Structural Dynamics 構造動力学 (8)

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6.6 Equations of Motion for Damped MDOF System in General Coordinate (基準座標で表し た減衰系の運動方程式)

 Some form of damping (energy dissipation (エネル ギー吸収)) always exists in all real engineering structures.

 Similar to the equations of motion for a SDOF system given by Eq. (2.10), Eq. (6.12) takes the following form with the inclusion of viscous damping

 $[M]{\ddot{u}}+[C]{\dot{u}}+[K]{u}=\{P\}$ (6.45) in which [C] is a square damping matrix (減衰マト リックス、減衰行列).

 $m\ddot{v}(t) + c\dot{v}(t) + k\overline{v}(t) = p(t) \qquad (2.10)$ $\{P\} - [M]\{\ddot{u}\} = [K]\{u_r\} \qquad (6.12)$

• The elements c_{ij} of the damping matrix [C] have a physical significance analogous to the elements k_{ij} of [K] matrix.

• The damping coefficient, c_{ij} , is defined as the force exerted on the i-th mass by the dashpots due to a unit velocity of the j-th mass when the velocities of all the other masses equal to zero.



• Similar to Eq. (6.34), we assume that

$$\{ \dot{u}(x,t) \} = \begin{cases} \dot{u}_1(x,t) \\ \vdots \\ \dot{u}_i(x,t) \\ \dot{u}_j(x,t) \\ \vdots \\ \dot{u}_n(x,t) \end{cases} = [\Phi(x)] \{ \dot{q}(t) \} \qquad (6.47)$$

$$\{ u(x,t) \} = [\Phi(x)] \{ q(t) \} \qquad (6.34)$$

$$\{ \ddot{u}(x,t) \} = [\Phi(x)] \{ \ddot{q}(t) \} \qquad (6.34)$$

• Similar to Eqs. (6.38) and (6.39), $[\Phi]\{\dot{q}\}$ may be written as

$$[\Phi]\{\dot{q}\} = \{\phi_1\}\dot{q}_1 + \{\phi_2\}\dot{q}_2 + \dots + \{\phi_r\}\dot{q}_r + \dots + \{\phi_n\}\dot{q}_n$$

(6.48)

$$[\Phi] \{q\} = \{\phi_1\} q_1 + \{\phi_2\} q_2 + \dots + \{\phi_r\} q_r + \dots + \{\phi_n\} q_n$$

$$(6.38)$$

$$[\Phi] \{\ddot{q}\} = \{\phi_1\} \ddot{q}_1 + \{\phi_2\} \ddot{q}_2 + \dots + \{\phi_r\} \ddot{q}_r + \dots + \{\phi_n\} \ddot{q}_n$$

$$(6.39)_5$$

• Hence, we have

$$\{u\} = [\Phi]\{q\} \\
\{\dot{u}\} = [\Phi]\{\dot{q}\} \\
(6.49) \\
\{\ddot{u}\} = [\Phi]\{\ddot{q}\} \\
(6.49) \text{ into Eq. (6.45), one obtains} \\
[M][\Phi]\{\ddot{q}\} + [C][\Phi]\{\dot{q}\} + [K][\Phi]\{q\} = \{P\} \\
(6.50) \\
(6.50) \\
(6.51) \\
[M][\dot{u}\} + [C]\{\dot{u}\} + [K][u] = \{P\} \\
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• Because, ϕ_1^T ϕ_2^{T} $[\Phi]^T = [\phi_1 \ \phi_2 \ \cdots \ \phi_n]^T = \left| \cdot \right|$ ϕ_n^T we have ϕ_1^{\prime} $\begin{bmatrix} \Phi \end{bmatrix}^T \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} \Phi \end{bmatrix} = \begin{vmatrix} \phi_2^T \\ \cdot \end{vmatrix} \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix}$ T_{\star}

• Hence we have $[\Phi]^T[M][\Phi] =$ $[M] [\phi_1 \phi_2 \cdots \phi_n]$ M_1 0 $\cdot \cdot \{\phi_1\}^T [M] \{\phi_n\}$ М ٠ $\cdot \{\phi_{2}\}^{T}[M]\{\phi_{n}\}$ $[M]{}_{m}$. _ $[M] \{\phi_1\} = \{\phi_n\}^T [M] \{\phi_2\}$ •

• Hence we obtain

$$\begin{bmatrix} \Phi \end{bmatrix}^{T} \llbracket M \rrbracket \Phi \rrbracket = \begin{bmatrix} M_{1}^{*} & 0 & \cdot & \cdot & 0 \\ 0 & M_{2}^{*} & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 0 & M_{n}^{*} \end{bmatrix}$$
(6.52)

• in which M_r^* is the generalized mass for the r-th mode (一般化された r 次の質量) and is given by Eq. (6.42a)

$$M_{r}^{*} = \{\phi_{r}\}^{T} [M] \{\phi_{r}\}$$
(6.42a)

• Similarly,

$$\begin{bmatrix} \Phi \end{bmatrix}^{T} [K] \llbracket \Phi \end{bmatrix} = \begin{bmatrix} K_{1}^{*} & 0 & \cdot & \cdot & 0 \\ 0 & K_{2}^{*} & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & K_{n}^{*} \end{bmatrix}$$
(6.53)

in which K_r^* is the generalized stiffness for the r-th mode (一般化された 次のばね定数), and is given by Eq. (6.42b)

 $K_{r}^{*} = \{\phi_{r}\}^{T} [K] \{\phi_{r}\}$ (6.42b)



• Orthogonality condition does not exist for damping matrix $\varphi_{\rm l}$ $[\Phi]^T[C]\![\Phi]$ $[C] [\phi_1 \ \phi_2 \ \cdots \ \phi_n]$ (6.55)**≠**0 $\cdot \quad \{\phi_1\}^T [C] \{\phi_n\}$ ϕ_2 · $\{\phi_{\gamma}\}^{T}$ $\cdot \{\phi_{2}\}^{T}[C]\{\phi_{n}\}$ $\{\phi_{\mathcal{P}}\}^T [C] \{\phi_1\}$ $C(\phi)$ ____ $[C]\{\phi_1\} \quad \{\phi_n\}^T [C]\{\phi_2\} \quad \cdot \quad \cdot \quad \{\phi_n\}^T$

• Since the orthogonal condition does not exist for the damping matrix, the generalized coordinates are coupled in the dynamic equations and complete independence which exists in the generalized coordinates $(q_r \text{ and } \ddot{q}_r)$ of undamped system (refer to Eq. (6.43)) is not possible. (減衰行列に対しては振動 モードの直交性が成立しないため、式(6.43)のような非 減衰系に対して成立した各次振動モードごとに一般化さ れた座標が独立であるという仮定は成立しない)

 $M_r^* \ddot{q}_r + K_r^* q_r = P_r^*$ (6.43)

Raleigh showed that if the damping matrix is a linear combination of the mass matrix and the stiffness matrix, the damping matrix can also have the orthogonality condition (Lord Raleigh: Theory of Sound, Macmillan, 1937, pp. 130) (Raleighは減衰マトリックスを質量行列と剛性行列の1次結合として表すと、減衰行列も質量行列と剛性行列と同様に直交条件を満足し、したがって、非減衰系と同様に各次振動モードごとに一般化された座標が独立であるという仮定が成立することを示した)

 That is, if we represent the damping matrix in the form of

$$[C] = \alpha[M] + \beta[K] \qquad (6.56)$$

The damping system represented by Eq. (6.56) is called Laleigh damping (レーリー減衰)

● Similar to the mass matrix and stiffness matrix, if we can diagonalize the damping matrix (もし減衰行列 を直交化できると仮定すると) as follows



in which $C_r^* = \{\phi_r\}^T [C] \{\phi_r\}$ is the generalized r-th damping coefficient (一般化されたr次の減衰 係数)

• From Eqs. (6.42), (6.56) and (6.57), one obtains $C_{r}^{*} = \{\phi_{r}\}^{T} [C] \{\phi_{r}\} = \alpha M_{r}^{*} + \beta K_{r}^{*}$ (6.58) $M_{r}^{*} = \{\phi_{r}\}^{T} [M] \{\phi_{r}\}$ (6.42a) $K_r^* = \{\phi_r\}^T [K]\{\phi_r\}$ (6.42b) $[C] = \alpha[M] + \beta[K] \tag{6.56}$ $C_1^* \quad 0 \quad \cdot \quad \cdot \quad 0$ $0 \quad C_2^* \quad 0 \quad \cdot \quad \cdot$ · 0 · · · $[\Phi]^T[C]\![\Phi] = |$ (6.57)· · · · · 0 $0 \cdot \cdot \cdot 0$



• From Eq. (6.59), we have n-sets of equation of motion for SDOF system as

$$M_{r}^{*}\ddot{q}_{r} + C_{r}^{*}\dot{q}_{r} + K_{r}^{*}q_{r} = P_{r}^{*}$$
(6.60)

• By dividing both sides of Eq. (6.60) by M_r^* , and denoting Eq. (6.44) and

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$$\frac{C_r}{M_r^*} = 2\xi_r \omega_r \tag{6.61}$$

we have

*

$$\ddot{q}_r + 2\xi_r \omega_r \dot{q}_r + \omega_r^2 q_r = \frac{P_r}{M_r^*}$$
 (6.62)

$$\omega_{r} = \sqrt{\frac{K_{r}^{*}}{M_{r}^{*}}} = \sqrt{\frac{\{\phi_{r}\}^{T} [K]\{\phi_{r}\}}{\{\phi_{r}\}^{T} [M]\{\phi_{r}\}}} \quad (6.44)$$

• Eq. (6.62) can be solved based on the method shown in "6.3 Step-by-Step Dynamic response Analysis."

• Once we can compute the generalized displacement, velocity and acceleration vectors $(\{q(t)\}, \{\dot{q}(t)\}\)$ and $\{\ddot{q}(t)\},$ we can evaluate the relative response displacement $\{u(x,t)\},$ relative response velocity $\{\dot{u}(x,t)\}$ and relative response acceleration $\{\ddot{u}(x,t)\}$ from Eq. (6.49)

• The restoring force $\{P\}$ can be computed by Eq. (6.8)

