7 類 V クラス 微分積分学演習第一 第 4 回 （逆井　2012 年 6 月 4 日）

演習問題 (テーマ: 偏微分, 合成函数の微分, 極座標変換)

[1] 次の合成函数 $f(x(t), y(t))$ に対して $\dfrac{df}{dt}(t)$ を合成函数の微分の公式を用いて求めよ.

(1) $f(x, y) = x^7 y + 2x^3 y^2,\ \ x = \cos t,\ \ y = e^t$.

(2) $f(x, y) = \log(x + y),\ \ x = \tan t,\ \ y = \sin t.$　ただし $0 < t < \pi/2$ とする.

[2] 領域 $\{(x, y) \in \mathbb{R}^2 \mid 0 < x\}$ における極座標変換

$$x = r\cos\theta, \quad y = r\sin\theta \quad (r > 0,\ -\pi/2 < \theta < \pi/2)$$

について次に答えよ.

(1) $\dfrac{\partial x}{\partial r}, \dfrac{\partial x}{\partial \theta}, \dfrac{\partial y}{\partial r}, \dfrac{\partial y}{\partial \theta}$ を求めよ.

(2) r, θ をそれぞれ x, y の函数として表せ.

(3) $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}, \dfrac{\partial \theta}{\partial x}, \dfrac{\partial \theta}{\partial y}$ を求めよ.

(4) $\dfrac{\partial r}{\partial x}\dfrac{\partial x}{\partial r},\ \ \dfrac{\partial \theta}{\partial x}\dfrac{\partial x}{\partial \theta},\ \ \dfrac{\partial r}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial \theta}{\partial x}\dfrac{\partial x}{\partial \theta},\ \ \dfrac{\partial r}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial y}{\partial \theta}$ を計算せよ.

[3] $f(x, y)$ を $\mathbb{R}^2$ 上の C^2-級函数とする. 極座標変換 $x = r\cos\theta,\ y = r\sin\theta$ に対し, 等式

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

が成り立つことを示せ.

[4] $f(x, y)$ を $\mathbb{R}^2$ 上の C^2-級函数とする. x, y が t に関する C^2-級函数であるとき, 合成函数 $F(t) = f(x(t), y(t))$ について

$$\frac{d^2 F}{dt^2} = \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial x}\frac{d^2 x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2 y}{dt^2}$$

が成り立つことを示せ.

[5] $f(x,y,z)$ を $\mathbb{R}^3$ 上の C^1-級函数とする. 極座標変換

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta \quad (r \geq 0,\ 0 \leq \varphi \leq 2\pi,\ 0 \leq \theta \leq \pi)$$

に対し, $\dfrac{\partial f}{\partial r}, \dfrac{\partial f}{\partial \theta}, \dfrac{\partial f}{\partial \varphi}$ のそれぞれを $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ を用いて表せ.

参考 (合成函数の微分)

- 一般に函数 $f = f(x_1, x_2, \ldots, x_n)$ と $x_i = x_i(t_1, t_2, \ldots, t_m)$ $(i = 1, 2, \ldots, n)$ の合成函数

$$F(t_1, t_2, \ldots, t_m) = f(x_1(t_1, t_2, \ldots, t_m),\ x_2(t_1, t_2, \ldots, t_m),\ \ldots,\ x_n(t_1, t_2, \ldots, t_m))$$

に対して

$$\boxed{\frac{\partial F}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\frac{\partial x_k}{\partial t_j}}$$

が成り立つ.

以上.

お知らせ

- 次回は **6 月 11 日 (月)** で, **線形代数**の演習を行います.
- 演習課題は OCW (http://www.ocw.titech.ac.jp/) にアップロードします. 演習問題や小テスト問題の略解もありますので必要に応じてダウンロードして下さい.

7 類 V クラス 微分積分学演習第一 第 4 回（逆井　2012 年 6 月 4 日）

小テスト問題

次の問題の解答を記入して下さい. 問題に断りがない限り, 答えのみを記したものは解答とは認めません.

[1] $F(x, y) = \tan^{-1}\dfrac{y}{x}$ とするとき,

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$$

を計算せよ.

学籍番号 _____ 氏名 _____ _____

[2] C^∞-級函数 $f(x,y)=\sin(3x+2y)$ について, 偏導函数

$$\frac{\partial^{p+q} f}{\partial x^p \partial y^q}(x,y)$$

を求めよ. ただし p, q は 0 以上の整数とする.

以上.

7 類 V クラス 微分積分学演習第一 第 4 回
演習課題解答

（逆井　2012 年 6 月 4 日出題）

小テスト解答

[1] (2 点)

$\dfrac{\partial}{\partial x}(\tan^{-1} x) = \dfrac{1}{1+x^2}$ であることを用いて計算をすると

$$\begin{aligned}\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-yx^{-2}}{1+y^2x^{-2}}\right) + \frac{\partial}{\partial y}\left(\frac{x^{-1}}{1+y^2x^{-2}}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-y(-2x)}{(x^2+y^2)^2} + \frac{x(-2y)}{(x^2+y^2)^2} = 0.\end{aligned}$$

[2] (2 点)

解答のみ記す. $p+q$ に関する帰納法などを使って証明すればよい.

$$\frac{\partial^{p+q} f}{\partial x^p \partial y^q}(x,y) = \begin{cases} (-1)^{\frac{p+q}{2}} 3^p 2^q \sin(3x+2y) & p+q \text{ が偶数のとき} \\ (-1)^{\frac{p+q-1}{2}} 3^p 2^q \cos(3x+2y) & p+q \text{ が奇数のとき} \end{cases}.$$

注意 $\dfrac{\partial^{p+q} f}{\partial x^p \partial y^q}(x,y) = 3^p 2^q \sin\left(3x+2y+\dfrac{(p+q)\pi}{2}\right)$ とすると場合分けは不要.

演習問題略解

[1] (1) $\dfrac{\partial f}{\partial x} = 7x^6y + 6x^2y^2,\ \dfrac{\partial f}{\partial y} = x^7 + 4x^3y$ に注意して合成函数の微分の公式を用いると

$$\begin{aligned}\frac{df}{dt}(t) &= \frac{\partial f}{\partial x}(\cos t, e^t)\cdot\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\cos t, e^t)\cdot\frac{dy}{dt}(t)\\ &= (7e^t\cos^6 t + 6e^{2t}\cos^2 t)(-\sin t) + (\cos^7 t + 4e^t\cos^3 t)e^t.\end{aligned}$$

(2) $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = \dfrac{1}{x+y}$ に注意して合成函数の微分の公式を用いると

$$\begin{aligned}\frac{df}{dt}(t) &= \frac{\partial f}{\partial x}(\tan t, \sin t)\cdot\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\tan t, \sin t)\cdot\frac{dy}{dt}(t)\\ &= \frac{1}{(\tan t + \sin t)\cos^2 t} + \frac{\cos t}{\tan t + \sin t}.\end{aligned}$$

[2] (1) $\dfrac{\partial x}{\partial r} = \cos\theta, \quad \dfrac{\partial x}{\partial \theta} = -r\sin\theta, \quad \dfrac{\partial y}{\partial r} = \sin\theta, \quad \dfrac{\partial y}{\partial \theta} = r\cos\theta.$

(2) $r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\dfrac{y}{x}.$

(3) $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r} = \cos\theta, \qquad \dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{y}{r} = \sin\theta,$

$\dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2} = \dfrac{-y}{r^2} = \dfrac{-\sin\theta}{r}, \qquad \dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{x}{r^2} = \dfrac{\cos\theta}{r}.$

(4) $\dfrac{\partial r}{\partial x}\dfrac{\partial x}{\partial r} = \cos^2\theta, \qquad \dfrac{\partial \theta}{\partial x}\dfrac{\partial x}{\partial \theta} = \sin^2\theta,$

$\dfrac{\partial r}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial \theta}{\partial x}\dfrac{\partial x}{\partial \theta} = \cos^2\theta + \sin^2\theta = 1, \qquad \dfrac{\partial r}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial y}{\partial \theta} = 1.$

[3] 合成函数の微分の公式を用いると

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta,$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta).$$

よって,

$$\begin{aligned}
\frac{\partial^2 f}{\partial r^2} &= \left\{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\right\}\cos\theta + \left\{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)\right\}\sin\theta \\
&= \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right)\sin\theta \\
&= \frac{\partial^2 f}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta, \\
\frac{\partial^2 f}{\partial \theta^2} &= \left\{\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)\right\}(-r\sin\theta) + \frac{\partial f}{\partial x}\left\{\frac{\partial}{\partial \theta}(-r\sin\theta)\right\} \\
&\quad + \left\{\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)\right\}(r\cos\theta) + \frac{\partial f}{\partial y}\left\{\frac{\partial}{\partial \theta}(r\cos\theta)\right\} \\
&= \left\{\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial \theta}\right\}(-r\sin\theta) + \frac{\partial f}{\partial x}(-r\cos\theta) \\
&\quad + \left\{\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right\}(r\cos\theta) - \frac{\partial f}{\partial y}(r\sin\theta) \\
&= \left\{\frac{\partial^2 f}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 f}{\partial x\partial y}(r\cos\theta)\right\}(-r\sin\theta) + \frac{\partial f}{\partial x}(-r\cos\theta) \\
&\quad + \left\{\frac{\partial^2 f}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 f}{\partial y^2}(r\cos\theta)\right\}(r\cos\theta) - \frac{\partial f}{\partial y}(r\sin\theta) \\
&= r^2\frac{\partial^2 f}{\partial x^2}\sin^2\theta - 2r^2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta + r^2\frac{\partial^2 f}{\partial y^2}\cos^2\theta - r\frac{\partial f}{\partial x}\cos\theta - r\frac{\partial f}{\partial y}\sin\theta.
\end{aligned}$$

以上を用いて $\dfrac{\partial^2 f}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial f}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial \theta^2}$ を計算すると, $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$ と一致する.

[4]　合成函数の微分の公式より

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

となる. 両辺を変数 t に関してもう一度微分すると, 積の微分の公式より

$$\begin{aligned}\frac{d^2F}{dt^2} &= \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\frac{dy}{dt}\right) \\ &= \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d}{dt}\left(\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d}{dt}\left(\frac{dy}{dt}\right)\end{aligned}$$

となるが, ここで再び合成函数の微分の公式より

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dt} = \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}, \\ \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt} = \frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\end{aligned}$$

が成り立つので,

$$\frac{d^2F}{dt^2} = \left(\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}\right)\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \left(\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right)\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}$$

が得られる. あとはこの等式を整理すればよい.

[5]　合成函数の微分の公式より

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = \sin\theta\cos\varphi\,\frac{\partial f}{\partial x} + \sin\theta\sin\varphi\,\frac{\partial f}{\partial y} + \cos\theta\,\frac{\partial f}{\partial z}, \\ \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} = r\cos\theta\cos\varphi\,\frac{\partial f}{\partial x} + r\cos\theta\sin\varphi\,\frac{\partial f}{\partial y} - r\sin\theta\,\frac{\partial f}{\partial z}, \\ \frac{\partial f}{\partial \varphi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \varphi} = -r\sin\theta\sin\varphi\,\frac{\partial f}{\partial x} + r\sin\theta\cos\varphi\,\frac{\partial f}{\partial y}\end{aligned}$$

となる.

レポート問題解答 (5 月 21 日出題)

[C1] (1) 合成函数の微分の公式より,

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y},$$
$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

となるので, $\left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\right)(2,-1) = 2\frac{\partial f}{\partial y}(2-(-1), 2+(-1)) = 2\frac{\partial f}{\partial y}(3,1) = 6.$

(2) 合成函数の微分の公式を再び使うと,

$$\begin{aligned}
\frac{\partial^2 F}{\partial u^2} &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) \\
&= \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u} \\
&= \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}, \\
\frac{\partial^2 F}{\partial v^2} &= \frac{\partial}{\partial v}\left(-\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) \\
&= -\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial v} - \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial v} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial v} \\
&= \frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

となるので,

$$\left(\frac{\partial^2 F}{\partial u^2} - \frac{\partial^2 F}{\partial v^2}\right)(2,-1) = 4\frac{\partial^2 f}{\partial x \partial y}(2-(-1), 2+(-1)) = 4\frac{\partial^2 f}{\partial x \partial y}(3,1) = -8.$$

注意 $\frac{\partial^2 f}{\partial x \partial y}$ と $\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)$ は意味が異なるものです.

参考 $f(x,y) = -2xy + 9y$ は問題文の条件をみたす. この $f(x,y)$ に対して, $F(u,v) = -2u^2+2v^2+9u+9v$ となるが, $\left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\right)(2,-1)$, $\left(\frac{\partial^2 F}{\partial u^2} - \frac{\partial^2 F}{\partial v^2}\right)(2,-1)$ を計算すると, 確かに上で求めたものと一致する.