, \{u_i\}]$ iff $\forall$ i= 1, … ,I,

(i) $u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \;\; \forall s_i, s'_i \in S^+_i$

(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \;\; \forall s_i \in S^+_i, \; \forall s'_i \notin S^+_i$

Pf : $\rightarrow$) First show that $\forall$ i= 1, … ,I

$$u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \;\; \forall s_i \in S^+_i, \; \underline{\forall s'_i \in S_i}$$

Suppose not, i.e., $\exists$ i, $s_i \in S^+_i$, $s'_i \in S_i$ s.t. $u_i(s'_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i})$.

Let σ'_i be s.t.

$$\sigma'_i(s''_i) = \begin{cases} \sigma_i(s''_i) & \text{for } s''_i \neq s_i, s'_i \\ \sigma_i(s'_i) + \sigma_i(s_i) & \text{for } s''_i = s'_i \\ 0 & \text{for } s''_i = s_i \end{cases}$$

Then $u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$, contradicting that σ is a Nash eq.

Mixed Strategy Nash equilibrium

Proposition 8.D.1: $S^+_i \subseteq S_i$ set of pure str. played with positive prob. in $\sigma = (\sigma_1, \ldots, \sigma_I)$. σ is a Nash eq. in $\Gamma_N = [N=\{0,1,\ldots,I\}, \{\Delta(S_i)\}, \{u_i\}]$ iff $\forall$ i= 1, … ,I,

$$\text{(i)}\ \ u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \ \ \forall\, s_i, s'_i \in S^+_i$$

$$\text{(ii)}\ u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \ \ \forall\, s_i \in S^+_{i,}\ \forall\, s'_i \notin S^+_i$$

Pf : $\rightarrow$) Next show that $\forall$ i= 1, … ,I

$$u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \ \ \forall\, s_i , s'_i \in S^+_{i,}$$

This is clear from the fact shown above:

$$u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \ \ \forall\, s_i \in S^+_{i,}\ \underline{\forall\, s'_i \in S_i}$$

Mixed Strategy Nash equilibrium

<u>Proposition 8.D.1</u>: $S^+_i \subseteq S_i$ set of pure str. played with positive prob. in $\sigma = (\sigma_1, \ldots, \sigma_I)$. σ is a Nash eq. in $\Gamma_N = [N=\{0,1,\ldots,I\}, \{\Delta(S_i)\}, \{u_i\}]$ iff $\forall$ i= 1, … ,I,

$$\text{(i)}\quad u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \quad \forall\, s_i, s'_i \in S^+_i$$

$$\text{(ii)}\quad u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \quad \forall\, s_i \in S^+_i,\ \forall\, s'_i \notin S^+_i$$

<u>Pf</u> : $\longleftarrow$) Suppose that σ is not a Nash eq. .
Then $\exists$ i, $\sigma'_i \in \Delta(S_i)$ s.t. $u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$.
Then $\exists\, s'_i \in S_i$ s.t. $u_i(s'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$ with $\sigma'_i(s'_i) > 0$.
From (i), $u_i(s_i, \sigma_{-i}) = u_i(\sigma_i, \sigma_{-i})$ for all $s_i \in S^+_i$.
Thus $s'_i \notin S^+_i$, contradicting (ii).

<u>Note</u>: To see a Nash eq. or not,
it suffices to check deviations to pure strategies.

Mixed Strategy Nash equilibrium

<u>Corollary 8.D.1</u>:

$s = (s_1, \ldots , s_I)$ is a Nash eq. of $\Gamma_N = [N=\{0,1,\ldots,I\}, \{S_i\}, \{u_i\}]$

iff it is a Nash eq. of $\Gamma'_N = [N=\{0,1,\ldots,I,\} \{\Delta(S_i)\}, \{u_i\}]$

<u>Pf</u> : $\leftarrow)$ clear.

$\rightarrow)$ Since s is a Nash eq. of $\Gamma_N = [N=\{0,1,\ldots,I\}, \{S_i\}, \{u_i\}]$,

$$\forall\, i= 1, \ldots ,I \quad u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \forall\, s'_i \notin S^+_i$$

Thus (i), (ii) in Proposition 8.D.1 trivially hold since $S^+_i = \{s_i\}$

Thus by Prop.8.D.1,

$$s \text{ is a Nash eq. of } \Gamma'_N = [N=\{0,1,\ldots,I,\} \{\Delta(S_i)\}, \{u_i\}].$$

Mixed Strategy Nash equilibrium

Example 8.D.5:

S's mixed strategy: $(\sigma_s, 1-\sigma_s)$

T: play E $\rightarrow$ $1000\sigma_s$

play C $\rightarrow$ $100(1-\sigma_s)$

S

T		E	C
	E	1000, 1000	0, 0
	C	0, 0	100, 100

Suppose T's mixed strategy $(\sigma_T, 1-\sigma_T)$ satisfies $0 < \sigma_T < 1$.

Then $S^+_T=\{E,C\}$.

Prop. 8.D.1 $\rightarrow$ $1000\sigma_s=100(1-\sigma_s)$

$\rightarrow$ $\sigma s = 1/11$ $\rightarrow$ S's mixed strategy (1/11, 10/11)

Similarly, T's strategy (1/11, 10/11)

Nash eq. ((1/11, 10/11), (1/11, 10/11))

Mixed Strategy in Nash Equilibria ???

What is a mixed strategy in Nash equilibria?

It just makes the rival indifferent over his strategies

(The player has no preference over the probabilities.)

Is a mixed strategy useful ?

1 Players have a pure strategy that gives the same payoff.

→ why randomize them ?

→ Players may not actually randomize; but they make definite choices that are affected by signals.

2 Stability of mixed strategy Nash eq.

players do not have an incentive to use the exact probability

→ may not arise as a social convention,

but as a self-enforcing agreement

Correlated Strategies

Example 8.D.5:

Public signal $\theta \in [0,1]$

$\theta \geq 1/2 \;\rightarrow\;$ both play E

$\theta < 1/2 \;\rightarrow\;$ both play C

S

T	E	C
E	<u>1000</u>, <u>1000</u>	0, 0
C	0, 0	<u>100</u>, <u>100</u>

This is equilibrium.

If T (S) follows, then S (T) has no incentive to deviate.

Correlated equilibrium

Existence of Nash equilibrium

Proposition 8.D.2:

$\Gamma_N = [N=\{0,1,\ldots,I\}, \{\Delta(S_i)\}, \{u_i\}]$ in which $S_1, \ldots, S_I$ have a finite number of elements has a mixed strategy Nash eq.

Proposition 8.D.3:

A Nash eq. exists in $\Gamma_N = [N=\{0,1,\ldots,I\}, \{S_i\}, \{u_i\}]$ if $\forall i=1, \ldots, I$

(i) S_i is a nonempty, convex, and compact subset of $\Re^m$, and

(ii) $u_i(s_1, \ldots, s_I)$ is continuous in $(s_1, \ldots, s_I)$ and quasi-concave in s_i.

$u_i(s_1, \ldots, s_I)$ is quasi-concave in s_i

if $\forall s'_i, s''_i, \alpha\in[0,1]$

$$u_i(\alpha s'_i+(1-\alpha)s''_i, s_{-i}) \geq \min(u_i(s'_i, s_{-i}), u_i(s''_i, s_{-i}))$$

Assignments

Problem Set 4 (due May 24):

Exercises (pp.262-266):

1. 8.D.3, 8.D.4, 8.D.5, 8.D.9
2. Read (i) – (v) on the concept of Nash equilibrium (pp.248-249) and summarize them.

Reading Assignments:

Text Chapter 8, pp.253-257