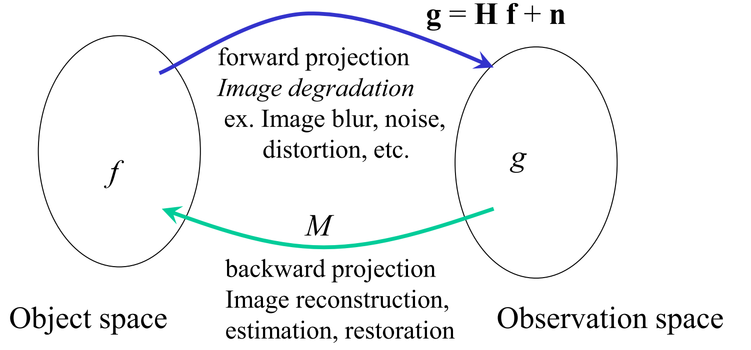
光画像工学
Optical imaging and image processing
(IX)

6. Image restoration
6. 画像復元

$g = h * f + n,\ G = H\,F + N$

$\mathbf{g} = \mathbf{H}\ \mathbf{f} + \mathbf{n}$



Linear ⇔ Nonlinear

- 画像劣化のモデル
  - 線形，シフトインバリアントな劣化　Linear, shift-invariant case

$$g(x,y) = \iint h(x-x', y-y') f(x',y') dx' dy'$$
$$= f(x,y) * h(x,y)$$

離散系による表記　$\mathbf{g} = \mathbf{H}\ \mathbf{f} + \mathbf{n}$

- 加法的ノイズ　Additive noise
  - $g(x,y) = f(x,y) * h(x,y) + n(x,y)$
- 背景光
  $g(x,y) = f(x,y) * h(x,y) + dc(x,y)$
  （暗電流ノイズ : dark current noise）
- 焦点はずれ　defocus
  - 円形開口　circular aperture　$h(x,y) = circ(\frac{\sqrt{x^2+y^2}}{D})$
- 流れ劣化　Motion blur
  - x方向　$h(x,y) = \mathrm{rect}(\frac{x}{a})$
- 幾何学的歪み　Geometrical distortion
  $g(x,y) = f(x',y')$　　$x' = X(x,y),\ y' = Y(x,y)$　　$h(x,y) = \delta(x-X, y-Y)$
- センサの非線形性　Sensor nonlinearlity
  $g'(x,y) = \phi\{g(x,y)\}$

6.1 Inverse filtering in continuous space
6.1　連続空間における逆フィルタ

- Linear, shift-invariant imaging system

$$g(x,y) = h(x,y) * f(x,y)$$
$$G(u,v) = H(u,v) F(u,v)$$

- Estimation of $f(x,y)$ from $g(x,y)$

$$\hat{F}(u,v) = \frac{1}{H(u,v)} G(u,v)$$

(where $|H(u,v)| \neq 0$)

- Additive noise case

$$g(x,y) = h(x,y) * f(x,y) + n(x,y)$$
$$G(u,v) = H(u,v) F(u,v) + N(u,v)$$
$$\hat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

⇨ Noise is amplified when $|H(u,v)|$ is small.

6.2 Wiener filtering in continuous space
6.2　連続空間におけるウィナーフィルタ

- Estimation of $f(x,y)$ by a linear, shift-invariant filter $m(x,y)$

$$\hat{f}(x,y) = m(x,y) * g(x,y)$$

- In Fourier domain,

$$\hat{F}(u,v) = M(u,v)G(u,v)$$

- Estimation error *er*,

$$er = | F(u,v) - M(u,v)G(u,v) |^2$$

- Ensemble average of *er*,

$$e = E\{| F(u,v) - M(u,v)G(u,v) |^2\}$$

- Differential of e by $M(u,v)$ becomes 0, if e is minimal.

$$\frac{\partial e}{\partial M(u,v)} = M^*(u,v)E\{| G(u,v) |^2\} - E\{G(u,v)F^*(u,v)\} = 0$$

$$\because \frac{\partial | M(u,v) |^2}{\partial M(u,v)} = M^*(u,v)$$

Wiener filtering in continuous space (continued)

- Now we have,

$$M(u,v) = \frac{E\{G^*(u,v)F(u,v)\}}{E\{| G(u,v) |^2\}}$$

- If the signal and the noise are statistically independent,

$$E\{F^*(u,v)N(u,v) |\} = E\{F(u,v)N^*(u,v) |\} = 0$$

$$E\{| G(u,v) |^2\} = E\{| H(u,v)F(u,v) + N(u,v) |^2\} = | H(u,v) |^2 \; E\{| F(u,v) |^2\} + E\{| N(u,v) |^2\}$$

– The spectral density is the Fourier transform of the autocorrelation function $R_f(x,y)$.

$$R_f(x,y) = f(x,y) \star f^*(x,y) \qquad R_n(x,y) = n(x,y) \star n^*(x,y)$$

$$\boldsymbol{F}[E\{f(x,y) \star f^*(x,y)\}] = E\{| F(u,v) |^2\} = S_f(u,v)$$

$$\boldsymbol{F}[E\{n(x,y) \star n^*(x,y)\}] = E\{| N(u,v) |^2\} = S_n(u,v)$$

Wiener filter for the 2D, linear, shift-invariant system

Spectral density of the signal: $S_f(u,v)$
Spectral density of the noise: $S_n(u,v)$

$$M(u,v) = \frac{H^*(u,v)S_f(u,v)}{| H(u,v) |^2 \; S_f(u,v) + S_n(u,v)} = \frac{H^*(u,v)}{| H(u,v) |^2 + \frac{S_n(u,v)}{S_f(u,v)}}$$

$$= \frac{| H(u,v) |^2}{| H(u,v) |^2 + \frac{1}{SNR(u,v)}} \frac{1}{H(u,v)} = \frac{H^*(u,v)}{| H(u,v) |^2 + \frac{1}{SNR(u,v)}} \qquad (*)$$

$SNR(u,v)$: Signal to noise ratio

For spatially independent, white random noise

$$S_n(u,v) = \sigma = const.$$

If the statistical characteristics of the signal and noise are unknown,

$$M(u,v) = \frac{H^*(u,v)}{| H(u,v) |^2 + \Gamma}$$ can be used instead of eq.(*).

Image restoration by Wiener filtering

- In the Fourier domain

$$\hat{F}(u,v) = M(u,v)G(u,v) = M(u,v)\{H(u,v)F(u,v) + N(u,v)\}$$

$$= \frac{| H(u,v) |^2}{| H(u,v) |^2 + \frac{S_n(u,v)}{S_f(u,v)}} F(u,v) + \frac{H^*(u,v)N(u,v)}{| H(u,v) |^2 + \frac{S_n(u,v)}{S_f(u,v)}} \qquad (*)$$

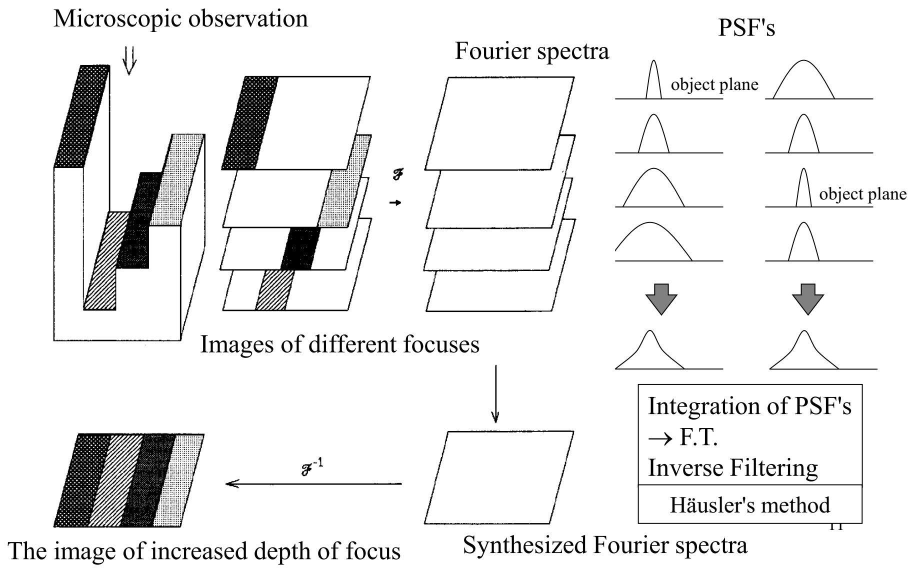
- By setting

$$\frac{S_n(u,v)}{S_f(u,v)} = 0$$

eq.(*) becomes the inverse filter.

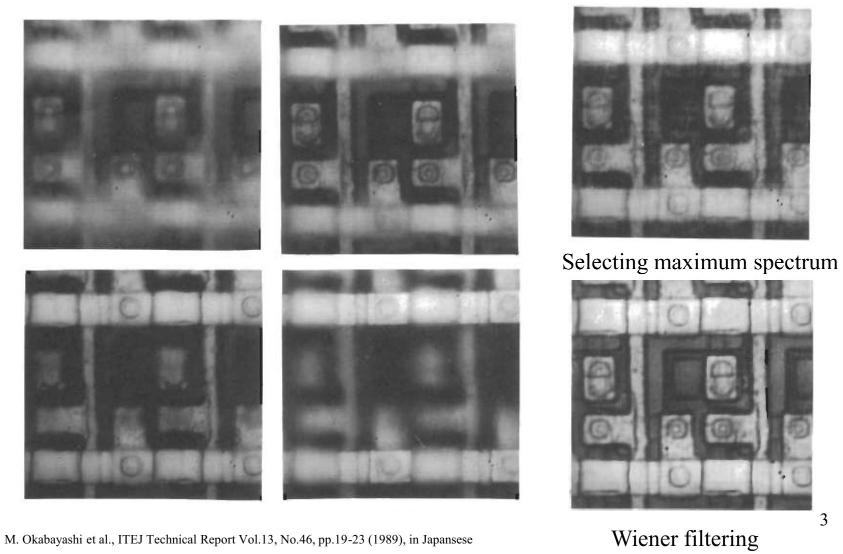
- In the Wiener filtering, the amplification of noise is suppressed by the term $\frac{S_n(u,v)}{S_f(u,v)}$ even when $|H(u,v)|$ is small.

Increasing the depth of field in microscopic observations



Wiener filtering for increasing depth of focus

$$e = E\{|\sum_{l=0}^{L} F_l(u,v) - \sum_{k=1}^{K} M_k(u,v)G_k(u,v)|^2\}$$ l:depth of object, k: focus position



M. Okabayashi et al., ITEJ Technical Report Vol.13, No.46, pp.19-23 (1989), in Japansese

3

6.3 Image restoration in discrete space
6.3　離散空間における画像復元

- Vector-matrix model of an imaging system

  $\mathbf{g} = \mathbf{H}\,\mathbf{f}$

  $\mathbf{g} = \mathbf{H}\,\mathbf{f} + \mathbf{n}$

  – The number of elements: $\mathbf{f}$: Q, $\mathbf{g}$, $\mathbf{n}$: P, $\mathbf{H}$: $P\times Q$

  Note, $Q > P$, in most cases.

- Linear estimation of the object from the observed image

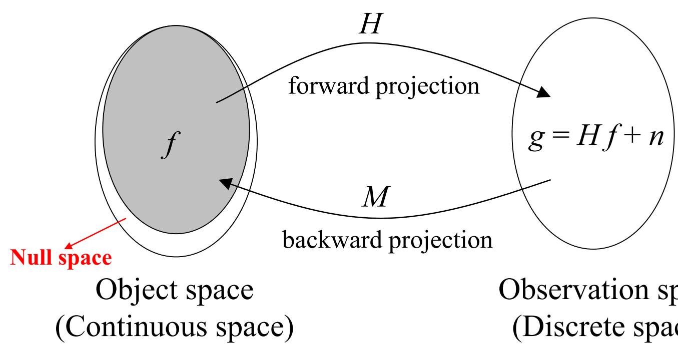
  $\hat{\mathbf{f}} = \mathbf{M}\mathbf{g}$

  $\mathbf{M}$: $Q\times P$ estimation matrix

  – If $Q > P$, ill-conditioned (ill-posed) problem.

  → Regularization techniques

Linear model of image observation and restoration / reconstruction



Sampled in Q elements

Observed by P elements

6.4 Pseudoinverse estimation
6.4 疑似逆推定

- General solution
- If $P = Q$

  $\hat{\mathbf{f}} = \mathbf{H}^{-1}\mathbf{g}$
- $P \neq Q$

  $\hat{\mathbf{f}} = \mathbf{H}^{+}\mathbf{g}$

  $\mathbf{H}^{+}$ Pseudoinverse matrix
- Moore-Penrose pseudoinverse $\mathbf{H}^{-}$

  $\mathbf{x}' = \mathbf{H}^{-}\mathbf{y}$ is one of the solutions to the simultaneous equations $\mathbf{y} = \mathbf{H}\,\mathbf{x}$

  $\mathbf{H}\,\mathbf{H}^{-}\,\mathbf{H} = \mathbf{H}$

  $\mathbf{H}^{-}\,\mathbf{H}\,\mathbf{H}^{-} = \mathbf{H}^{-}$

  $(\mathbf{H}\,\mathbf{H}^{-})^{t} = \mathbf{H}\,\mathbf{H}^{-}$

  $(\mathbf{H}^{-}\,\mathbf{H})^{t} = \mathbf{H}^{-}\,\mathbf{H}$
- then,

  $\hat{\mathbf{f}} = \mathbf{H}^{-}\mathbf{g} + (\mathbf{I} - \mathbf{H}^{-}\mathbf{H})\mathbf{v}$

  $\mathbf{v}$ is a Q-element arbitrary vector.

- If P > Q and $(\mathbf{H}^{t}\,\mathbf{H})$ is nonsingular (overdetermined)

  $\mathbf{H}^{-} = (\mathbf{H}^{t}\,\mathbf{H})^{-1}\,\mathbf{H}^{t}$
- If P < Q and $(\mathbf{H}\,\mathbf{H}^{t})$ is nonsingular (underdetermined)

  $\mathbf{H}^{-} = \mathbf{H}^{t}(\mathbf{H}\,\mathbf{H}^{t})^{-1}$

- Useful relationships for the generalized inverse

  $(\mathbf{H}^{t})^{-} = (\mathbf{H}^{-})^{t}$

  $(\mathbf{H}^{-})^{-} = \mathbf{H}$

  $\mathrm{rank}(\mathbf{H}^{-}) = \mathrm{rank}(\mathbf{H})$

  $(\mathbf{H}^{t}\,\mathbf{H})^{-} = \mathbf{H}^{-}(\mathbf{H}^{t})^{-}$

  $(\mathbf{A}\,\mathbf{B})^{-} = \mathbf{B}^{-}\,\mathbf{A}^{-}$

  – where $\mathbf{A}$ is a $P\times R$ matrix of rank R, and $\mathbf{B}$ is an $R\times P$ matrix of rank R.

  $(\mathbf{A}\,\mathbf{H}\,\mathbf{B})^{-} = \mathbf{B}^{t}\,\mathbf{H}^{-}\,\mathbf{A}^{t}$

  – where $\mathbf{A}$ is a $P\times P$ orthogonal matrix, and $\mathbf{B}$ is a $Q\times Q$ orthogonal matrix.

  $(a\,\mathbf{H})^{-} = a^{-1}\,\mathbf{H}^{-}$

Note: Substitute the transpose to the complex conjugate for complex number matrices

Minimum-norm least-squares generalized inverse

$\hat{\mathbf{f}} = \mathbf{M}\mathbf{g}$

- Estimation error e_g in the observation space

  $e_g = \mathbf{g} - \hat{\mathbf{g}} = \mathbf{g} - \mathbf{H}\hat{\mathbf{f}} = \mathbf{g} - \mathbf{H}\mathbf{M}\mathbf{g} = (\mathbf{I} - \mathbf{H}\mathbf{M})\mathbf{g}$
- Norm of estimation

  $e_o^{\,2} = \|\hat{\mathbf{f}}\|^2 = \hat{\mathbf{f}}^t\hat{\mathbf{f}}$
- Least square error
  - Minimize e_o^2 where $e_g = 0$

  - If P > Q and $(\mathbf{H}^{t}\,\mathbf{H})$ is nonsingular (overdetermined)

  $\mathbf{M} = (\mathbf{H}^{t}\,\mathbf{H})^{-1}\,\mathbf{H}^{t}$
  - If P < Q and $(\mathbf{H}\,\mathbf{H}^{t})$ is nonsingular (underdetermined)

  $\mathbf{M} = \mathbf{H}^{t}(\mathbf{H}\,\mathbf{H}^{t})^{-1}$

→ Minimum norm solution = Moore-Penrose pseudoinverse

Minimum-norm estimate in the presence of noise

- Minimize $e_g^{\,2} + \omega e_o^{\,2}$:

  $e^2 = e_g^{\,2} + \omega e_o^{\,2} = \|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 + \omega\|\hat{\mathbf{f}}\|^2$
- then we have,

  $\mathbf{M} = (\mathbf{H}^{t}\,\mathbf{H} + \omega\mathbf{I})^{-1}\,\mathbf{H}^{t} = \mathbf{H}^{t}(\mathbf{H}\,\mathbf{H}^{t} + \omega\mathbf{I})^{-1}$
- $\omega\mathbf{I}$: Regularization parameter
  - ω is large → Suppress noise (low resolution)
  - ω is small → High resolution (amplify noise)

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SVD (singular value decomposition) pseudoinverse

$\mathbf{H} = \mathbf{V}\,\Lambda\,\mathbf{U}^t$

- $\mathbf{H}$: $P\times Q$, $\mathbf{V}$: $P\times P$ orthogonal matrix, $\mathbf{U}$: $Q\times Q$ orthogonal matrix,
- Λ: $P\times Q$ diagonal matrix

- **V**, **Λ**, and **U** are determined by eigenvalue decomposition

$\mathbf{H}\mathbf{H}^t\,\mathbf{V} = \mathbf{V}\,\Lambda^2$

$\mathbf{H}^t\mathbf{H}\,\mathbf{U} = \Lambda^2\,\mathbf{U}$

- SVD pseudoinverse is given by

$\mathbf{H}^+ = \mathbf{U}^t\,\Lambda^{-1}\,\mathbf{V}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_{\min((P,Q)} \end{pmatrix} \qquad \Lambda^{-1} = \begin{pmatrix} 1/\lambda_1 & & & \\ & 1/\lambda_2 & & \\ & & \ddots & \\ & & & 1/\lambda_{\min(P,Q)} \end{pmatrix}$$

If λ is too small, 1/λ is substituted by 0, to avoid the noise amplification

SVD is also useful for the analysis of the imaging systems.

4.5 Wiener estimation in discrete space
4.5 離散空間におけるウィナー推定

- Estimation error e_f in the object space

$$e_f = \mathbf{f} - \hat{\mathbf{f}} = \mathbf{f} - \mathbf{MHf} = (\mathbf{I} - \mathbf{MH})\mathbf{f}$$

- Ensemble average of estimation error in the object space

$$E\{e_f^{\,2}\} = E\{\|\,\mathbf{f} - \hat{\mathbf{f}}\,\|^2\} = E\{\|\,\mathbf{f} - \mathbf{Mg}\,\|^2\}$$

- Minimize $E\{\,e_f^2\,\}$

$$E\{e_f^{\,2}\} = E\{tr(\mathbf{f} - \hat{\mathbf{f}})(\mathbf{f} - \hat{\mathbf{f}})^t\} = E\{tr(\mathbf{f} - \mathbf{Mg})(\mathbf{f} - \mathbf{Mg})^t\} \qquad (\#)$$

- We have,

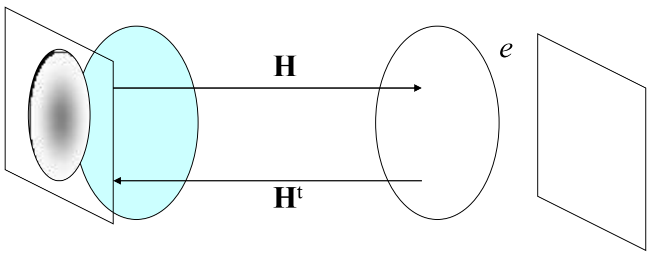
$$\mathbf{M} = E\{\mathbf{fg}^t\}E\{\mathbf{gg}^t\}^{-1} = \mathbf{R}_{fg}(\mathbf{R}_{gg})^{-1}$$

or,

$$\mathbf{M} = \mathbf{R}_f\mathbf{H}^t(\mathbf{HR}_f\mathbf{H}^t + \mathbf{R}_n)^+ \qquad (\#\#)$$

$\mathbf{R}_{fg}$: Correlation matrix of **f** and **g**
$\mathbf{R}_{gg}$: Autocorrelation matrix of **g**
$\mathbf{R}_f$: Autocorrelation matrix of **f**
$\mathbf{R}_n$: Autocorrelation matrix of **n**

Iterative image restoration methods



Estimated image

* Constraint in the object space
-- Nonnegativity, Object support, etc.,

Observed data

* Reduce the estimation error

- Jacobi method
- POCS (Projection onto convex sets)
- EM (expectation maximization)
- Gerchberg-Papoulis
- ...

Practical issues in image restoration / reconstruction

- Estimation of the system matrix (or PSF)
  - Measurement of the imaging system
  - Estimation from the observed image
- Estimation of the noise characteristics
- Estimation of the autocorrelation matrix of **f**
- Sampling in the object space
- Computing cost
  - For $M\times N$ image, inverse of $(MN)\times(MN)$ matrix is needed.
- Numerical error in the calculation
- Trade-offs between the noise suppression and the image resolution

The image restoration / reconstruction method should be addressed in the design of an imaging system.