Basic Mathematics

Norimasa Kobayashi @ Tokyo Institute of Technology

1 Logic

$\wedge$ denotes "and"

$\vee$ denotes "or"

$\neg$ denotes "not"

$p \Rightarrow q$ denotes "if p then q"

$p \Leftarrow q$ denotes "if q then p"

$p \Leftrightarrow q$ is defined by $(p \Rightarrow q) \wedge (p \Leftarrow p)$ and is read "p if and only if (iff) q"

$\forall$ denotes "for all"

$\exists$ denotes "exists"

2 Sets and Functions

Definition 2.1 (Power Set) *The power set of a set X is the set of all subsets of X denoted $\mathcal{P}(X) = \{A|A \subset X\}$*

Definition 2.2 (Binary Relation) *A binary relation R between an element in set X and an element in Y is a subset of the Cartesian product $X \times Y$, that is $R \subset X \times Y$.*
The statement $(x, y) \in R$ is read "x is R-related to y" and is denoted xRy.
When $X = Y$, binary relation $R \subset X^2$ is said to be defined on set X.

Definition 2.3 (Function) *A function $f : X \to Y$ is a binary relation $f \subset X \times Y$ that associates to each element $x \in X$ exactly one element $y \in Y$, that is:*

- $(\forall x \in X \exists y \in Y)(x, y) \in f$
- $(\forall x \in X \forall y, y' \in Y)(x, y), (x, y') \in f \Rightarrow y = y'$

(x, y) is denoted $y = f(x)$.

Definition 2.4 (Image and Preimage (Inverse Image)) *Let $f : X \to Y$ be a function.*

Image *For $\forall A \subset X$, $f(A) := \{f(x)|x \in A\}$*

Preimage *For $\forall B \subset Y$, $f^{-1}(B) := \{x \in X|f(x) \in B\}$*

3 Vectors

Following notations are used for vectors and cartesian products. Particularly, vectors are denoted with normal fonts.

- $x = (x_i)_{i \in N} = (x_1, \cdots, x_N) \in X = \times_{i \in N} X_i$
- $x_{-i} := (x_j)_{j \in N \setminus \{i\}} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i} = \times_{j \in N \setminus \{i\}} X_j$

4 Probability

Definition 4.1 *Denote $\Delta(X)$ a set of probability distributions over set X.*
If X is finite, $\Delta(X) = \{\phi \in \Re^X | \sum_{x \in X} \phi(x) = 1 \wedge (\forall x \in X)\phi(x) \geq 0\}$ is a simplex.

Definition 4.2 (Support) *Support of a probability distribution $\phi \in \Delta(X)$ is*

$$\text{supp } \phi = \{x \in X | \phi(x) \neq 0\}$$

Definition 4.3 (Restriction) *Probability $\phi \in \Delta(X)$ is restricted to $Y \subset X$ iff $\text{supp } \phi \subset Y$.*

5 Real Number and its Cartesian Products

Definition 5.1 *Denote $\Re$ the set of real numbers and $\Re_+$ the set of nonnegative real numbers.*

Definition 5.2 (Pareto Order) *For $x, y \in \Re^N$:*

- $x \geq y \Leftrightarrow (\forall i \in N)(x_i \geq y_i)$
- $x > y \Leftrightarrow x \geq y \wedge x \neq y$
- $x \gg y \Leftrightarrow (\forall i \in N)(x_i > y_i)$

Definition 5.3 (Pareto Efficiency (Optimality)) *$x \in S \subset \Re^N$ is*

- *(weakly) Pareto efficient iff not $(\exists y \in S)(y \gg x)$*
- *strongly Pareto efficient iff not $(\exists y \in S)(y > x)$*