Structural Dynamics
構造動力学
(6)

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CHAPTER 6 FORMULATION OF THE MULTI-DEGREE-OF-FREEDOM EQUATIONS OF MOTION (多自由度系の運動方程式)

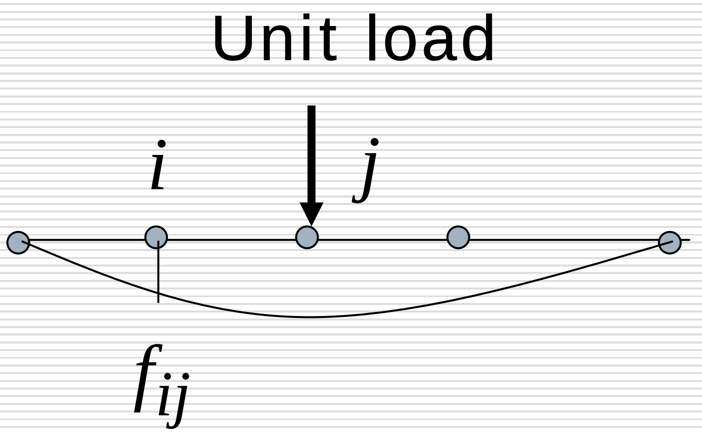
"Dynamics of Structures" by Sheldon Cherry and Note by Kawashima are used.

6．1 Stiffness Matrix and Flexibility Matrix

1） Flexibility Matrix (フレキシビリティー行列)

●Discrete a structure into a n-degree of freedom system.

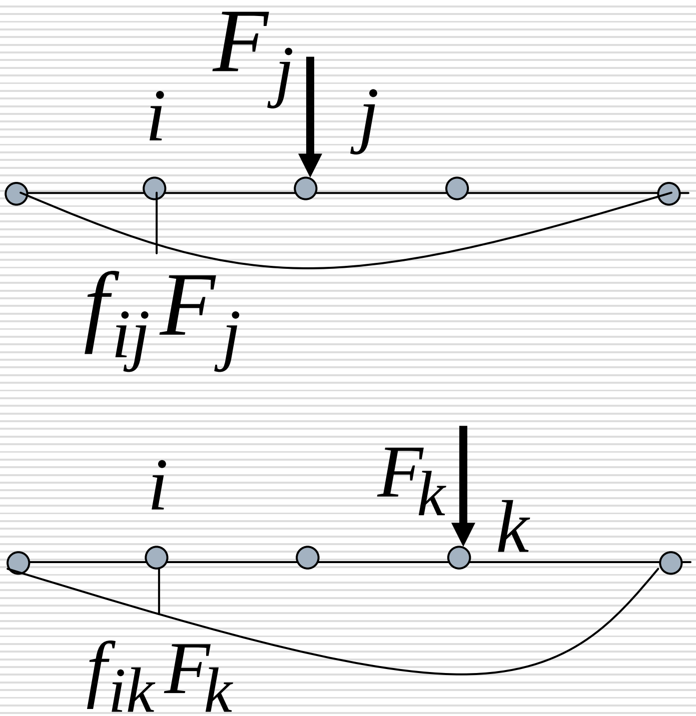
f_{ij} = deflection of coordinate i due to unit load applied to coordinate j (接点jに単位の荷重を作用させた場合に接点iに生じるたわみ)



f_{ij} is called flexibility influence coefficient (フレキシビリティ影響係数), or simply, coefficient of flexibility matrix (フレキシビリティマトリックスの係数).

● Deflection at point i due to any combination of loads F_j may be expressed as

$$u_i = f_{i1}F_1 + f_{i2}F_2 + \cdots\cdots\cdots + f_{in}F_n \qquad (6.1)$$



Hence

$$
\begin{aligned}
u_1 &= f_{11}F_1 + f_{12}F_2 + \cdots\cdots\cdots\cdots + f_{1n}F_n \\
u_2 &= f_{21}F_1 + f_{22}F_2 + \cdots\cdots\cdots\cdots + f_{2n}F_n \\
&\bullet \\
&\bullet \\
u_n &= f_{n1}F_1 + f_{n2}F_2 + \cdots\cdots\cdots\cdots + f_{nn}F_n
\end{aligned}
\tag{6.2}
$$

This can be written in the matrix form as

$$
\begin{Bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{Bmatrix}
=
\begin{bmatrix}
f_{11} & f_{12} & \cdot & \cdot & f_{1n} \\
f_{21} & f_{22} & \cdot & \cdot & f_{2n} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
f_{n1} & \cdot & \cdot & \cdot & f_{nn}
\end{bmatrix}
\begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ F_n \end{Bmatrix}
\tag{6.3}
$$

or,

$$\{u\}=[F]\{P\} \quad (6.4)$$

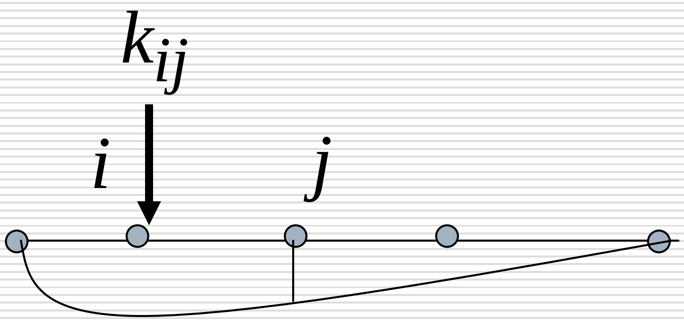
Load vector**荷重ベクトル**

Displacement vector**変位ベクトル**

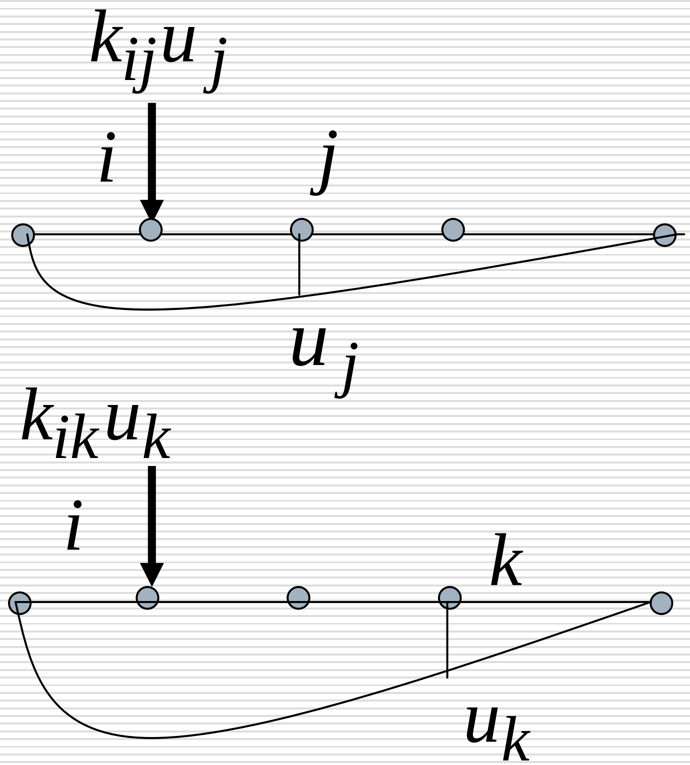
Flexibility matrix **フレキシビリティマトリックス**

2) Stiffness Matrix (剛性行列)

k_{ij} = restoring force (復元力) corresponding to coordinate *i* due to a unit displacement of coordinate *j* (節点 (座標) jに単位の変位が生じたときに接点 (座標) iに生じる力)



Unit deflection



● Load (荷重)(=restoring force (復元力)) at point *i* may be written as

$$p_i = k_{i1}u_1 + k_{i2}u_2 + \cdots\cdots\cdots + k_{in}u_n \tag{6.5}$$

$$
\begin{aligned}
p_1 &= k_{11}u_1 + k_{12}u_2 + \cdots\cdots\cdots + k_{1n}u_n \\
p_2 &= k_{21}u_1 + k_{22}u_2 + \cdots\cdots\cdots + k_{2n}u_n \\
&\quad\cdot \\
p_i &= k_{i1}u_1 + k_{i2}u_2 + \cdots\cdots\cdots + k_{in}u_n \\
&\quad\cdot \\
p_n &= k_{n1}u_1 + k_{n2}u_2 + \cdots\cdots\cdots + k_{nn}u_n
\end{aligned}
\qquad (6.6)
$$

This can be written in the matrix form as

$$
\begin{Bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ p_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdot & \cdot & k_{1n} \\ k_{21} & k_{22} & \cdot & \cdot & k_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{n1} & k_{n2} & \cdot & \cdot & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{Bmatrix} \qquad (6.7)
$$

or,

$$\{P\} = [K]\{u\} \qquad (6.8)$$

External force (=restoring force) vector

Displacement vector

剛性行列
Stiffness Matrix

3) Relation between Stiffness Matrix and Flexibility matrix (剛性行列とフレキシビリティ行列の関係)

- Pre-multiplying $[K]^{-1}$ to both sides of Eq. (6.8),

$$[K]^{-1}\{P\}=[K]^{-1}[K]\{u\}=[I]\{u\}=\{u\}$$

- Comparing this with Eq. (6.4), it is evident that

$$[F]=[K]^{-1} \qquad (6.9a)$$

$$[K]=[F]^{-1} \qquad (6.9b)$$

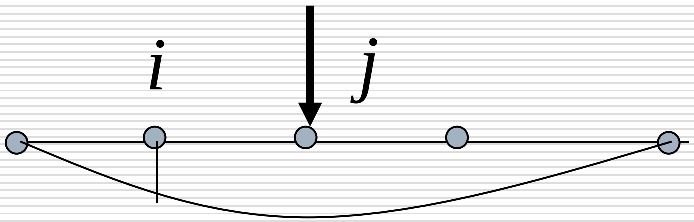
Hence,

$$[K][F]=[F][K]=[I] \qquad (6.10)$$

$\{u\}=[F]\{P\}$	(6.4)
$\{P\}=[K]\{u\}$	(6.8)

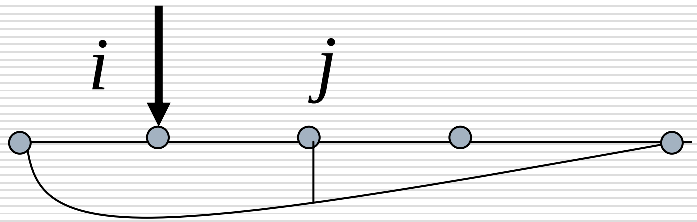
4） Maxwell and Betti's Reciprocal Theorem
(マクスウェルとベッティの相反作用の原理)

Unit load



f_{ij}

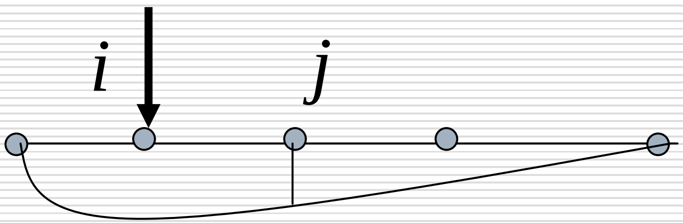
Unit Load



f_{ji}

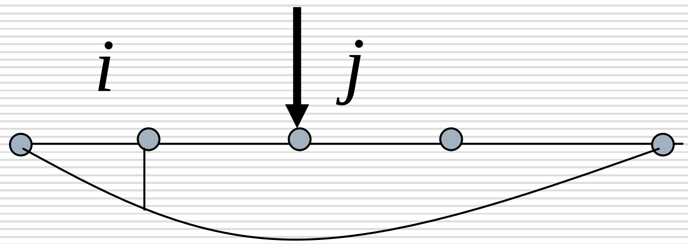
$$f_{ij} = f_{ji}$$

k_{ij}



Unit Displacement

k_{ji}



Unit Displacement

$$k_{ij} = k_{ji}$$

Both stiffness matrix and flexibility matrix are symmetric.

6.2 Equations of Motion for Multi-Degree-of-Freedom System without Damping (多自由度系 (非減衰系) の運動方程式)

- It is not realistic to idealize a complex structure by a single-degree-of-freedom system. In such a case, a complex structure is generally idealized by a multi-degree-of-freedom system (MDOF system).
- Equations of motion for MDOF system is developed in this section.

1) Equations of Motion of a MDOF System based on d'Alembert's Principle (ダランベールの法則に基づく運動方程式)

● Idealize a structure by a discrete MDOF system.

●Assume that each lumped mass (**凝縮マス**) has a single degree of freedom in the lateral direction (**水平方向**).

●As we studied in "2.4 Influence of Support Excitation," consider a MDOF system subjected to a ground motion $\ddot{u}_g(t)$ at its base and lateral external force (**水平方向外力**) $p_i(t)$.

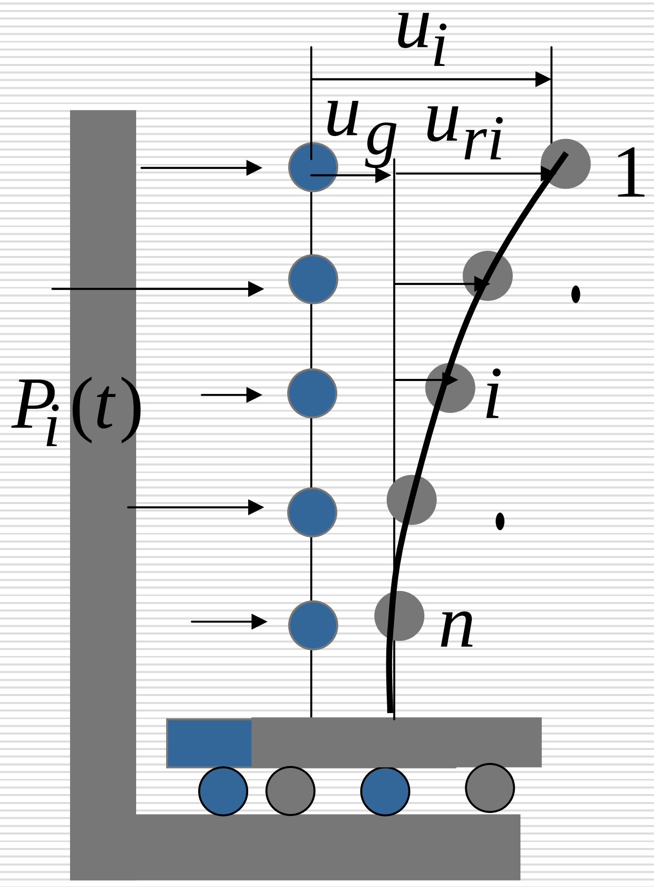


Fig. 6.1(a)

● Based on d'Alembert's Principle, the equilibrium of point i may be written from Eq. (6.6) as

$$p_i(t) - m_i\ddot{u}_i = k_{i1}u_{r1} + k_{i2}u_{r2} + \cdots\cdots\cdots + k_{in}u_{rn}$$

Note here that because the restoring force is proportional to the relative displacement, subscript "r" is attached in the right hand side.

● The above equation can be extended to n-set of equations, and using Eq. (6.7), one obtains

$$\boxed{p_i = k_{i1}u_1 + k_{i2}u_2 + \cdots\cdots\cdots + k_{in}u_n \quad (6.6)}$$

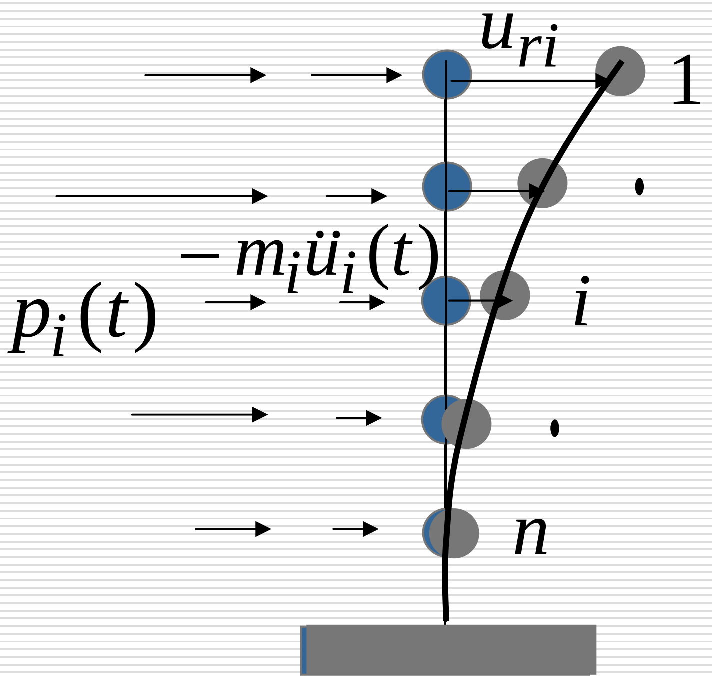


Fig. 6.1 (b)

$$\begin{Bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ p_n \end{Bmatrix} - \begin{bmatrix} m_1 & 0 & \cdot & \cdot & 0 \\ 0 & m_2 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \cdot \\ \cdot \\ \ddot{u}_n \end{Bmatrix}$$

$$= \begin{bmatrix} k_{11} & k_{12} & \cdot & \cdot & k_{1n} \\ k_{21} & k_{22} & \cdot & \cdot & k_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{n1} & k_{n2} & \cdot & \cdot & k_{nn} \end{bmatrix} \begin{Bmatrix} u_{r1} \\ u_{r2} \\ \cdot \\ \cdot \\ u_{rn} \end{Bmatrix} \qquad (6.11)$$

Hence,

$$\{P\} - [M]\{\ddot{u}\} = [K]\{u_r\} \qquad (6.12)$$

- Expanding Eq. (2.15), we separate the absolute displacement at point , u_i, into the relative displacement u_{ri} and ground displacement u_g as

$$u_i = u_{ri} + u_g \qquad (6.13)$$

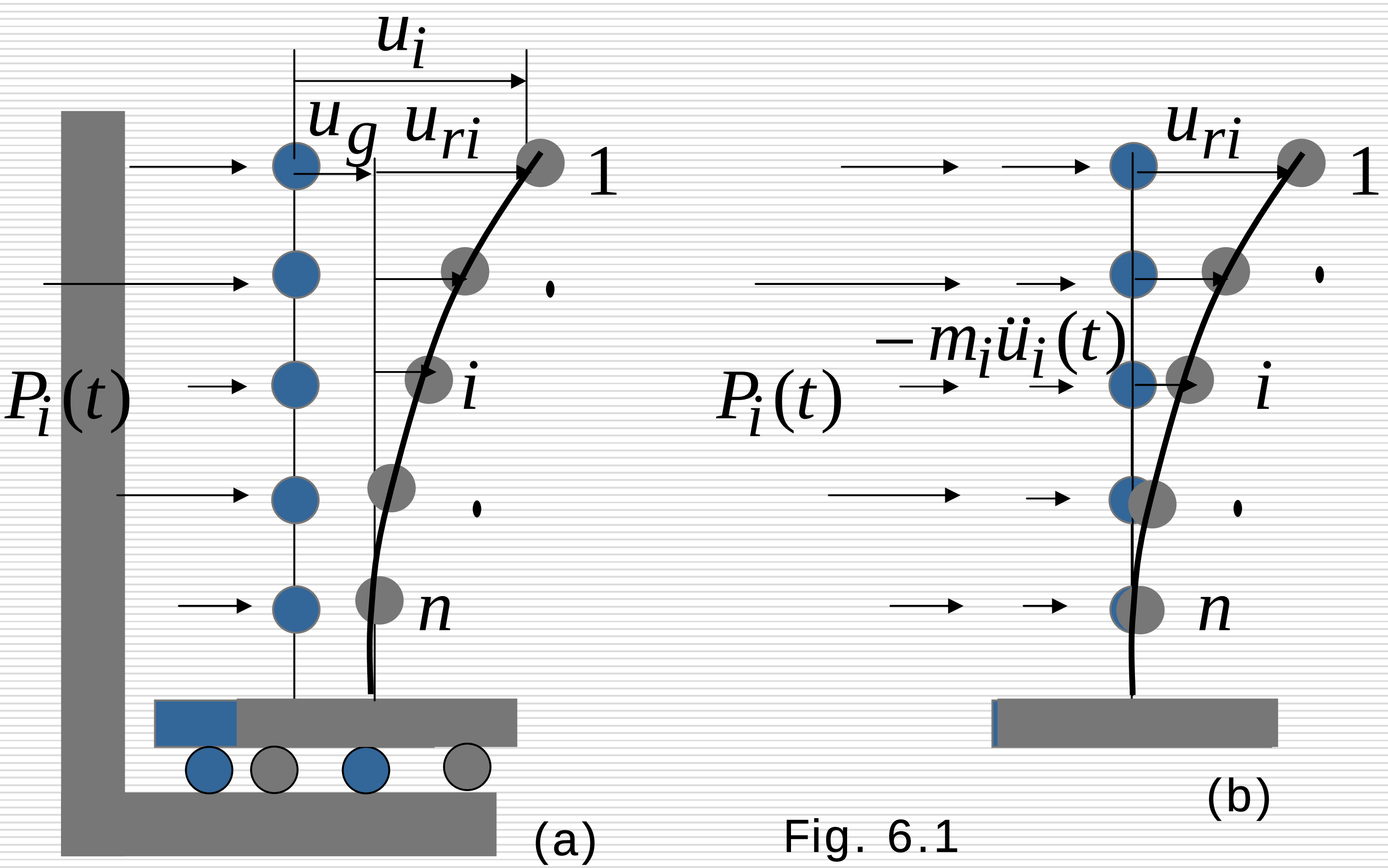


Fig. 6.1

$$v^t(t) = v_g(t) + v(t) \qquad (6.15)$$

- By expressing Eq. (6.13) in the matrix form, one obtains that

$$\{u\} = \{u_r\} + u_g \{I\} \qquad (6.14a)$$

$$\{\ddot{u}\} = \{\ddot{u}_r\} + \ddot{u}_g \{I\} \qquad (6.14b)$$

- Substitution of Eq. (6.14) into Eq. (6.12) leads to

$$\{P\} - [M]\{\ddot{u}_r\} - \ddot{u}_g [M]\{I\} = [K]\{u_r\}$$

- Hence, the equations of motion for a MDOF system can be written

$$[M]\{\ddot{u}_r\} + [K]\{u_r\} = \{P\} - \ddot{u}_g [M]\{I\} \qquad (6.15)$$

$$u_i = u_{ri} + u_g \qquad (6.13)$$

$$\{P\} - [M]\{\ddot{u}\} = [K]\{u_r\} \qquad (6.12)$$

- For simplicity of notation, the subscript “r” which represents the response “relative” to the base (**基礎に対する相対応答**) is eliminated hereinafter.
- Hence, Eq. (6.16) is written as

$$[M]\{\ddot{u}\}+[K]\{u\}=\{P\}-\ddot{u}_g[M]\{I\} \qquad (6.16)$$

$$\boxed{[M]\{\ddot{u}_r\}+[K]\{u_r\}=\{P\}-\ddot{u}_g[M]\{I\} \qquad (6.15)}$$

6.3 Natural Frequencies and Natural Mode Shapes (固有振動数と固有振動モード)

- The equations of motion for free vibration is obtained by assuming that the external force and the foundation (ground) displacement of the right hand side of Eq. (6.16) are zero

$$[M]\{\ddot{u}\}+[K]\{u\}=\{0\} \qquad (6.17)$$

- Assume the displacement and acceleration vectors as

$$\{u\}=\{A\}\sin\omega t$$
$$\{\ddot{u}\}=-\omega^2\{A\}\sin\omega t=-\omega^2\{u\} \qquad (6.18)$$

in which $\{A\}$ represents an unknown amplitude vector with displacement amplitudes of the mass points, and ω is an unknown angular frequency.

$$\boxed{[M]\{\ddot{u}\}+[K]\{u\}=\{P\}-\ddot{u}_g[M]\{I\} \qquad (6.16)}$$

- Substituting Eq. (6.18) into Eq. (6.17) leads to

$$-\omega^2[M]\{A\}+[K]\{A\}=\{0\}$$

Thus, rearranging, we have

$$\left[[K]-\omega^2[M]\right]\{A\}=\{0\} \qquad (6.19)$$

$$[M]\{\ddot{u}\}+[K]\{u\}=\{0\} \qquad (6.17)$$

$$\{u\}=\{A\}\sin\omega t$$

$$\{\ddot{u}\}=-\omega^2\{A\}\sin\omega t=-\omega^2\{u\} \qquad (6.18)$$

● For illustrative propose, it may be easy to represent Eq. (6.19) in the form of a set of equations

$$\left(k_{11}-\omega^2 m_{11}\right)A_1+k_{12}A_2+\cdots\cdots+k_{1n}A_n=0$$

$$k_{21}A_1+\left(k_{22}-\omega^2 m_{22}\right)A_2+\cdots\cdots+k_{2n}A_n=0$$

$$\vdots \qquad (6.20)$$

$$k_{n1}A_1+k_{n2}A_2\cdots\cdots+\left(k_{nn}-\omega^2 m_{nn}\right)A_n=0$$

$$\boxed{\left[[K]-\omega^2[M]\right]\{A\}=\{0\} \qquad (6.19)}$$

● For having a non trivial solution, $\{A\} \neq \{0\}$, it is necessary that

$$\left| [K] - \omega^2 [M] \right| = 0 \qquad (6.21)$$

● Eq. (6.21) is called characteristic equation (**特性方程式**), or eigen value equation (**固有値解析方程式**). By solving Eq. (6.21), one obtains n-set of eigen values, or angular natural frequencies ω_i (n**組の固有値すなわち角固有振動数**).

● In practice, natural frequencies f_i (the i-th natural frequency, **第i次固有振動数**(Hz)) or natural periods T_i (the i-th natural period, **第i次固有周期**(s)) are more often used which are defined as

$$T_i = \frac{2\pi}{\omega_i} \qquad (6.22)$$

$$f_i = \frac{1}{T_i} = \frac{\omega_i}{2\pi} \qquad (6.23)$$

- Once n-set of ω (ω_1, ω_2, ω_3, ….., ω_n) are known, the associated eigen vectors $\{A_1\}, \{A_2\}, \{A_3\}, \ldots, \{A_n\}$ can be found by substituting ω_i into Eq. (6.19).
- However, since the right hand sides of Eq. (6.19) are zero, only the ratios or relative values of the elements of the eigen vector $\{A_i\}$ can be found.
- While the eigen value problem does not fix the absolute amplitude of the eigen vector $\{A_i\}$, the mode shape is uniquely defined in terms of the amplitude ratios.
- It is generally the practice to normalize each mode vector by arbitrarily assigning a value of unity to the component of greatest value.

$$\left|[K] - \omega^2 [M]\right| \{A\} = \{0\} \qquad (6.19)$$

● The elements of $\{A_i\}$ can be normalized by an arbitral component, and if it is A_{1i}, the $\{A_i\}$ vector may be written

$$\{A_i\} = \begin{Bmatrix} A_{1i} \\ A_{2i} \\ . \\ A_{ii} \\ . \\ A_{ni} \end{Bmatrix} \qquad \{\phi_i\} = \begin{Bmatrix} A_{1i}/A_{1i} \\ A_{2i}/A_{1i} \\ . \\ A_{ii}/A_{1i} \\ . \\ A_{ni}/A_{1i} \end{Bmatrix} \equiv \begin{Bmatrix} \phi_{1i} \\ \phi_{2i} \\ . \\ \phi_{ii} \\ . \\ \phi_{ni} \end{Bmatrix} \qquad (6.24)$$

● The $\{\phi_i\}$ vector which is associated with ω_i is generally called the i-th mode shape (i次固有振動モード、あるいは、i次振動モード).

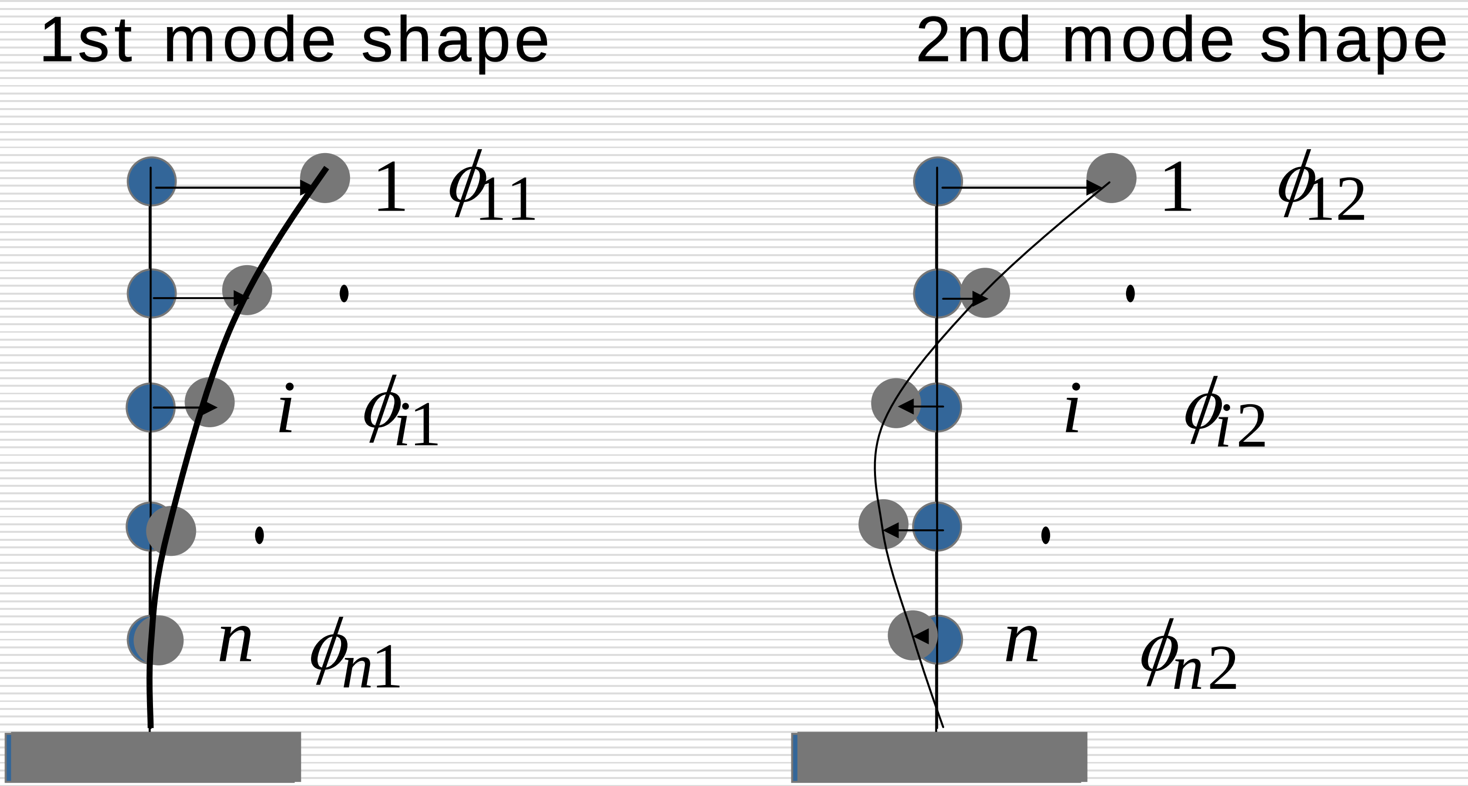
● The analysis of Eq. (6.19) determines as many natural frequencies and independent mode shapes as there are degrees of freedom of the structure, which corresponds to the number of lumped masses assumed in the MDOF system shown in Fig. 6.1.

● The frequency is generally counted from the lowest value as the 1st, the 2nd, ..., the n-th natural frequency (1**次固有振動数**、2**次固有振動数**、...、n**次固有振動数**) and its associated mode shapes are called the 1st mode shape, the 2nd mode shape, ..., the n-th mode shape (1**次固有振動モード**、2**次固有振動モード**、...、n**次固有振動モード**).

●The 1st natural frequency and mode shape are often called the fundamental frequency（**基本固有振動数**） and the fundamental natural mode shape (**基本固有振動モード**).

$$\left|[K]-\omega^2[M]\right|\{A\}=\{0\} \qquad (6.19)$$

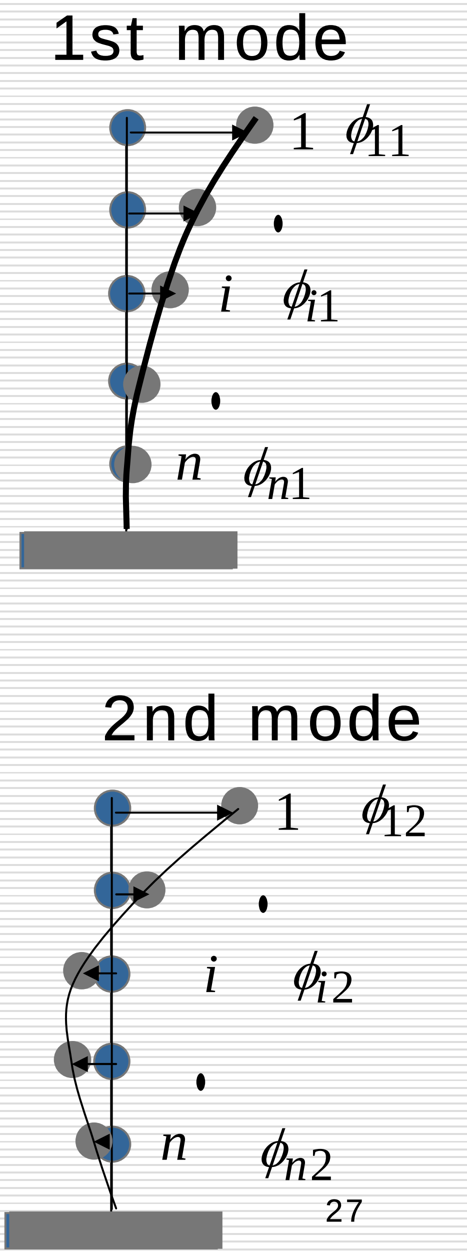
● Example of mode shapes: refer to Example 1 on p.34



- By collecting n-sets natural mode shape vector $\{\phi_i\}\ (i=1,2,\cdots,n)$ we obtain

$$[\Phi]=[\phi_1 \quad \phi_2 \quad \cdot \quad \cdot \quad \phi_n]$$

$$=\begin{array}{c} \left[\begin{array}{c|c|cc|c} \phi_{11} & \phi_{12} & \cdot & \cdot & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdot & \cdot & \phi_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{n1} & \cdot & \cdot & \cdot & \phi_{nn} \end{array}\right] \\ \begin{array}{cc} \phi_1 & \phi_2 \end{array} \end{array} \qquad (6.25)$$



- $[\Phi]$ in Eq. (6.25) is called the modal matrix (**モ-ダルマトリックス**).

● In a similar way, by collecting n-sets of the natural frequency, we have

$$[\Omega]=\begin{bmatrix} \omega_1 & 0 & \cdot & \cdot & 0 \\ 0 & \omega_2 & 0 & \cdot & 0 \\ \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ 0 & \cdot & 0 & 0 & \omega_n \end{bmatrix} \qquad (6.26)$$

● $[\Omega]$ in Eq. (6.26) is called the frequency matrix (**振動数マトリックス**).

● As shown in Fig. (6.22), the natural period (the i-th natural period), $T_i = 2\pi / \omega_i$, is generally used in practice.

●The lowest natural period T_1 is often called the fundamental natural period (**基本固有周期**).

6.4 Orthogonality Condition of Mode Shapes (振動モードの直交性)

● From Eq. (6.19), we have the following relation for the i-th mode.

$$\omega_i^{\ 2}[M]\{\phi_i\} = [K]\{\phi_i\} \qquad (6.27)$$

● Because Eq. (6.27) is valid for any two sets of mode, Eq. (6.27) can be written for the r-th and the s-th modes

$$\omega_r^{\ 2}[M]\{\phi_r\} = [K]\{\phi_r\} \qquad (6.28a)$$

$$\omega_s^{\ 2}[M]\{\phi_s\} = [K]\{\phi_s\} \qquad (6.28b)$$

$$\boxed{\left[[K] - \omega^2[M]\right]\{A\} = \{0\} \qquad (6.19)}$$

- Transposing Eq. (6.28b) leads to

$$\omega_s^{\,2}\{\phi_s\}^T[M]^T = \{\phi_s\}^T[K]^T \qquad (6.29)$$

- Pre-multiplying $\{\phi_s\}^T$ to Eq. (6.28a), one obtains

$$\omega_r^{\,2}\{\phi_s\}^T[M]\{\phi_r\} = \{\phi_s\}^T[K]\{\phi_r\} \qquad (6.30a)$$

- Post-multiplying $\{\phi_r\}$ to Eq. (6.29) leads to

$$\omega_s^{\,2}\{\phi_s\}^T[M]^T\{\phi_r\} = \{\phi_s\}^T[K]^T\{\phi_r\} \qquad (6.30b)$$

- Because both $[M]$ and $[K]$ are symmetric,

$$[M]^T = [M] \qquad [K]^T = [K] \qquad (6.31)$$

$$\boxed{\begin{array}{ll} \omega_r^{\,2}[M]\{\phi_r\} = [K]\{\phi_r\} & (6.28a) \\ \omega_s^{\,2}[M]\{\phi_s\} = [K]\{\phi_s\} & (6.28b) \end{array}}$$

- Subtracting Eq. (6.30b) from Eq. (6.30a) taking account of Eq. (6.32) leads to

$$(\omega_r{}^2 - \omega_s{}^2)\{\phi_s\}^T[M]\{\phi_r\} = 0 \qquad (6.32)$$

- Hence, if $\omega_r \neq \omega_s$, we obtain

$$\{\phi_s\}^T[M]\{\phi_r\} = 0 \qquad (6.33a)$$

and from Eq. (6.30b), we also have

$$\{\phi_s\}^T[K]\{\phi_r\} = 0 \qquad (6.33b)$$

- Note that two modes having the same frequency (r=s) are not necessarily orthogonal.

$$\omega_r{}^2\{\phi_s\}^T[M]\{\phi_r\} = \{\phi_s\}^T[K]\{\phi_r\} \qquad (6.30a)$$

$$\omega_s{}^2\{\phi_s\}^T[M]^T\{\phi_r\} = \{\phi_s\}^T[K]^T\{\phi_r\} \qquad (6.30b)$$

$$[M]^T = [M] \qquad [K]^T = [K] \qquad (6.31)$$

$$\{\phi_s\}^T[M]\{\phi_r\}$$

$$= \{\phi_{1s}, \phi_{2s}, \cdot, \phi_{is}, \cdot, \phi_{ns}\} \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & m_i & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & m_n \end{bmatrix} \begin{Bmatrix} \phi_{1r} \\ \phi_{2r} \\ \cdot \\ \phi_{ir} \\ \cdot \\ \phi_{nr} \end{Bmatrix}$$

$$= \sum_{i=1}^{n} m_i \phi_{ir} \phi_{is} = 0$$

Orthogonality relation of two vectors

- If two vectors

$$\{a\} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} \qquad \{b\} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$

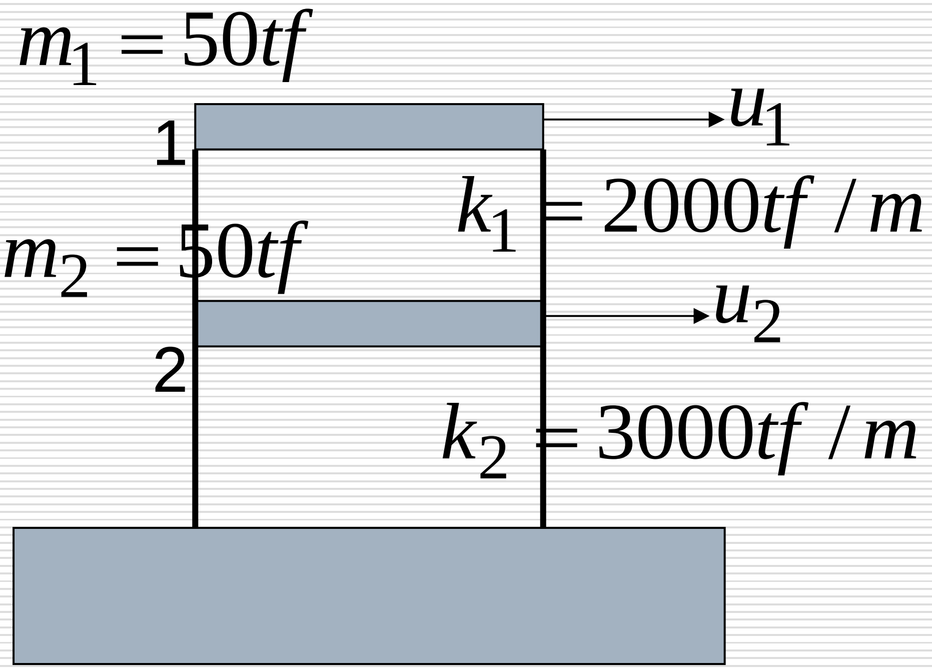
satisfy the following condition, $\{a\}$ and $\{b\}$ vectors are orthogonal.

$$\{a\}^T \{b\} = a_x b_x + a_y b_y + a_z b_z = 0$$

$$\sum_{i=1}^{n} m_i \phi_{ir} \phi_{is} = 0 \longrightarrow$$

$m^{1/2}\phi_s$ and $m^{1/2}\phi_r$ are orthogonal where $m^{1/2}$ is called weighting coefficient （重み係数）.

Example 1: Analyze the natural frequencies and natural mode shapes of a 2DOF system



$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[K] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}$$

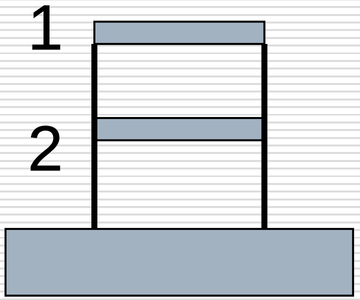
$$= \begin{bmatrix} 2000 & -2000 \\ -2000 & 5000 \end{bmatrix}$$

- Based on Eq. (6.20) or Eq. (6.21),

$$\begin{bmatrix} 2000-\omega^2\dfrac{50}{9.8} & -2000 \\ -2000 & 5000-\omega^2\dfrac{50}{9.8} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Thus,

$$\begin{vmatrix} 2000-\omega^2\dfrac{50}{9.8} & -2000 \\ -2000 & 5000-\omega^2\dfrac{50}{9.8} \end{vmatrix} = 0$$



$$\boxed{\left|[K]-\omega^2[M]\right| = 0 \qquad (6.21)}$$

- Solving the characteristic equations, we have

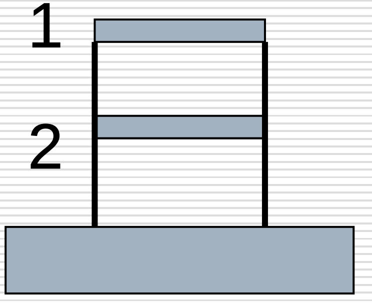
$$\omega_1^{\,2} = 196 \longrightarrow \omega_1 = 14.0 rad/s$$

$$\omega_2^{\,2} = 1176 \longrightarrow \omega_2 = 34.3 rad/s$$

- Hence

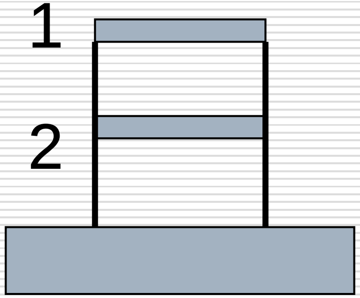
$$T_1 = \frac{2\pi}{\omega_1} = 0.449s$$

$$T_2 = \frac{2\pi}{\omega_2} = 0.183s$$



- Mode shapes (amplitude ratios) can be obtained by substituting the computed natural frequency into the first equation

$$\{A_1\} = \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix}$$



- For ω_1

$$(2000 - \omega_1^2 \frac{50}{9.8}) A_{11} - 2000 A_{21} = 0$$

$$\frac{A_{21}}{A_{11}} = \frac{2000 - 5.1\omega_1^2}{2000} = \frac{1}{2}$$

$$\begin{bmatrix} 2000 - \omega^2 \dfrac{50}{9.8} & -2000 \\ -2000 & 5000 - \omega^2 \dfrac{50}{9.8} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

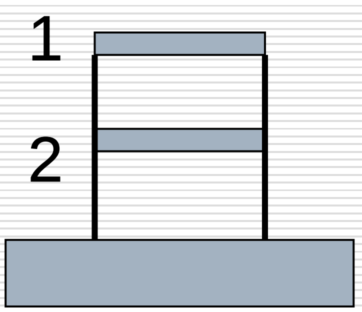
- Note that if we substitute ω_1^2 in the second equation, we can obtain the same mode shape as

$$-2000A_{11} + (5000 - \omega_1^2 \frac{50}{9.8})A_{21} = 0$$

$$\frac{A_{21}}{A_{11}} = \frac{2000}{5000 - 5.1\omega_1^2} = \frac{1}{2}$$

- Thus the same 1st mode shape is obtained as

$$\{\phi_1\} = \begin{Bmatrix} A_{11}/A_{11} \\ A_{21}/A_{11} \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.5 \end{Bmatrix}$$



For ω_2

From the first equation

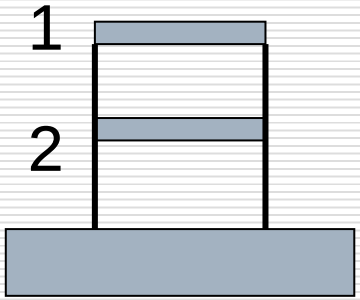
$$(2000 - \omega_2^{\,2}\,\frac{50}{9.8})A_{12} - 2000A_{22} = 0$$

$$\frac{A_{22}}{A_{12}} = \frac{2000 - 5.1\omega_2^{\,2}}{2000} = -2.0$$

Hence,

$$\{\phi_2\} = \begin{Bmatrix} A_{12}/A_{22} \\ A_{22}/A_{22} \end{Bmatrix} = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 1.0 \end{Bmatrix}$$

$$\{A_2\} = \begin{Bmatrix} A_{12} \\ A_{22} \end{Bmatrix}$$

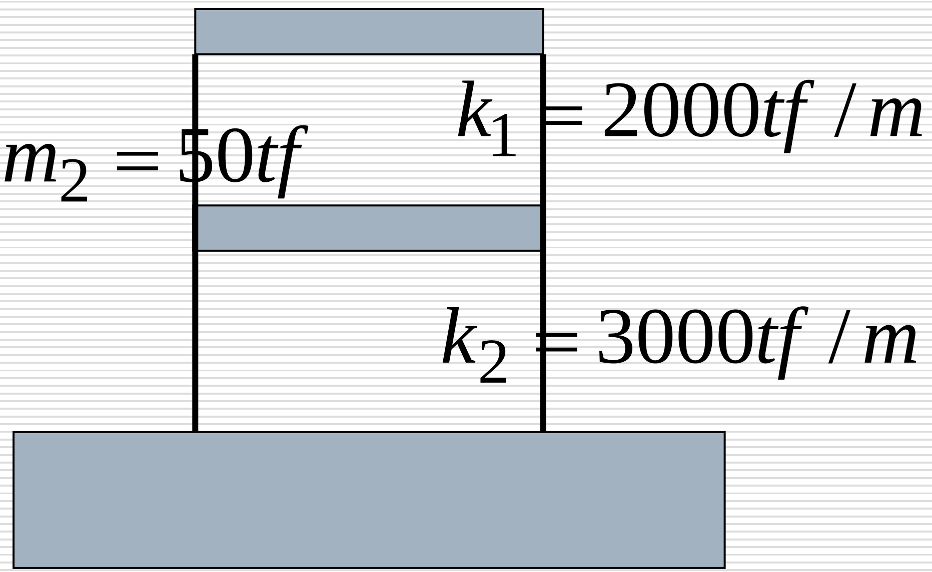


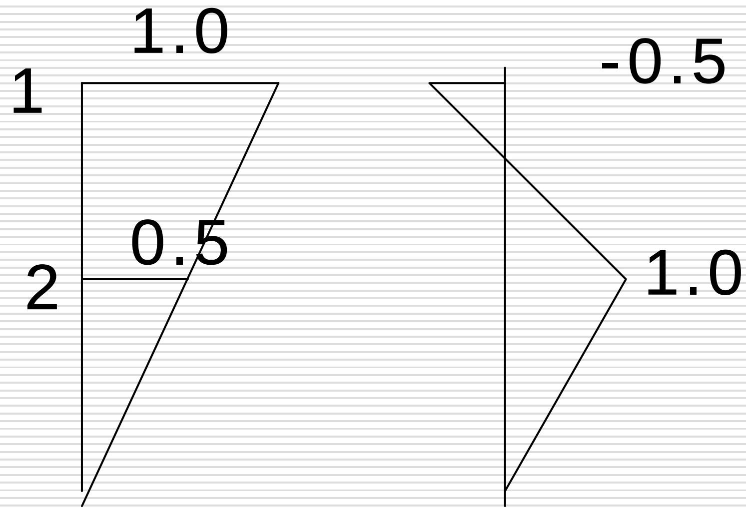
$$\begin{bmatrix} 2000 - \omega^2 \frac{50}{9.8} & -2000 \\ -2000 & 5000 - \omega^2 \frac{50}{9.8} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

● Thus, the 1st and the 2nd mode shapes (refer to Eq.(6.24)) are

$$\{\phi_1\} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.5 \end{Bmatrix}$$

$$\{\phi_2\} = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 1.0 \end{Bmatrix}$$





1st mode 2nd mode

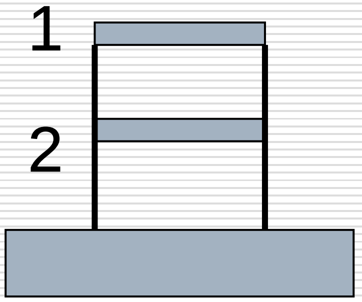
- Note that the i-th mode shape merely represents the amplitude ratios among the n-set of values. Therefore, the first mode can be expressed in various form as

$$\{\phi_1\} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.5 \end{Bmatrix} = \begin{Bmatrix} 2.0 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 0.333 \\ 0.167 \end{Bmatrix}$$

- The modal matrix by Eq. (6.25) is written as

$$[\Phi] = [\phi_1 \; \phi_2] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & -0.414 \\ 0.414 & 1.0 \end{bmatrix}$$



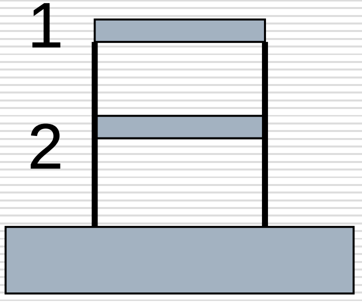
● We can know that two modes satisfy the orthogonal condition as

$$\{\phi_1\}^T[M]\{\phi_2\}$$

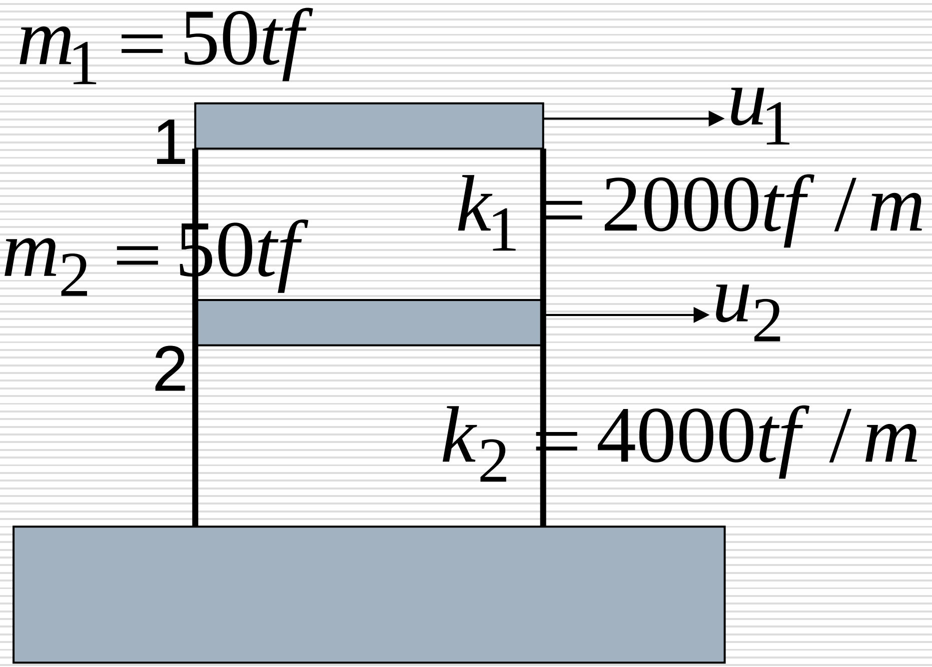
$$=\{1.0, 0.5\}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} -0.5 \\ 1.0 \end{Bmatrix} = m\{1.0, 0.5\}\begin{Bmatrix} -0.5 \\ 1.0 \end{Bmatrix} = 0$$

$$\{\phi_1\}^T[K]\{\phi_2\}$$

$$=\{1.0, 0.5\}\begin{bmatrix} 2000 & -2000 \\ -2000 & 5000 \end{bmatrix}\begin{Bmatrix} -0.5 \\ 1.0 \end{Bmatrix} = 0$$



Example 2: Analyze the natural frequencies and natural mode shapes of a 2DOFS which has the same masses and k_1 stiffness with those of Example 1, but k_2 stiffness of 4000tf/m.



$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[K] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}$$

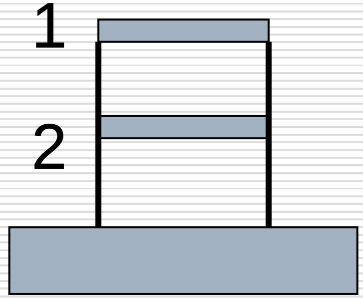
$$= \begin{bmatrix} 2000 & -2000 \\ -2000 & 6000 \end{bmatrix}$$

- Based on Eq. (6.20) or Eq. (6.21),

$$\begin{bmatrix} 2000-\omega^2\dfrac{50}{9.8} & -2000 \\ -2000 & 6000-\omega^2\dfrac{50}{9.8} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Thus,

$$\begin{vmatrix} 2000-\omega^2\dfrac{50}{9.8} & -2000 \\ -2000 & 6000-\omega^2\dfrac{50}{9.8} \end{vmatrix} = 0$$



- Solving the characteristic equations, we have

$$\omega_1^{\ 2} = 229 \qquad \omega_2^{\ 2} = 1338$$

$$\omega_1 = 15.15 rad/s \qquad \omega_2 = 36.58 rad/s$$

$$T_1 = 0.415s \qquad T_2 = 0.172s \quad \longleftrightarrow$$

Example 1
$T_1 = 0.449s$
$T_2 = 0.183s$

- Mode shapes (amplitude ratios) can be obtained by substituting the computed natural frequencies

For ω_1

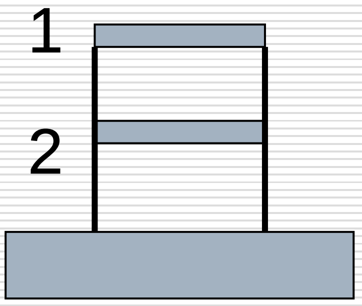
$$(2000-\omega_1^2\frac{50}{9.8})A_{11}-2000A_{21}=0$$

$$\frac{A_{21}}{A_{11}}=\frac{2000-5.1\omega_1^2}{2000}=0.416$$

For ω_2

$$(2000-\omega_2^2\frac{50}{9.8})A_{12}-2000A_{22}=0$$

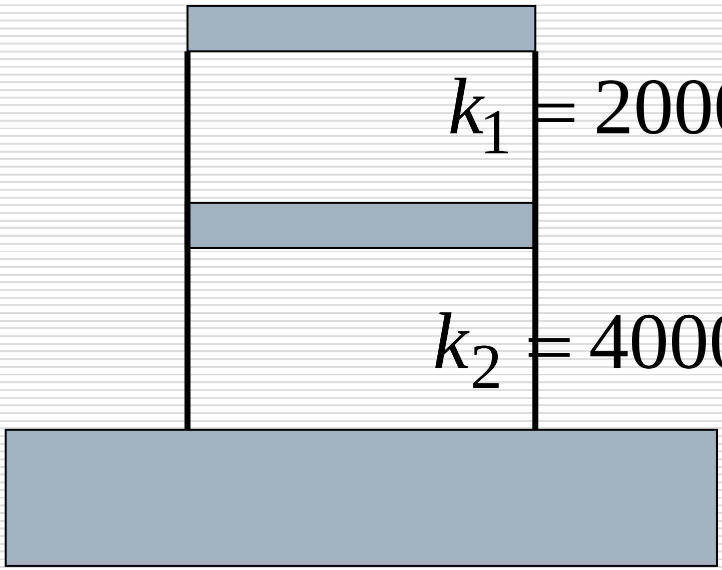
$$\frac{A_{22}}{A_{12}}=\frac{2000-5.1\omega_2^2}{2000}=-2.41$$



- Hence,

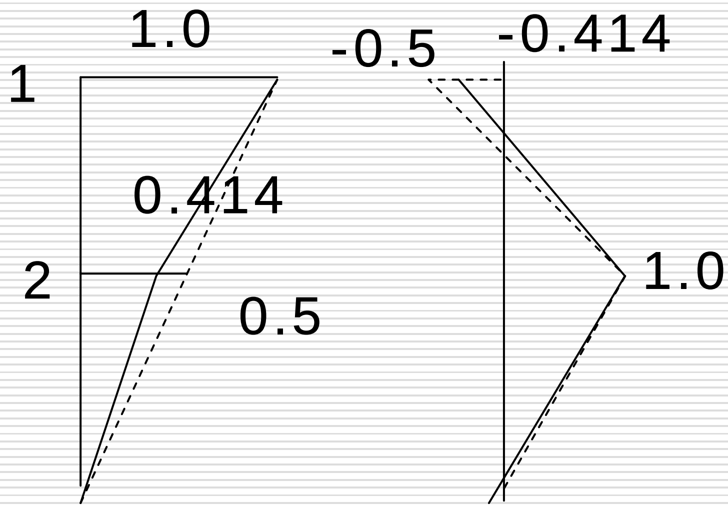
$$\{\phi_1\} = \begin{Bmatrix} A_{11}/A_{11} \\ A_{21}/A_{11} \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.414 \end{Bmatrix}$$

$$\{\phi_2\} = \begin{Bmatrix} A_{12}/A_{22} \\ A_{22}/A_{22} \end{Bmatrix} = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} -0.414 \\ 1.0 \end{Bmatrix}$$



$k_1 = 2000tf/m$

$k_2 = 4000tf/m$



$k_2 = 3000tf/m$

1st mode

2nd mode

- Check of orthogonal condition

$$\{\phi_1\}^T[M]\{\phi_2\}$$

$$= \{1.0, 0.414\}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} -0.414 \\ 1.0 \end{Bmatrix}$$

$$= m\{1.0, 0.414\}\begin{Bmatrix} -0.414 \\ 1.0 \end{Bmatrix} = 0$$

$$\{\phi_1\}^T[K]\{\phi_2\}$$

$$= \{1.0, 0.414\}\begin{bmatrix} 2000 & -2000 \\ -2000 & 5000 \end{bmatrix}\begin{Bmatrix} -0.414 \\ 1.0 \end{Bmatrix} = 0$$