Literature

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Anti-Dual TU Games (N, (-*v*)*)

- ***v**** is a **dual** of ***v*** :

$$v^*(S) := v(N) - v(N \setminus S) \quad \forall S \subseteq N$$

- **(-*v*)* is the anti-dual of *v***

:= the **dual** of (- ***v***)

$$(-v)^*(S) := -v(N) + v(N \setminus S) \quad \forall S \subseteq N$$

Airport game v_A and bidder collusion game v_C

$$v_A(S) = -\max_{i \in S} c_i, \quad \forall S \subseteq N$$

with $c_1 > c_2 > \cdots > c_n > 0$

$$v_C(S) = \begin{cases} c_1 - \max_{j \in N \setminus S} c_j & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S \end{cases}$$

with $c_1 > c_2 > \ldots > c_n > 0$

$(\max_{j \in N \setminus N} c_j := 0)$

v_A and v_C are anti-duals each other

- $(-v_A)^* = v_C$:

$$\begin{aligned} &(-v_A)^*(S) \\ &= -v_A(N) + v_A(N \setminus S) \\ &= \max_{i \in N} c_i - \max_{i \in N \setminus S} c_i \\ &= \begin{cases} c_1 - \max_{i \in N \setminus S} c_i, & \text{if } 1 \in S, \\ 0, & \text{if } 1 \notin S. \end{cases} \\ &= v_C(S) \end{aligned}$$

… continued

- $(-v_C)^* = v_A$:

$$
\begin{aligned}
&(-v_C)^*(S)\\
&= -v_C(N) + v_C(N \setminus S)\\
&= \begin{cases} -c_1 + c_1 - \max_{i\in S} c_i & = -\max_{i\in S} c_i, \ \text{if } 1 \notin S,\\ -c_1 & = -\max_{i\in S} c_i, \ \text{if } 1 \in S \end{cases}\\
&= v_A(S)
\end{aligned}
$$

Airport game v_A is convex

$$c_{i(S)} := \max_{j\in S} c_j, \quad v_A(S) = -c_{i(S)} \quad \forall S \subseteq N$$

(1) $i(S \cup T) \in S \cap T$

$$\Rightarrow v_A(S) = v_A(T) = v_A(S \cup T) = v_A(S \cap T)$$

(2) $i(S \cup T) \in S \setminus T$

$$\Rightarrow v_A(S) = v_A(S \cup T); \quad v_A(T) \le v_A(S \cap T)$$

(3) $i(S \cup T) \in T \setminus S$

$$\Rightarrow v_A(T) = v_A(S \cup T); \quad v_A(S) \le v_A(S \cap T)$$

$\therefore \ v_A(S)+v_A(T) \le v_A(S\cup T)+v_A(S\cap T) \ \ \forall S \subseteq N$

Lemma 1. *Let v be any game and let a be any additive game defined by $a(S) = \sum_{i\in S} a_i$ for all $S \subseteq N$. Then,* $(-((-v)^* + a))^* = v - a$.

Prove this. (Problem antidual 1)

Remark: Letting $a \equiv 0$, $(-(-v)^*)^* = v$.

Anti-Dual Convexity

- v is convex
  $\iff (-v)^*$ is convex

$\therefore \ v_A$ and v_C are both convex

Proof of anti-dual convexity:

Let $S, T \subseteq N$ and assume that v is convex. Then,

$$
\begin{aligned}
&(-v)^*(S) + (-v)^*(T) \\
&= -[v(N) - v(N \setminus S)] - [v(N) - v(N \setminus T)] \\
&= v(N \setminus S) + v(N \setminus T) - 2v(N) \\
&\leq v((N \setminus S) \cup (N \setminus T)) + v((N \setminus S) \cap (N \setminus T)) - 2v(N) \\
&= v(N \setminus (S \cap T)) + v(N \setminus (S \cup T)) - 2v(N) \\
&= (-v)^*(S \cap T) + (-v)^*(S \cup T)
\end{aligned}
$$

The converse follows from Lemma 1 by taking $a \equiv 0$.

Anti-Dual Core

For any pre-imputation x,

$$
\begin{aligned}
& x(S) \geq v(S) \quad \forall S \subseteq N \\
\iff \quad & x(N \setminus S) \geq v(N \setminus S) \quad \forall S \subseteq N \\
\iff \quad & v^*(S) \geq x(S) \quad \forall S \subseteq N \\
\iff \quad & -x(S) \geq -v^*(S) \quad \forall S \subseteq N \\
\iff \quad & -x(S) \geq (-v)^*(S) \quad \forall S \subseteq N
\end{aligned}
$$

Therefore

$$x \in Core(v) \iff -x \in Core((-v)^*)$$

Anti-Dual Nucleolus

- $(-v)^*$ is the anti-dual of v

$$
\begin{aligned}
(-v)^*(S) &:= (-v)(N) - (-v)(N \setminus S) \\
&= -v(N) + v(N \setminus S), \quad \forall S \subseteq N
\end{aligned}
$$

- the nucleolus of v : $\mu(v)$
- **If v and $(-v)^*$ are both super additive, then**

$$\mu((-v)^*) = -\mu(v)$$

Proof of $\mu((-v)^*) = -\mu(v)$

$$
\begin{aligned}
v(S) - x(S) &= v(N) + (-v(N) + v(S)) - x(S) \\
&= v(N) + (-v)^*(N \setminus S) - x(S) \\
&= (-v)^*(N \setminus S) - (-x(N \setminus S)) \\
&\qquad \forall S \subseteq N
\end{aligned}
$$

-x is a pre-imputation of anti-dual $(-v)^*$. Hence, the vectors of dissatisfaction in game v and $(-v)^*$ coincide each other.

$\mu(v_A) = -\mu(v_C)$

$$v_A = (-v_C)^* \text{ and } v_C = (-v_A)^*$$

- v_A and v_C are both **convex**;

  hence, super additive

$\therefore \quad \mu(v_A) = -\mu(v_C)$

Public good game

$$v(S) = \max\left(0\ ,\ \sum_{i \in S} B_i - C\right) \quad \forall S \subseteq N$$

- $B_i > 0$: i's utility
- $C > 0$: cost of the public good

Bankruptcy game

$$v(S) = \max\left(0,\ E - \sum_{j \in N \setminus S} d_j\right) \quad \forall S \subseteq N$$

- E : estate of a bankrupt
- d_j : debt to $j \in N$

  $E \leq \sum_{j \in N} d_j$
- $v(S)$: **amount guaranteed to S**

Strategically Equivalent Anti-Dual

d^o is an additive game such that
$d^o(S) = \sum_{i \in S} d_i$ (for all $S \subseteq N$)

For public good game v_p and bankruptcy game v_B :

$$(-v_B)^* = v_P - d^\circ \text{ and } (-v_P)^* = v_B - d^\circ$$

$$\text{where } C = E,\ B_i = d_i \quad (\forall i \in N)$$

Hence, $(-v_B)^*(S) = v_P(S) - d^\circ(S)$;

$(-v_P)^*(S) = v_B(S) - d^\circ(S), \quad \forall S \subseteq N$

Public good game v_P and Bankruptcy game v_B

$$(-v_B)^* = v_P - d^\circ \quad \text{and} \quad (-v_P)^* = v_B - d^\circ$$

where $C = E,\ B_i = d_i \quad (\forall i \in N)$

- v_P and v_B are convex; so that super additive

$\therefore \quad \mu(v_P) = \mu(v_P - d^o) + d = \mu((-v_B)^*) + d = -\mu(v_B) + d$

Anti-Dual Shapley Value $\phi((-v)^*)$

$$\phi((-v)^*) = -\phi(v)$$

Proof First of all,

$$\phi_i(-v) = \frac{1}{n!}\sum_{S \subseteq N\setminus\{i\}} |S|!(n-|S|-1)!\big(-v(S\cup\{i\}) - (-v(S))\big)$$
$$= -\frac{1}{n!}\sum_{S \subseteq N\setminus\{i\}} |S|!(n-|S|-1)!\big(v(S\cup\{i\}) - v(S)\big)$$
$$= -\phi_i(v)$$

Proof of $\phi((-v)^*) = -\phi(v)$, continued

$$(-v)^*(S) := -v^*(S) = -(v(N) - v(N\setminus S)) \quad \forall S \subseteq N$$

$$v^*(S\cup\{i\}) - v^*(S) = v(N\setminus S) - v(N\setminus(S\cup\{i\})) = v(N\setminus S) - v((N\setminus S)\setminus\{i\}) \quad \forall S \not\ni i.$$

$$\phi_i(v^*) = \frac{1}{n!}\sum_{S \subseteq N\setminus\{i\}} |S|!(n-|S|-1)!\big(v^*(S\cup\{i\}) - v^*(S)\big)$$
$$= \frac{1}{n!}\sum_{N\setminus S \subseteq N} (n-|S|-1)!|S|!\big(v(N\setminus S) - v((N\setminus S)\setminus\{i\})\big)$$
$$= \phi_i(v)$$

Anti-Dual Shapley Value

$$\therefore \quad \phi((-v)^*) = \phi(-v^*) = -\phi(v^*) = -\phi(v)$$

Compare :
anti-dual nucleolus $\mu((-v)^*)$ and core C

$$\mu((-v)^*) = -\mu(v)$$
$$x \in C(v) \iff (-x) \in C((-v)^*)$$

Airport game v_A and bidder collusion game v_C

$$v_A(S) = -\max_{i \in S} c_i, \quad \forall S \subseteq N$$

with $c_1 > c_2 > \cdots > c_n > c_{n+1} = 0$

$$v_C(S) = \begin{cases} c_1 - \max_{j \in N \setminus S} c_j & \text{if} \quad 1 \in S \\ 0 & \text{if} \quad 1 \notin S \end{cases}$$

with $c_1 > c_2 > \ldots > c_n > c_{n+1} = 0$

$(\max_{j \in N \setminus N} c_j := 0)$

Shapley value of airport game v_A

$$\phi(v_A)_j = -\phi(v_C)_j = -\sum_{i=j}^{n} \frac{c_i - c_{i+1}}{i}, \quad \forall j \in N$$

where

$$c_1 > c_2 > \cdots > c_n > c_{n+1} = 0$$

Shapley value of airport game v_A : Interpretation

$$\phi(v_A)_n = -\frac{c_n - c_{n+1}}{n}$$

$$\phi(v_A)_{n-1} = -\frac{c_{n-1} - c_n}{n-1} - \frac{c_n - c_{n+1}}{n}$$

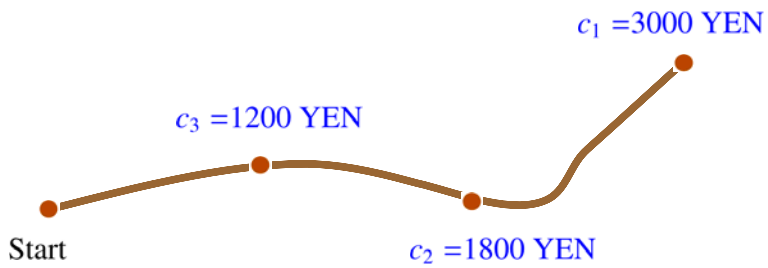
$$\phi(v_A)_{n-2} = -\frac{c_{n-2} - c_{n-1}}{n-2} - \frac{c_{n-1} - c_n}{n-1} - \frac{c_n - c_{n+1}}{n}$$

$\ldots$

$$\phi(v_A)_1 = -(c_1 - c_2) - \sum_{i=2}^{n} \frac{c_i - c_{i+1}}{i}$$

Shapley value of airport game v_A : Application

Sharing a taxi fare



c_i := the fare for the *sole* passenger $i \in N$

Sharing the taxi fare c_1 among n passengers

$$\phi(v_A)_i = -\sum_{i=1}^{n} \frac{c_i - c_{i+1}}{i}$$

$$-\phi(v_A)_3 = \frac{1200}{3} = 400$$
$$-\phi(v_A)_2 = \frac{1800-1200}{2} + \frac{1200}{3} = 700$$
$$-\phi(v_A)_1 = \frac{3000-1800}{1} + \frac{1800-1200}{2} + \frac{1200}{3} = 1900$$

Sharing the taxi fare c_1 among n passengers (2)

Sharing by the nucleolus $\mu(v_A)$ gives

$$-\mu(v_A)_3 = 600$$
$$-\mu(v_A)_2 = 600$$
$$-\mu(v_A)_1 = 1800$$

$$\max_{S\neq N,\emptyset} (v_A(S) - \varphi(v_A)(S)) = -400$$
$$> -600 = \max_{S\neq N,\emptyset} (v_A(S) - \mu(v_A)(S))$$

Prove these facts. (Problem antidual 2)

Shapley value of bidder collusion game v_C

$$\phi(v_C)_j = -\phi(v_A)_j = \sum_{i=j}^{n} \frac{c_i - c_{i+1}}{i}, \quad \forall j \in N$$

where

$$c_1 > c_2 > \cdots > c_n > c_{n+1} = 0$$

Shapley value of bidder collusion game v_C

Ring and Knockout

- $A(S)$:= English auction among the participants $S \subseteq N$
- $R \subseteq N$:= bidder collusion = *ring*, holding the ownership of the commodity
- $R' \subsetneq R$ *knockouts* $R \setminus R'$ with a sole bidder $k \in R'$ defeating any of the member of $R \setminus R'$ in $A(\{k\} \cup R \setminus R')$

Shapley value of bidder collusion game v_C

$$\phi(v_C)_j = -\phi(v_A)_j = \sum_{i=j}^{n} \frac{c_i - c_{i+1}}{i}, \quad \forall j \in N$$

- N, the initial ring.
- $A(\{n-j\} \cup \{n-j+1\})$: the j–th knockout by $\{1, 2, \ldots, n-j\}$ against $\{n-j+1\}$ with the lowest bid c_{n-j}, for each $j = 1, \ldots, n-1$.
- equal division of increment $c_i - c_{i+1} > 0$ in the $(n-i)$-th knockout, for each $i = n, n-1, \ldots, 1$.

Proof of φ(v_A)

$$\phi(v_C)_j = -\phi(v_A)_j = \sum_{i=j}^{n} \frac{c_i - c_{i+1}}{i}, \quad \forall j \in N$$

where

$$c_1 > c_2 > \cdots > c_n > c_{n+1} = 0$$

Proof (continued)

$$-v_A(S) := C(S) = \max_{i \in S} c_i = \sum_{i=1}^{n} (c_i - c_{i+1}) V_i(S) \quad \forall S \subseteq N$$

where

$$V_i(S) = \begin{cases} 0 & \text{if } S \cap \{1, \ldots, i-1, i\} = \emptyset \\ 1 & \text{if } S \cap \{1, \ldots, i-1, i\} \neq \emptyset \end{cases}$$

$$\therefore \phi(C)_j = \sum_{i=1}^{n} \phi((c_i - c_{i+1})V_i)_j = \sum_{i=1}^{n} (c_i - c_{i+1}) \phi(V_i)_j$$

Proof (continued)

In game V_i,

- $\forall k, l \in \{1, \ldots, i-1, i\}$ are *substitutes*
- $\forall h \in \{i+1, \ldots, n\}$ is *null*

Hence, by the corresponding axioms

$$\phi(V_i)_j = \begin{cases} \frac{V_i(N)}{i} = \frac{1}{i} & \forall j \leq i \\ 0 & \forall j > i \end{cases}$$

$$\therefore \phi(C)_j = \sum_{i=1}^{n} (c_i - c_{i+1}) \phi(V_i)_j = \sum_{i=j}^{n} \frac{c_i - c_{i+1}}{i}, \quad \forall j \in N$$

Big Boss Games

(N, v_{BB}) is a *Big Boss game* if it is *monotonic*, and satisfies

1. $v_{BB}(S) = 0$ if $1 \notin S$
2. $v_{BB}(N) - v_{BB}\big(N \setminus (N \setminus S)\big) \geq \sum_{i \in N \setminus S} m_i$ if $1 \in S$

   where $m_i := v_{BB}(N) - v_{BB}(N \setminus \{i\}) \quad \forall i \in N.$

Remark $m_i \geq 0 \;\; \forall i \in N$; v_{BB} is super additive.

Example of Big Boss Games

- (N, v_B^1): bankruptcy game with one big claimant :

$$v_B^1(S) = \begin{cases} E - d(N \setminus S) & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S \end{cases}$$

where $d_1 \geq E,\ d_2 + \cdots + d_n < E$

- (N, v_P^1): public good game with one big agent :

$$v_P^1(S) = \begin{cases} B(S) - C & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S \end{cases}$$

where $B_1 > C,\ B_2 + \cdots + B_n \leq C$

Anti-Dual of Big Boss Games

The anti-dual (N, v_L) of a Big Boss game satisfies

$$v_L(S) \begin{cases} = v_L(N) & \text{if } 1 \in S \\ \leq \sum_{i \in S} v_L(\{i\}) & \text{if } 1 \notin S, \end{cases}$$

which might be called the *leader game*.

Remark $0 \geq v_L(\{i\}) \geq v_L(N)$; nevertheless, $v_L(N) \geq v_L(\{1\}) + \sum_{i \in N \setminus \{1\}} v_L(\{i\})$.

Nucleolus of Big Boss Games and Leader Games

$\mu(v)$: the nucleolus of (N, v)

$\mu(v_{BB}) = -\mu((-v_{BB})^*) = -\mu(v_L)$

$m_i := v_{BB}(N) - v_{BB}(N \setminus \{i\}) = -v_L(\{i\}) \quad \forall i \in N.$

$$\mu(v_{BB}) = \begin{cases} v_{BB}(N) - \frac{1}{2} \sum_{j \in N \setminus \{1\}} m_j & \text{if } i = 1 \\ \frac{1}{2} m_i & \text{if } i \neq 1 \end{cases}$$

$$\mu(v_L) = \begin{cases} v_L(N) - \frac{1}{2} \sum_{j \in N \setminus \{1\}} v_L(\{j\}) & \text{if } i = 1 \\ \frac{1}{2} v_L(\{i\}) & \text{if } i \neq 1 \end{cases}$$

Nucleolus of Big Boss Games and Leader Games

Proof. Let $z = \mu(v_L)$. Then

$$v_L(\{i\}) - z_i = \frac{1}{2}v_L(\{i\}) \qquad \text{if } i \neq 1$$

$$v_L(N \setminus \{i\}) - z(N \setminus \{i\}) = v_L(N \setminus \{i\}) - z(N) + z_i$$

$$= z_i = \frac{1}{2}v_L(\{i\}) \qquad \text{if } i \neq 1$$

$$v_L(S) - z(S) \leq \sum_{j\in S} \frac{1}{2}v_L(\{j\}) \leq \frac{1}{2}v_L(\{j\}) \text{ if } S \not\ni 1,\ j \in S$$

$$v_L(S) - z(S) = \sum_{j\in N\setminus S} \frac{1}{2}v_L(\{j\}) \leq \frac{1}{2}v_L(\{j\}) \text{ if } N \supsetneq S \ni 1,\ j \notin S$$

Taking any $x \neq z$, we necessarily have $x_i > z_i$ or $x_i < z_i$ for some $i \neq 1$, which leads to the conclusion.

The Nucleolus and the Shapley Value of Convex Big Boss Games

$\phi(v)$: Shapley value of game v

Proposition : If the leader game v_L is super additive, then

$$\mu(v_L) = \phi(v_L) \quad \text{and} \quad \mu(v_{BB}) = \phi(v_{BB})$$

Proof: Obtain $\phi(v_L)_i = \frac{1}{2}v_L(\{i\})$ for $i \neq 1$ by direct calculation, where v_L is given, due to the super additivity, as follows.

$$v_L(S) = \begin{cases} v_L(N) & \text{if } 1 \in S, \\ \sum_{i\in S} v_L(\{i\}) & \text{if } 1 \notin S. \end{cases}$$

Try to complete the proof (Problem antidual 3).

The Bankruptcy Game and the Self-Duality of the Nucleolus

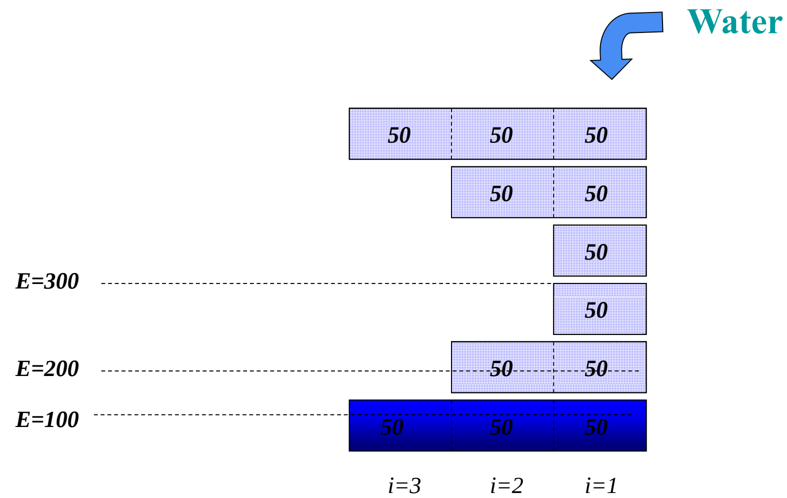
Dvision Rule from the Talmud

	$d_1 = 300$	$d_2 = 200$	$d_3 = 100$
$(a) : E = 100$	$100/3$	$100/3$	$100/3$
$(b) : E = 200$	75	75	50
$(c) : E = 300$	150	100	50

(a) Equal division (c) Proportional division

(b) Unknown

The Nucleolus



The Bankruptcy Game

$$v_{E;d}(S) = \left(E - \sum_{j \in N \setminus S} d_j\right)_+ \quad \forall S \subseteq N$$

- $\boldsymbol{E}$: estate of the bankrupt
- $\boldsymbol{d_j}$: debt to creditor $\boldsymbol{j \in N}$

$$E \leq \sum_{j \in N} d_j := D; \quad d_1 \geq \cdots \geq d_n$$

- **$v_{E;d}(S)$: amount S secures for itself**

The Bankruptcy Game and the Self-Duality of the Nucleolus

$$v_{D-E;d}(S) = \left(D - E - d(N \setminus S)\right)_+$$
$$= \left(d(S) - E\right)_+, \quad \forall S \subseteq N$$

: public good game !

$$v_{D-E;d} = (-v_{E;d})^* + d^\circ$$

Hence, the self-duality :

$$\mu(v_{E;d}) = d - \mu(v_{D-E;d})$$

The Nucleolus of the Bankruptcy Game

Assumption 1. $E \leq \frac{D}{2}$ i.e., cases 1 and 2 below

Remark 1. *The case:* $E \geq \frac{D}{2}$ *can be obtained by the self-duality,* $\mu(v_{E;d}) = d - \mu(v_{D-E;d})$.

case 1: $E \leq \frac{nd_n}{2}$

$$\mu(v_{E;d})_i = \frac{E}{n}, \quad i = 1, \ldots, n.$$

case 2: For $m = 0, 1, \ldots, n-2$, if

$$\frac{1}{2}\left(D - \sum_{j=1}^{n-m}(d_j - d_{n-m})\right) \leq E \leq \frac{1}{2}\left(D - \sum_{j=1}^{n-m-1}(d_j - d_{n-m-1})\right)$$

then,

$$\mu(v_{E;d})_i = \frac{d_i}{2}, \quad i = n, n-1, \ldots, n-m$$

$$\mu(v_{E;d})_i = \frac{d_{n-m}}{2} + \frac{1}{n-m-1}\left(E - \frac{D - \sum_{j=1}^{n-m}(d_j - d_{n-m})}{2}\right),$$
$$i = n-m-1, n-m-2, \ldots, 1.$$