3.2.6. Poisson Process and Poisson Distribution

Poisson Process

1) An event can occur *randomly* at any time (or any point).
2) The occurrence is *statistically independent*
3) The probability of occurrence in a small interval Δt is $\nu\,\Delta t$,
   and the probability of two or more occurrences in Δt is negligible.

Poisson Distribution

probability that the event occur exactly x times in t

$$P(X_t = x) = \frac{(\nu t)^x}{x!} e^{-\nu t} = \frac{\lambda^x}{x!} e^{-\nu t} \qquad x = 0, 1, 2, ..., n \qquad (3.34)$$

ν: mean occurrence rate per unit time

λ : mean occurrence time in t

$$P(X_t = x) = \frac{\nu t}{x} \frac{(\nu t)^{x-1}}{(x-1)!} e^{-\nu t} = \frac{\nu t}{x} P(X_t = x-1)$$

ポアソン過程の事例

- 1時間に特定の交差点を通過する車両の台数。
- 1mlの希釈された水試料中に含まれる特定の細菌の数。
- 1ページの文章を入力するとき、綴りを間違える回数。
- 1日に受け取る電子メールの件数。
- 1分間のWebサーバへのアクセス数。
- 1マイルあたりのある通り沿いのレストランの軒数。
- 1ヘクタールあたりのエゾマツの本数。
- 1立方光年あたりの恒星の個数。

Mean and Variance

$$\mathbf{E(X)} = \sum_{\mathbf{x=0}}^{\infty} \mathbf{x\, f_x} = \sum_{\mathbf{x=0}}^{\infty} \mathbf{x} \frac{(\nu \mathbf{t})^{\mathbf{x}}}{\mathbf{x!}} \mathbf{e}^{-\nu \mathbf{t}} = \sum_{\mathbf{x=1}}^{\infty} \frac{\nu \mathbf{t} (\nu \mathbf{t})^{\mathbf{x}-1}}{(\mathbf{x}-1)!} \mathbf{e}^{-\nu \mathbf{t}}$$

$$= \nu \mathbf{t} \sum_{\mathbf{x=1}}^{\infty} \frac{(\nu \mathbf{t})^{\mathbf{x}-1}}{(\mathbf{x}-1)!} \mathbf{e}^{-\nu \mathbf{t}} = \nu \mathbf{t}$$

$$\mathbf{E(X^2)} = \sum_{\mathbf{x=0}}^{\infty} \mathbf{x}^2\, \mathbf{f_x} = \sum_{\mathbf{x=0}}^{\infty} \{\mathbf{x}(\mathbf{x}-1) + \mathbf{x}\}\, \mathbf{f_x}$$

$$= \sum_{\mathbf{x=0}}^{\infty} \mathbf{x(x-1)\, f_x} + \sum_{\mathbf{x=0}}^{\infty} \mathbf{x\, f_x} = \mathbf{R} + \mu$$

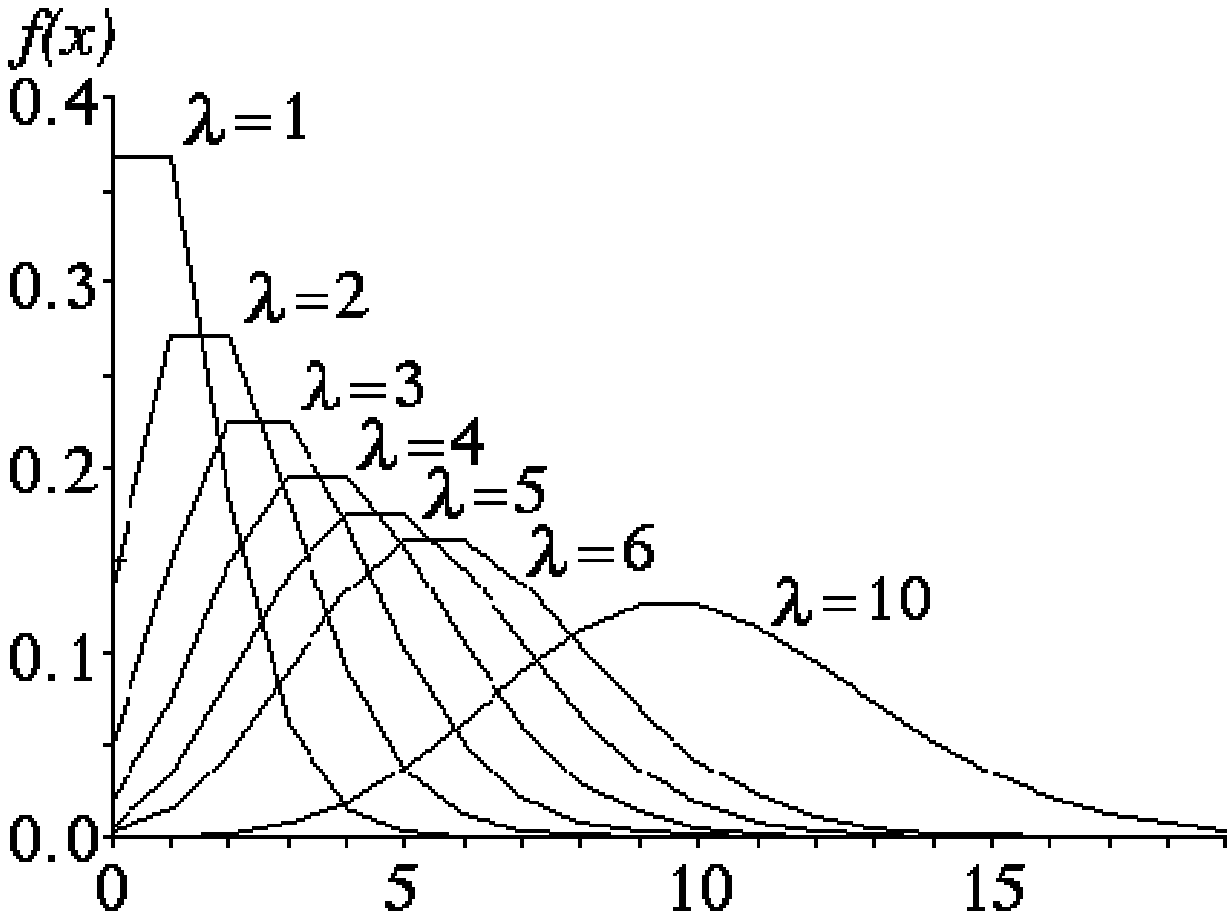
$$\mathbf{R} = \sum_{\mathbf{x=2}}^{\infty} \mathbf{x(x-1)\, f_x} = \sum_{\mathbf{x=2}}^{\infty} \mathbf{x(x-1)} \frac{(\nu \mathbf{t})^{\mathbf{x}}}{\mathbf{x!}} \mathbf{e}^{-\nu \mathbf{t}} = (\nu \mathbf{t})^2 \sum_{\mathbf{x=2}}^{\infty} \frac{(\nu \mathbf{t})^{\mathbf{x}-2}}{(\mathbf{x}-2)!} \mathbf{e}^{-\nu \mathbf{t}} = (\nu \mathbf{t})^2$$

$$\mathbf{E(X^2)} = (\nu \mathbf{t})^2 + \nu \mathbf{t}$$

$$\sigma^2 = (\nu \mathbf{t})^2 + \nu \mathbf{t} - (\nu \mathbf{t})^2 = \nu \mathbf{t}$$

Shape of Poisson Distribution

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$



The Binomial Distribution and The Poisson Distribution

An average of 60 cars/hour makes left turns at an intersection. What is the probability that exactly 10 cars will makes left turns in a 10-minute interval?

Approximate Solution

1. Divide the hour into 120 30-second intervals.
2. Calculate the probability of a left turn in any 30-second interval.

   $p=60/120=0.5$
3. If we assume that <u>no more than one car</u> can make left turns in the interval, the problems is reduced to the binomial probability of 10 events in 20 trials.

   $P(10LT \text{ in } 10min.)={}_{20}C_{10}(0.5)^{10}(1-0.5)^{20-10} = 0.176$
4. Change the interval to 10-second interval

   $p=60/360=1/6$

   $P(10LT \text{ in } 10min.)={}_{60}C_{10}(1/6)^{10}(1-1/6)^{60-10} = 0.137$

Approximate Solution

5. Change the interval to t-second interval

$p=60/(3600/t)=t/60$

$P(10LT \text{ in } 10min.)={}_{600/t}C_{10}(t/60)^{10}(1-t/60)^{600/t-10}$

6. Generalize the above equation,

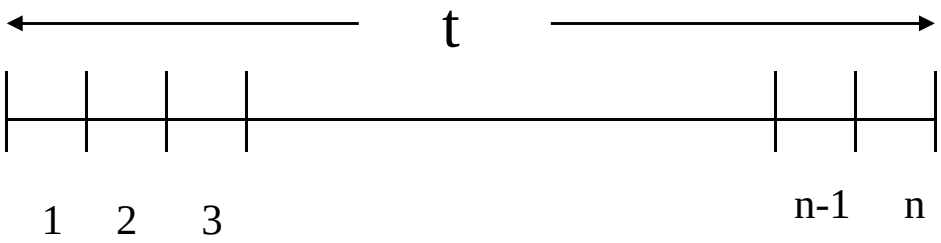
where

n: the number of intervals	600/t-sec
x: the number of events that occurs in time t	10 Left Turn
λ: the average number of events in time t	10/10min
T: the time when the events happen	10min

$p=\lambda/n$

$P(x \text{ LT in } T)={}_nC_x(\lambda/n)^x(1-\lambda/n)^{n-x}$

Bernoulli ----> Poisson



ν: occurrence per unit time

νt: average in t

$$p = \frac{\nu t}{n} = \frac{\lambda}{n}$$

P(x **events in t)**

$$= \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} = \lim_{n\to\infty} \binom{n}{x} (\frac{\lambda}{n})^x (1-\frac{\lambda}{n})^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1).....(n-x+1)}{x(x-1)...2\cdot 1} (\frac{\lambda}{n})^x (1-\frac{\lambda}{n})^{n-x}$$

$$= \lim_{n\to\infty} \frac{1(1-\frac{1}{n}).....(1-\frac{x-1}{n})}{x(x-1)...2\cdot 1} (\lambda)^x (1-\frac{\lambda}{n})^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} (1-\frac{\lambda}{n})^{n-x} = \frac{\lambda^x}{x!} \lim_{n\to\infty} (1-\frac{\lambda}{n})^{n}$$

$$= \frac{\lambda^x}{x!} (1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+.....) = \frac{\lambda^x}{x!} e^{-\lambda} = \frac{(\nu t)^x}{x!} e^{-\nu t}$$

Ex. 3.20 4 rainstorms per year in last 20 years

No rainstorms next year

4 rainstorms next year

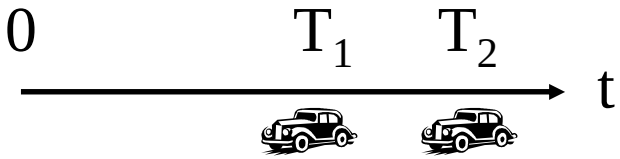
2 or more rainstorms next year

3.2.7 Exponential Distribution

$$P(T_1 > t) = P(Xt = 0 \text{ in } t) = \frac{(\nu t)^0}{0!} e^{-\nu t} = e^{-\nu t}$$

$$F_T(t) = P(T_1 \le t) = 1 - P(T_1 > t) = 1 - e^{-\nu t} \qquad (3.35)$$

$$f_T(t) = \frac{dF_T(t)}{dt} = \nu e^{-\nu t} \qquad (3.36)$$



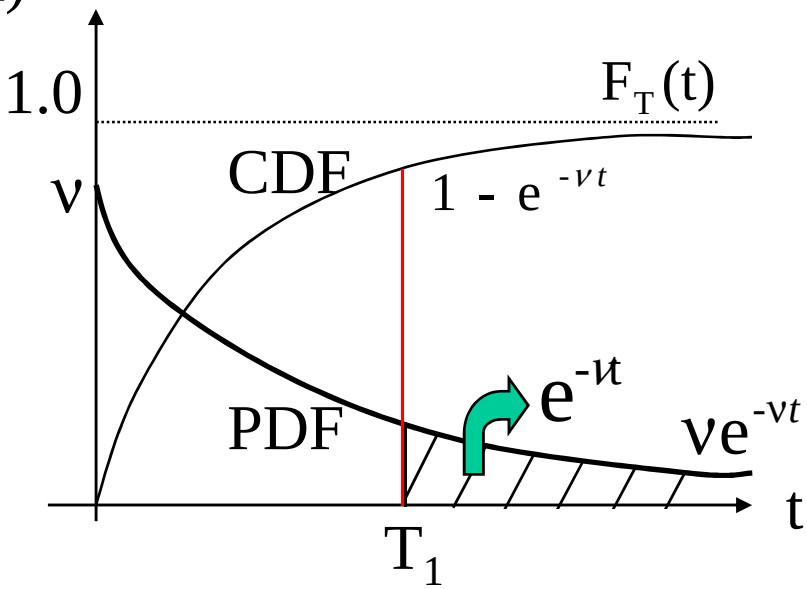
T_1: first occurrence time (recurrence time)

Mean Recurrence Time (Return Period)

$$\mu = E(T_1) = \int_0^\infty t\, f_T(t)dt = \int_0^\infty t\, \nu e^{-\nu t} dt = \frac{1}{\nu} \qquad (3.37)$$

$$E(T_1^2) = \int_0^\infty t^2\, f_T(t)dt = \frac{2}{\nu^2}$$

$$\sigma^2 = \frac{2}{\nu^2} - (\frac{1}{\nu})^2 = \frac{1}{\nu^2}$$



Ex. 3.25

16 earthquakes, 125 years

$$\nu = \frac{16}{125} = 0.128$$

$$P(T \le 2) = 1 - e^{-0.128 \times 2} = 0.226$$

$$P(T > 10) = e^{-0.128 \times 10} = 0.278$$

Return Period

$$E(T) = \frac{1}{\nu} = 7.8 \quad \text{years}$$

Event occurrence within return period

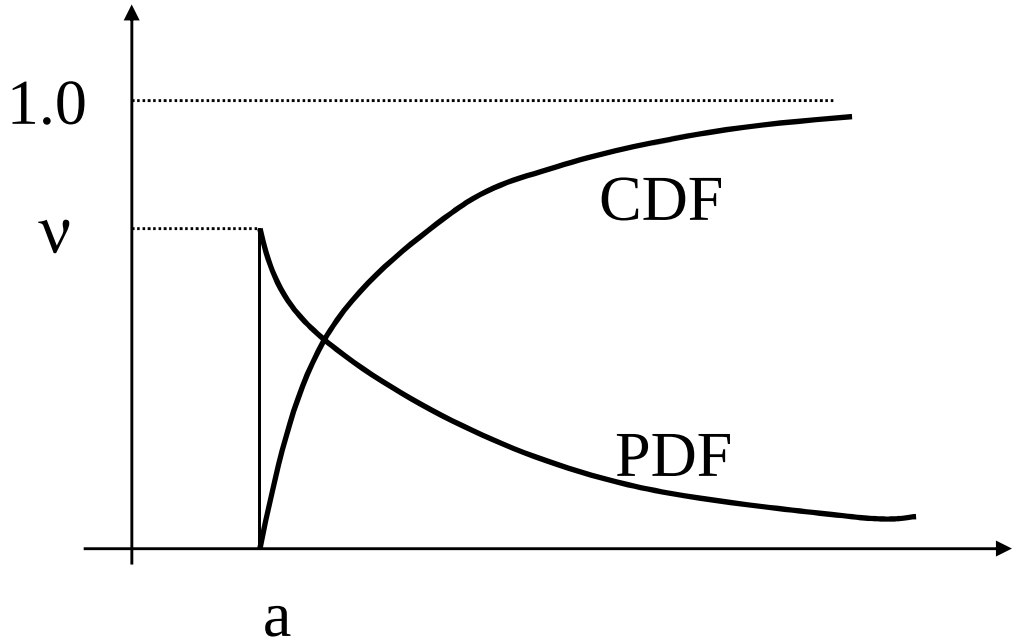
$$P(T_1 \le E(T_1)) = 1 - e^{-E(T_1)\cdot\nu} = 1 - e^{-\frac{1}{\nu}\cdot\nu} = 1 - e^{-1}$$

$$= 0.632$$

Shifted Negative Exponential Distribution

$$\text{PDF} \quad f_T(t) = \begin{cases} \nu e^{-\nu(t-a)} & t \geq a \\ 0 & t < a \end{cases} \qquad (3.40)$$

$$\text{CDF} \quad F_T(t) = \begin{cases} 1 - e^{-\nu(t-a)} & t \geq a \\ 0 & t < a \end{cases} \qquad (3.41)$$



$$\mu = \frac{1}{\nu} + a$$

$$\sigma^2 = \frac{1}{\nu^2}$$

Theory of Reliability

Ex. 3.26

4 identical diesel engines

At least 2 engines must start during an emergency

Mean of operational life: 15 years Mean life = $\frac{1}{E(T)}$

Probability that any engine start without any problem within four years

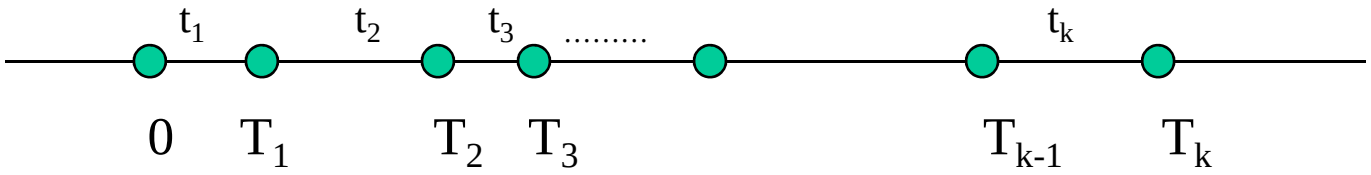
$$P(T > 4) = e^{-(4-2)/(15-2)} = 0.8574$$

Reliability of the backup system within 4 years

$$P(T \geq 2) = \sum_{n=2}^{4} \binom{4}{n} (0.8574)^n (0.1426)^{4-n}$$

$$= 1 - \binom{4}{0} (0.1426)^4 - \binom{4}{1} (0.8574)^1 (0.1426)^3 = 0.990$$

3.2.8 Gamma Distribution (Erlang Distribution)



T_k : the time till *k*th event occurs

$P(T_k \le t)$: Probability that k or more events occur in time t

$$P(T_k \le t) = F_T(t) = \sum_{x=k}^{\infty} P(Xt = x)$$

$$= \sum_{x=k}^{\infty} \frac{(\nu t)^x}{x!} e^{-\nu t} = 1 - \sum_{x=0}^{k-1} \frac{(\nu t)^x}{x!} e^{-\nu t}$$

$$f_T(t) = \frac{\nu (\nu t)^{k-1}}{(k-1)!} \mathrm{e}^{-\nu t} \quad (3.44)$$

$$\mu = k\left(\frac{1}{\nu}\right)$$

$$\sigma^2 = k\left(\frac{1}{\nu^2}\right)$$

Derivation of PDF

$$F_T(t) = 1 - \sum_{x=0}^{k-1} \frac{(\nu t)^x}{x!} e^{-\nu t} = 1 - \frac{(\nu t)^0}{0!} e^{-\nu t} - \sum_{x=1}^{k-1} \frac{(\nu t)^x}{x!} e^{-\nu t} = 1 - e^{-\nu t} - \sum_{x=1}^{k-1} \frac{(\nu t)^x}{x!} e^{-\nu t}$$

$$f_T(t) = \frac{dF_T(t)}{dt} = \nu e^{-\nu t} - \sum_{x=1}^{k-1} \left(\frac{x\nu(\nu t)^{x-1}}{x!} e^{-\nu t} + (-\nu) \frac{(\nu t)^x}{x!} e^{-\nu t} \right)$$

$$= \nu e^{-\nu t} - e^{-\nu t} \nu \sum_{x=1}^{k-1} \left(\frac{(\nu t)^{x-1}}{(x-1)!} - \frac{(\nu t)^x}{x!} \right)$$

$$= \nu e^{-\nu t} - e^{-\nu t} \nu \left\{ \left(\frac{(\nu t)^0}{0!} - \frac{(\nu t)^1}{1!} \right) + \left(\frac{(\nu t)^1}{1!} - \frac{(\nu t)^2}{2!} \right) + \cdots + \left(\frac{(\nu t)^{k-2}}{(k-2)!} - \frac{(\nu t)^{k-1}}{(k-1)!} \right) \right\}$$

$$= \nu e^{-\nu t} - e^{-\nu t} \nu \left\{ (1 \qquad\qquad\qquad\qquad) + (\qquad\qquad\qquad\qquad) + \cdots + (\qquad\qquad\qquad - \frac{(\nu t)^{k-1}}{(k-1)!}) \right\}$$

$$= e^{-\nu t} \nu \frac{(\nu t)^{k-1}}{(k-1)!}$$

$$f_T(t) = \frac{\nu(\nu t)^{k-1}}{(k-1)!} \mathrm{e}^{-\nu t}$$

Ex.3.28

Fatal accidents: once every 6 months $\nu = \frac{1}{6}$

First accident

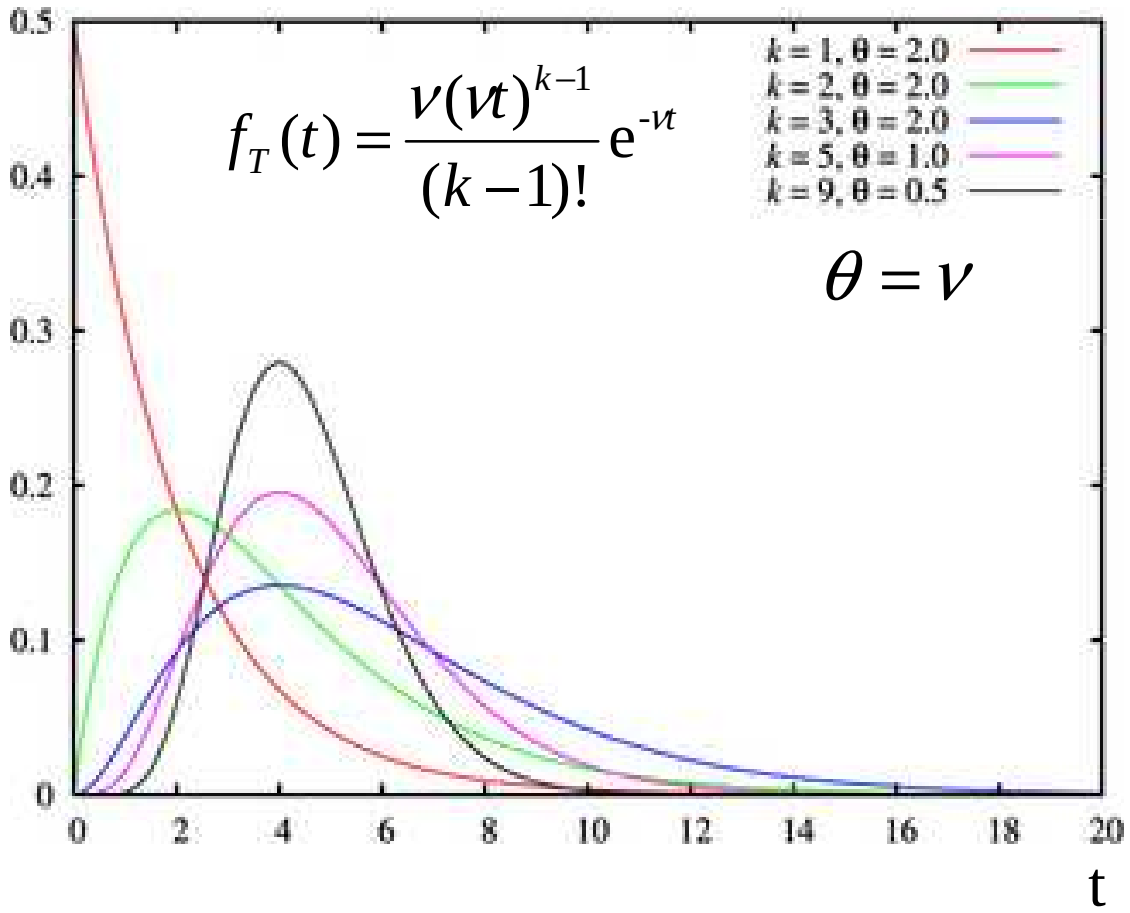
$$f_T(t) = \frac{1}{6} e^{-t/6}$$

Second accident

$$f_T(t) = \frac{1}{6}\left(\frac{t}{6}\right) e^{-t/6}$$

Third accident

$$f_T(t) = \frac{1}{2} \cdot \frac{1}{6}\left(\frac{t}{6}\right)^2 e^{-t/6}$$



3.2.9 The Hypergeometric Distribution

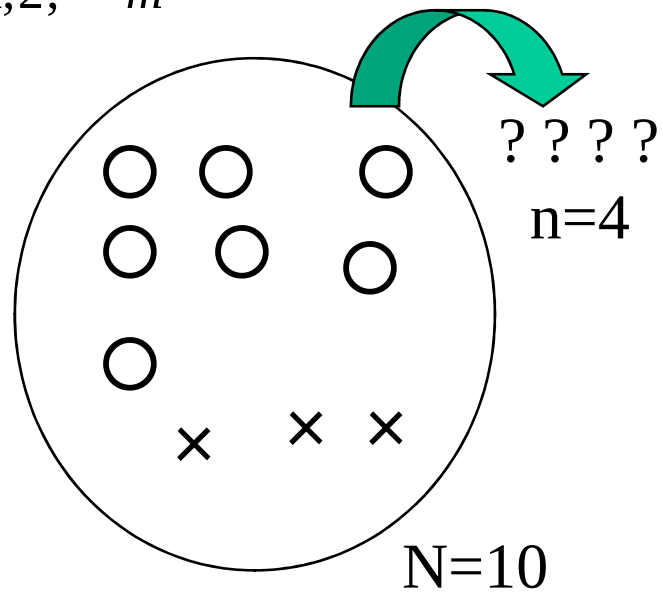
$$P(X = x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \qquad (3.46)$$

N: Number of items

m: Defective sample

n: Number of samples

$x = 1, 2, \cdots m$



N=10

× : Defective m=3

○ : Good N-m =7

P(X=0)=P(? ? ? ?= ○ ○ ○ ○)

$= {}_3C_0\ {}_7C_4 / {}_{10}C_4$

P(X=1)=P(? ? ? ?= ○ ○ ○ ×)

$= {}_3C_1\ {}_7C_3 / {}_{10}C_4$

P(X=2)=P(? ? ? ?= ○ ○ × ×)

$= {}_3C_2\ {}_7C_2 / {}_{10}C_4$

P(X=3)=P(? ? ? ?= ○ × × ×)

$= {}_3C_3\ {}_7C_1 / {}_{10}C_4$

P(X=4)=P(? ? ? ?= × × × ×)

$= 0$

*Sampling without Replacement

Ex.3.31 Concrete cylinders N = 100, n = 10, m/100 = d (%)

$$P(X>1) = 1\text{-P}(\text{X} \le 1) = 1 - \left[\frac{\binom{100d}{0}\binom{100(1-d)}{10}}{\binom{100}{10}} + \frac{\binom{100d}{1}\binom{100(1-d)}{9}}{\binom{100}{10}} \right]$$

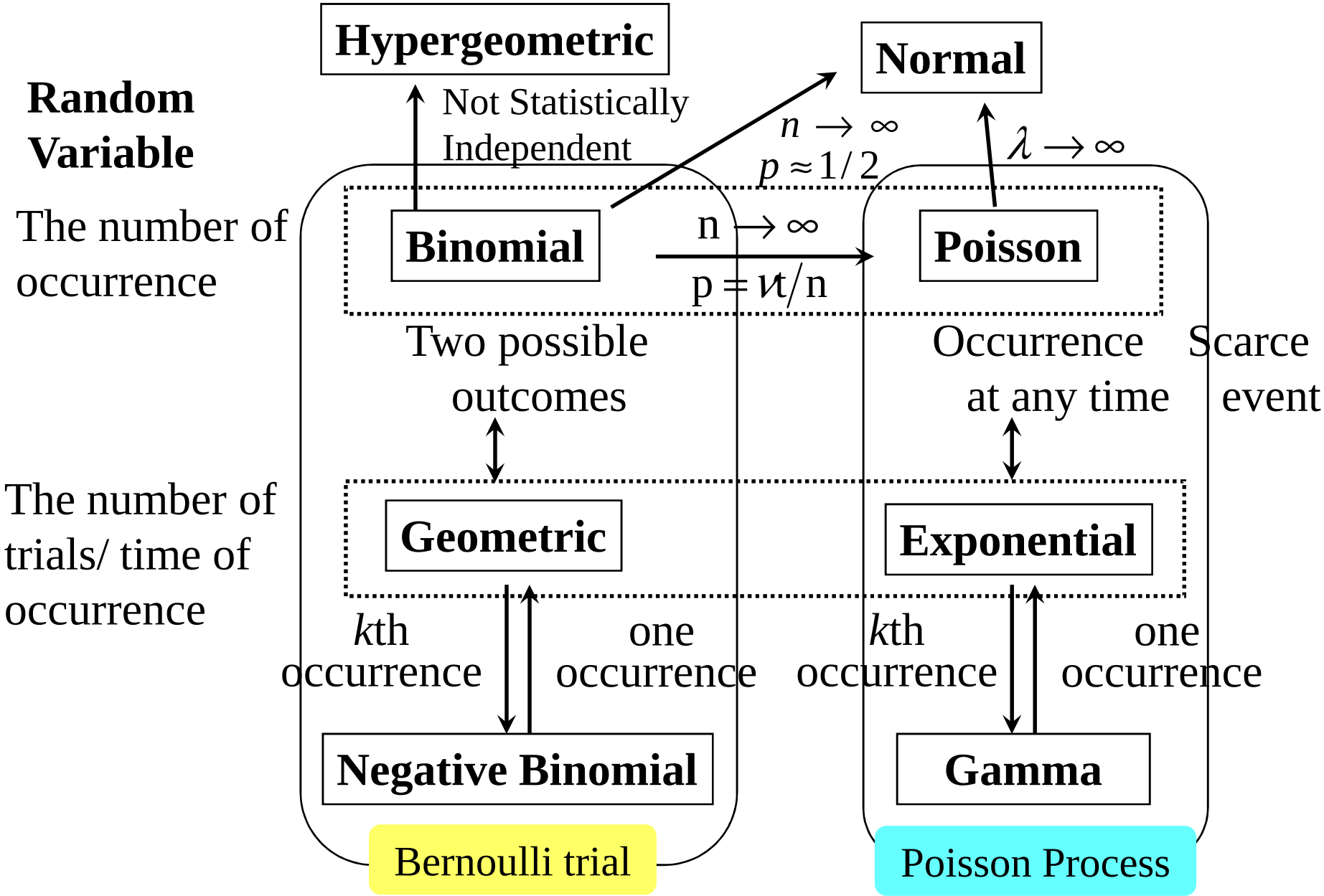
d = 5% $P(rej.) =$

$$1 - \left[\frac{\binom{95}{10}}{\binom{100}{10}} + \frac{\binom{5}{1}\binom{95}{9}}{\binom{100}{10}} \right] = 0.077$$

d = 2% P(rej.) =

$$1 - \left[\frac{\binom{98}{10}}{\binom{100}{10}} + \frac{\binom{2}{1}\binom{98}{9}}{\binom{100}{10}} \right] = 0.0091$$

Interrelation of Probability Distributions



Central Limit Theorem : 中心極限定理

$X_1,\ X_2,....,X_n$: Random variable with mean μ and variance σ^2

(*Any* probability distribution)

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + ... + X_n)$$

$$n \to \infty$$

$\overline{X}$: Normal Distribution $\quad \mathrm{N}(\mu_{\overline{X}} = \mu, \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}})$