4.1 How to Convert an LP to Standard From

Standard form

Each inequality constraint must be replaced by an equality constraint

$$\max \quad z = 4x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + x_2 \le 40 \quad (1)$$

$$2x_1 + x_2 \le 60 \quad (2)$$

$$x_1, x_2 \ge 0$$

Slack Variable ***si***

$$x_1 + x_2 + s_1 = 40$$

$$2x_1 + x_2 + s_2 = 60$$

convert

$$\max \quad z = 4x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + x_2 + s_1 = 40 \quad (1)$$

$$2x_1 + x_2 + s_2 = 60 \quad (2)$$

$$x_1, x_2, s_1, s_2 \ge 0$$

adding the sign restriction

Excess Variable e_i

$$\min \quad z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

s.t.

$$\begin{aligned}
400x_1 + 200x_2 + 150x_3 + 500x_4 &\geq 500 \\
3x_1 + \quad 2x_2 \qquad\qquad\qquad &\geq 6 \\
2x_1 + \quad 2x_2 + \quad 4x_3 + \quad 4x_4 &\geq 10 \\
2x_1 + \quad 4x_2 + \quad x_3 + \quad 5x_4 &\geq 8 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}
400x_1 + 200x_2 + 150x_3 + 500x_4 - e_1 &= 500 \\
3x_1 + \quad 2x_2 \qquad\qquad\qquad - e_2 &= 6 \\
2x_1 + \quad 2x_2 + 4x_3 \quad + \quad 4x_4 \quad - e_3 &= 10 \\
2x_1 + \quad 4x_2 + \ x_3 \quad + \quad 5x_4 \quad - e_4 &= 8 \\
x_i, e_i &\geq 0 \quad (i = 1,2,3,4)
\end{aligned}$$

adding the sing restriction

$a \leq$ constraint

-- adding a slack variable s_i

$a \geq$ constraint

-- subtracting a excess variable e_i

4.2 Preview of the Simplex Algorithm

$$\min \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$x_i \geq 0 \qquad (i = 1,2,....n)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots & \\ a_{m1} & a_{m2} \cdots a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$$

A$\mathbf{x}$ = $\mathbf{b}$

m linear equations

n variables

$n \geq m$

Basic variable (BV) m

Nonbasic variable (NBV) n-m : set variables = 0

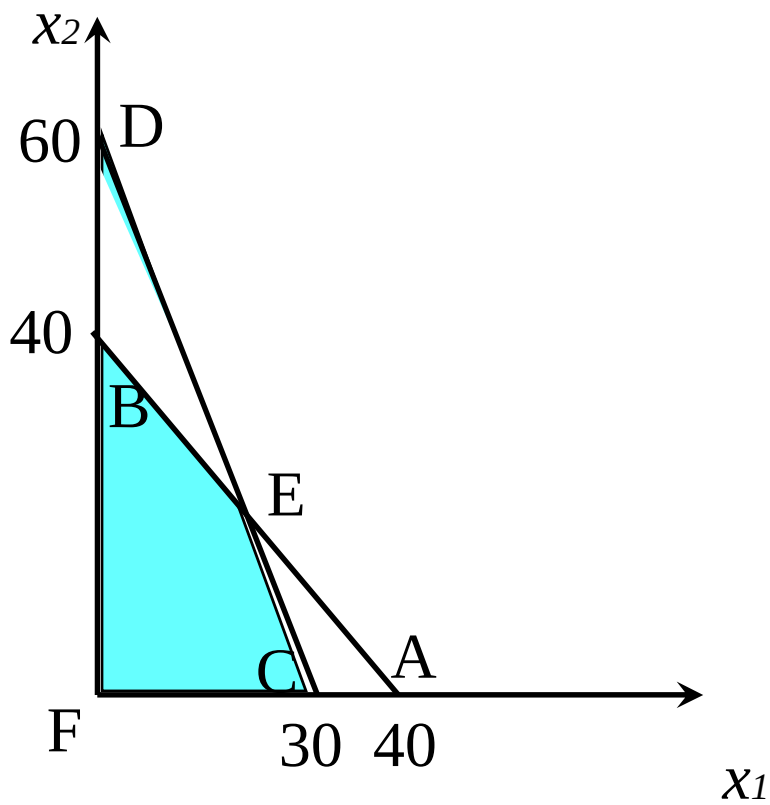
Basic feasible solution (bfs)

$$\max \quad z = 4x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + x_2 + s_1 = 40 \quad (1)$$

$$2x_1 + x_2 + s_2 = 60 \quad (2)$$

$$x_1, x_2, s_1, s_2 \geq 0$$



BV	**NBV**	**bfs** NBV = 0	
x_1, x_2	s_1, s_2	$s_1 = s_2 = 0, x_1 = x_2 = 20$	E
x_1, s_1	x_2, s_2	$x_2 = s_2 = 0, x_1 = 30, s_1 = 10$	C
x_1, s_2	x_2, s_1	$x_2 = s_1 = 0, x_1 = 40, s_2 = -20$	
x_2, s_1	x_1, s_2	$x_1 = s_2 = 0, s_1 = -20, x_2 = 60$	
x_2, s_2	x_1, s_1	$x_1 = s_1 = 0, x_2 = 40, s_2 = 20$	B
s_1, s_2	x_1, x_2	$x_1 = x_2 = 0, s_1 = 40, s_2 = 60$	F

Adjacent basic feasible solution

E&C, B&E, C&F, F&B

m-1

4.3 Simplex Algorithm

Maximization problems

Step 1 Convert the LP to standard form.

Step 2 Obtain bfs (if possible) from the standard form.

Step 3 Determine whether the current bfs is optimal.

Step 4 If the current bfs is not optimal, determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value.

Step 5 Use ero's to find the new bfs with the better objective function value. Go back to step 3.

Convert the LP to Standard Form

$$\begin{aligned} \max \quad & z = 60x_1 + 30x_2 + 20x_3 \\ \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \le 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 \le 8 \\ & x_2 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

convert

Standard form

$$\begin{aligned} & 8x_1 + 6x_2 + x_3 + s_1 = 48 \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8 \\ & x_2 + s_4 = 5 \\ & x_1, x_2, x_3, s_1, s_2, s_3, s_4 \ge 0 \end{aligned}$$

<u>Non negative</u>

Canonical Form 0 **BV**

$$\begin{array}{llll} \text{Row } 0 & z - 60x_1 - 30x_2 - 20x_3 = 0 & & Z = 0 \\ \text{Row } 1 & 8x_1 + 6x_2 + x_3 + s_1 = 48 & & s_1 = 48 \\ \text{Row } 2 & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 & & s_2 = 20 \\ \text{Row } 3 & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8 & & s_3 = 8 \\ \text{Row } 4 & x_2 + s_4 = 5 & & s_4 = 5 \end{array}$$

$BV = \{z, s_1, s_2, s_3, s_4\}$

$NBV = \{x_1, x_2, x_3\}$

Coefficient of BV = 1

Determine the Entering Variable

$$z = 60x_1 + 30x_2 + 20x_3$$

Most positive coefficient

What limits how large we can make x_1?

$s_1 = 48 - 8x_1 \geq 0$ for $x_1 \leq \frac{48}{8} = 6$

$s_2 = 20 - 4x_1 \geq 0$ for $x_1 \leq \frac{20}{4} = 5$

$s_3 = 8 - 2x_1 \geq 0$ for $x_1 \leq \frac{8}{2} = 4$

$s_4 \geq 0$ for all values of x_1

$$\text{Ratio} = \frac{\text{Right-hand side of row}}{\text{Coefficient of entering variable in row}}$$

Winner of the ratio test: How much increase x_1

ero step: Gauss-Jordan Method

Pivot in the Entering Variable
Canonical Form 1

			BV
Row 0'	$z + 15x_2 - 5x_3 + 30s_3$	$= 240$	$Z = 240$
Row 1'	$-x_3 + s_1 - 4s_3$	$= 16$	$s_1 = 16$
Row 2'	$-x_2 + 0.5x_3 + s_2 - 2s_3$	$= 4$	$s_2 = 4$
Row 3'	$x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3$	$= 4$	$x_1 = 4$
Row 4'	$x_2 + s_4$	$= 5$	$s_4 = 5$

$$BV = \{z, s_1, s_2, x_1, s_4\}$$

$$NBV = \{s_3, x_2, x_3\}$$

ero step: Gauss-Jordan Method

Pivot term: x_1 from most positive coefficient
Pivot row: Row 3 from ratio test

Row 0	$z - 60x_1 - 30x_2 - 20x_3 = 0$	$Z = 0$	
Row 1	$8x_1 + 6x_2 + x_3 + s_1 = 48$	$s_1 = 48$	
Row 2	$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$	$s_2 = 20$	Enter x_1 into the basis (x_1 become BV), leave s_3 from the basis (s_3 become NBV).
Row 3	$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$	$s_3 = 8$	
Row 4	$x_2 + s_4 = 5$	$s_4 = 5$	

To make new Canonical Form 1

ero1	Row 3'	$x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4$	(Row3 ÷ 2)
ero2	Row 0'	$z + 15x_2 - 5x_3 + 30s_3 = 240$	(Row0+Row3×30)
ero3	Row 1'	$-x_3 + s_1 - 4s_3 = 16$	(Row1 − Row3×4)
ero4	Row 2'	$-x_2 + 0.5x_3 + s_2 - 2s_3 = 4$	(Row2 − Row3×2)
	Row 4'	$x_2 + s_4 = 5$	

Iteration

Most positive coefficient

$$z = 240 - 15x_2 + 5x_3 - 30s_3$$

from row1': $s_1 = 16 + x_3$

from row2': $s_2 = 4 - 0.5x_3$

from row3': $x_1 = 4 - 0.25x_3$

from row4': $s_4 = 5$

Row1': no ratio $\quad x_3$: Negative coefficient

Row2': $\dfrac{4}{0.5} = 8$ Winner

Row3': $\dfrac{4}{0.25} = 16$

Row4': no ratio $\quad x_3$: Nonpositive coefficient

ero step

Canonical Form 2

			BV
Row 0''	$z + 5x_2 + 10s_2 + 10s_3$	$= 280$	$z = 280$
Row 1''	$-2x_2 + s_1 + 2s_2 - 8s_3$	$= 24$	$s_1 = 24$
Row 2''	$-2x_2 + x_3 + 2s_2 - 4s_3$	$= 8$	$x_3 = 8$
Row 3''	$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3$	$= 2$	$x_1 = 2$
Row 4''	$x_2 + s_4$	$= 5$	$s_4 = 5$

Optimal (Max Problem)

$$z + 5x_2 + 10s_2 + 10s_3 = 280$$

$$z = 280 - 5x_2 - 10s_2 - 10s_3$$

<u>Nonnegative coefficient of NBV in Row 0</u>

$$BV = \{z, s_1, x_3, x_1, s_4\}$$

$$NBV = \{s_2, s_3, x_2\}$$

Representing Simplex Tableaus

$$
\begin{aligned}
z + 3x_1 + x_2 \qquad\qquad &= 6 \\
x_1 \qquad + s_1 \qquad &= 4 \\
2x_1 + x_2 \qquad + s_2 &= 3
\end{aligned}
$$

Simplex Tableau

z	x_1	x_2	s_1	s_2	rhs	BV
1	3	1	0	0	6	z
0	1	0	1	0	4	s_1
0	2	1	0	1	3	s_2