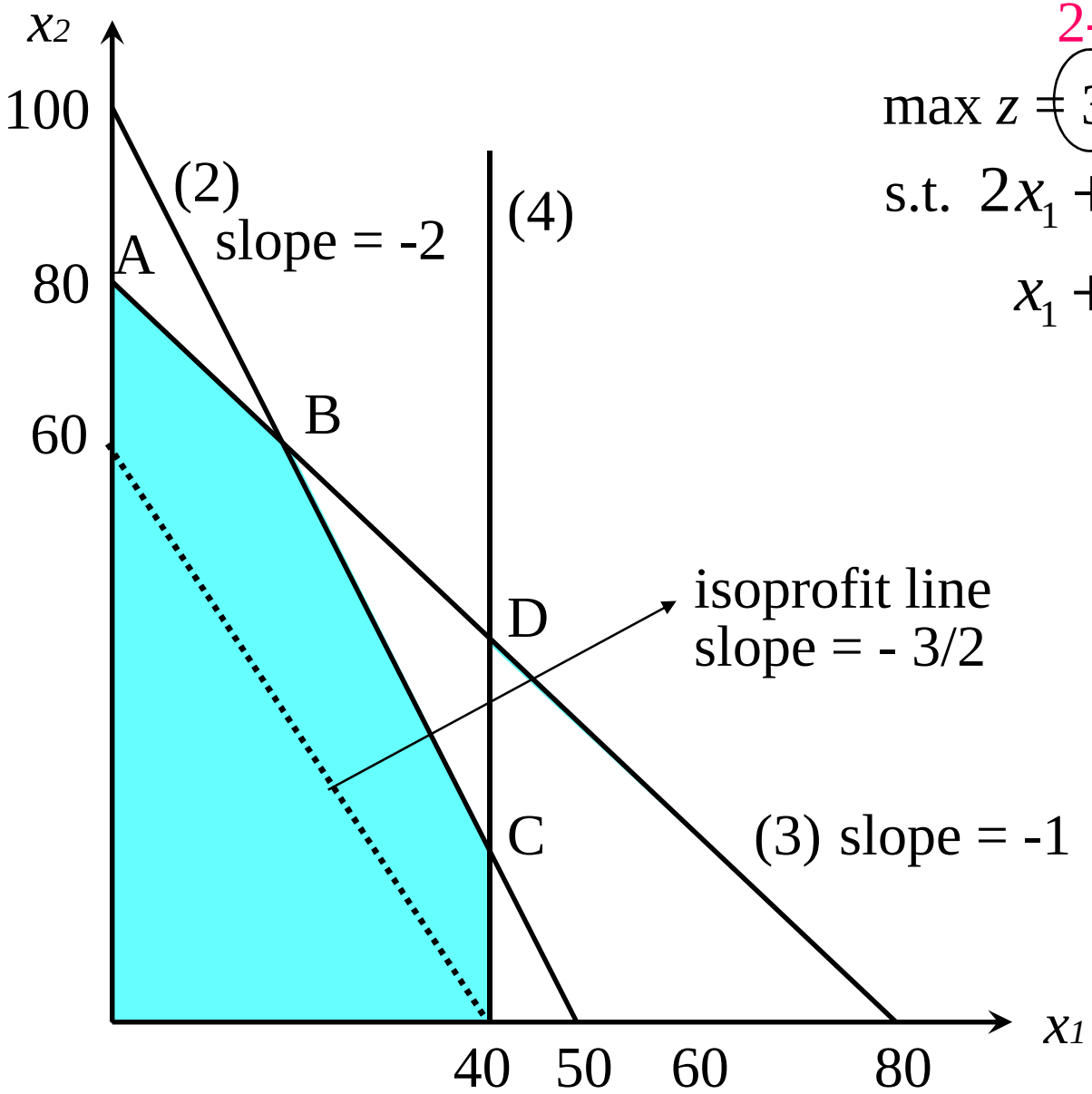
6.1 Graphical Introduction to Sensitivity Analysis



2-4

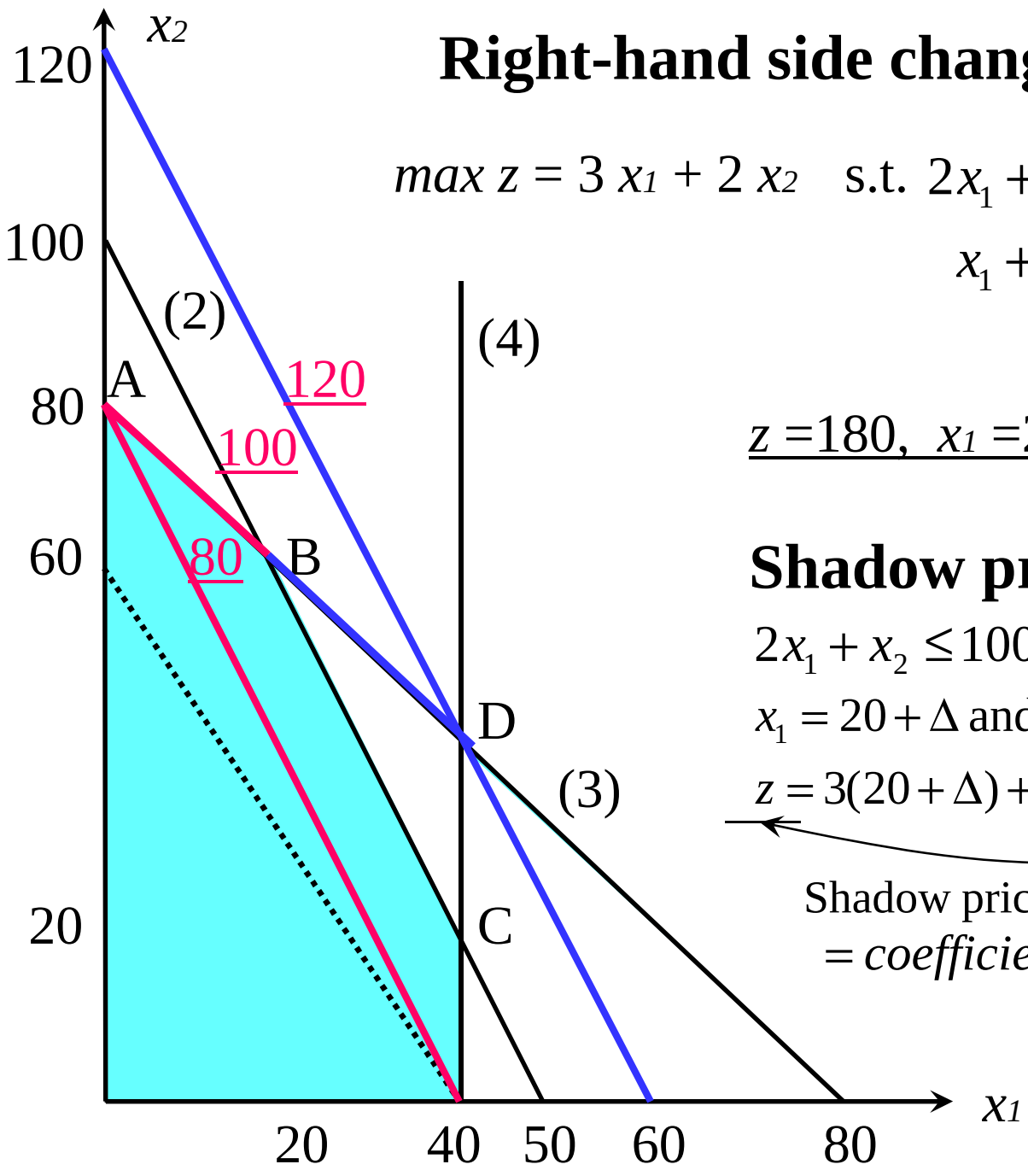
$$\max z = 3x_1 + 2x_2$$

$$\text{s.t. } 2x_1 + x_2 \le 100 \quad (2)$$

$$x_1 + x_2 \le 80 \quad (3)$$

$$x_1 \le 40 \quad (4)$$

Right-hand side change



80-120

$$max\ z = 3x_1 + 2x_2 \quad \text{s.t. } 2x_1 + x_2 \le 100 \quad (2)$$

$$x_1 + x_2 \le 80 \quad (3)$$

$$x_1 \le 40 \quad (4)$$

$z = 180,\ x_1 = 20,\ x_2 = 60$

Shadow price

$$2x_1 + x_2 \le 100 + \Delta \quad (2)$$

$$x_1 = 20 + \Delta \text{ and } x_2 = 60 - \Delta$$

$$z = 3(20 + \Delta) + 2(60 - \Delta) = 180 + \Delta$$

Shadow price of constraint (2) is \$1

$= \textit{coefficient}$ of Δ

6.2 Important Formulas

$$\max \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

s.t.
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$
$$x_i \geq 0 \qquad (i = 1,2,....n)$$

$$\max \quad z = 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3$$

s.t.
$$8x_1 + 6x_2 + x_3 + s_1 = 48$$
$$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$$
$$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

BV, NBV

$$\mathbf{x_{BV}} = \begin{bmatrix} x_{BV1} \\ x_{BV2} \\ .. \\ x_{BVm} \end{bmatrix}$$

$$\mathbf{x_{BV}} = \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} \qquad \mathbf{x_{NBV}} = \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$$

Definition

c_{BV} : $1 \times m$ row vector of the objective function coefficients

c_{NBV} : $1 \times (n-m)$ row vector of the objective function coefficients

B : $m \times m$ matrix of jth column for BVj

N : $m \times (n-m)$ matrix of the column for NBV

a_j : column for the variable x_j in constraints

b : $m \times 1$ column vector of right-hand side of constraints

Standard Form

$$z = \mathbf{c}_{BV}\mathbf{x}_{BV} + \mathbf{c}_{NBV}\mathbf{x}_{NBV}$$

$$\text{s.t.} \quad B\mathbf{x}_{BV} + N\mathbf{x}_{NBV} = \mathbf{b}$$

$$\mathbf{x}_{BV}, \mathbf{x}_{NBV} \geq 0$$

Constraints of Optimal Tableau

$$\mathbf{x}_{BV} + B^{-1}N\mathbf{x}_{NBV} = B^{-1}\mathbf{b}$$

$B^{-1}\mathbf{a_j}$ column for x_j in optimal tableau's constraints

$B^{-1}\mathbf{b}$ right-hand side of optimal tableau's constraints

Row 0 of Optimal Tableau

$$\begin{array}{rl} & \mathbf{c}_{BV}\mathbf{x}_{BV} + \mathbf{c}_{BV}B^{-1}N\mathbf{x}_{NBV} = \mathbf{c}_{BV}B^{-1}\mathbf{b} \\ +) & z - \mathbf{c}_{BV}\mathbf{x}_{BV} - \mathbf{c}_{NBV}\mathbf{x}_{NBV} = 0 \\ \hline & \mathbf{z} + (\mathbf{c}_{BV}B^{-1}N - \mathbf{c}_{NBV})\mathbf{x}_{NBV} = \mathbf{c}_{BV}B^{-1}\mathbf{b} \end{array}$$

Coefficient of x_j in the optimal tableau's row 0

$\mathbf{c}_{BV}B^{-1}\mathbf{a_j} - c_j = \bar{c}_j$ $\qquad$ c_j : column of C

Coefficient of s_i, e_i, a_i in the optimal tableau's row 0

ith element of $\mathbf{c}_{BV}B^{-1}$ $\quad$ - (ith element of $\mathbf{c}_{BV}B^{-1}$) $\qquad$ (ith element of $\mathbf{c}_{BV}B^{-1}$) $+ M$

Right-hand side of optimal tableau's row 0

$\mathbf{c}_{BV}B^{-1}\mathbf{b}$

Example 1

$$\max \quad z = x_1 + 4x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 \le 6 \qquad\qquad x_1 + 2x_2 + s_1 = 6$$

$$2x_1 + x_2 \le 8 \qquad\qquad 2x_1 + x_2 + s_2 = 8$$

$$x_1, x_2 \ge 0 \qquad\qquad \underline{BV = \{x_2, s_2\}}$$

$$B^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

$$\mathbf{c}_{BV} B^{-1}\mathbf{b} = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = 12$$

$$\mathbf{c}_{BV} B^{-1}\mathbf{a}_1 - c_j = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 = 1$$

$$B^{-1}\mathbf{b} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$B^{-1}\mathbf{a}_1 = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \quad B^{-1}s_1 = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$\mathbf{c}_{BV} B^{-1}\mathbf{b}$

optimal value z =
rhs of optimal tableau' row 0

$\mathbf{c}_{BV} B^{-1}\mathbf{a_j} - c_j$

Coefficient of x_j

in the optimal tableau's row 0

$B^{-1}\mathbf{b}$

BV of optimal solution =
rhs of optimal tableau

$B^{-1}\mathbf{a_j}$

column for x_j

in optimal tableau's constraints

Optimal Tableau

$$\begin{aligned} z + x_1 \qquad\qquad + 2s_1 \qquad &= 12 \\ 0.5x_1 + x_2 + 0.5s_1 \qquad &= 3 \\ 1.5x_1 \qquad - 0.5s_1 + s_2 &= 5 \end{aligned}$$

6.3 Sensitivity Analysis

$$\max \quad z = 60x_1 + 30x_2 + 20x_3$$

$$\text{s.t.} \quad 8x_1 + 6x_2 + x_3 \le 48$$

$$4x_1 + 2x_2 + 1.5x_3 \le 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \le 8$$

Initial Tableau

$$z - 60x_1 - 30x_2 - 20x_3 = 0$$

$$8x_1 + 6x_2 + x_3 + s_1 = 48$$

$$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$$

Optimal Tableau

$$z + 5x_2 + 10s_2 + 10s_3 = 280$$

$$-2x_2 + s_1 + 2s_2 - 8s_3 = 24$$

$$-2x_2 + x_3 + 2s_2 - 4s_3 = 8$$

$$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$$

$$BV = \{s_1, x_3, x_1\}, NBV = \{x_2, s_2, s_3\}$$

Parameter Change

1. Objective function coefficient of a NBV
2. Objective function coefficient of a BV
3. Right-hand side of a constraint
4. Column of a NBV
5. Add a new variable or activity
6. Add a new constraint

1. Changing objective function coefficient of a nonbasic variable

Suppose c_2 is change to $30+\Delta$

$$\bar{c}_2 = \mathbf{c}_{BV} B^{-1} \mathbf{a_2} - c_2 = 5 - \Delta \geq 0$$

if $\Delta \leq 5$, $\bar{c}_2 \geq 0$ remains optimal

if $\Delta > 5$, $\bar{c}_2 < 0$ no longer optimal

If BV remains optimal after a change in a nonbasic variable's objective function coefficient, <u>the values of the decision variables and the optimal value remain unchanged</u>.

If BV will no longer be optimal, this is not optimal solution (suboptimal).

The ***reduced cost*** for a nonbasic variable is the maximum amount by which the variable's objective function coefficient can be increased *before* the current basis becomes suboptimal and it becomes optimal for the nonbasic variable to enter the basis.

$$z = 280 - 5x_2 - 10s_2 - 10s_3$$

2. Changing objective function coefficient of a basic variable

Suppose c_1 is change to $60+\Delta$

$$c_{BV} = \begin{bmatrix} 0 & 20 & 60+\Delta \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

Coefficient of each nonbasic variable $\{x_2, s_2, s_3\}$

$$x_2,\ \bar{c}_2 = \mathbf{c}_{BV} B^{-1} \mathbf{a_2} - c_2 = 5 + 1.25\Delta \geq 0 \qquad \Delta \geq -4$$

$$s_2,\ \mathbf{c}_{BV} B^{-1} = 10 - 0.5\Delta \geq 0 \qquad \Delta \leq 20$$

$$s_3,\ \mathbf{c}_{BV} B^{-1} = 10 + 1.5\Delta \geq 0 \qquad \Delta \geq -20/3$$

Range of value on c_1 for which current basis remains optimal

$$-4 \leq \Delta \leq 20$$
$$56 \leq c_1 \leq 80$$

Value of the decision variables do not change, but z-value does changed.

<u>If any variable in row 0 has a negative coefficient, the current basis is no longer optimal</u>

3. Changing the right-hand side of a constraint

Suppose b_2 is change to $20+\Delta$

$$B^{-1}\mathbf{b} = B^{-1}\begin{bmatrix} 48 \\ 20+\Delta \\ 8 \end{bmatrix} = \begin{bmatrix} 24+2\Delta \\ 8+2\Delta \\ 2-0.5\Delta \end{bmatrix} \geq 0$$

$\Delta \geq -12$

$\Delta \geq -4$

$\Delta \leq 4$

Current basis remains optimal

$-4 \leq \Delta \leq 4$

$\Rightarrow$ $16 \leq b_2 \leq 24$

<u>If the right-hand side of each constraint in the tableau remains nonnegative, the current basis remains optimal.</u>

<u>If the right-hand side of any constraint is negative, the current basis is infeasible.</u>

Change of values of optimal solution (z-value) and the value of BVs

$$\text{new value of } z = \mathbf{c}_{BV} B^{-1}(\text{new } \mathbf{b})$$

$$\text{new value of BVs} = B^{-1}(\text{new } \mathbf{b})$$

4. Changing the column of a nonbasic variable

If the column of a nonbasic variable is changed,

the current basis remains optimal. if $\bar{c}_j \geq 0$

the current basis is no longer optimal if $\bar{c}_j < 0$

Price Out: Calculate the new coefficient of *x* in the optimal tableau row 0

5. Adding a new activity

Addition of the new column (new decision variables)

$$\bar{c}_4 = \mathbf{c}_{BV} B^{-1} \mathbf{a_4} - c_4$$

the current basis remains optimal. if $\bar{c}_j \geq 0$

the current basis is no longer optimal if $\bar{c}_j < 0$