## Structural Dynamics

構造動力学（4）

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## Chapter 4 Response to General Dynamic Loading：Superposition Methods一般的な荷重に対する応答重ね合わせ法に基づく）

This chapter is described based on＂Dynamics of Structures＂by Shelton Cherry．

## 4．1 Impulsive Excitation：The Impulse Function インパルス応答（インパルス関数）

－Consider an idealized SDOF viscous damped system（under critically damped system）at rest at $t=0$ at which time a force $F_{0}$ is applied to the mass for a very short interval $\Delta \mathrm{t}$ ．
－Therefore at time $\mathrm{t}=0$ ，the system is subjected to a single impulse $\mathrm{I}=\mathrm{F}_{0} \Delta \mathrm{t}$（力積）as shown in Fig． 4.1

$\Delta t$ Fig．4．1 Impulse force and impulse response

- From impulse-momentum principle, we have

$$
\begin{equation*}
F_{0} \Delta t=m \dot{v}_{0} \tag{4.1}
\end{equation*}
$$

- Eq. (4.1) implies that as a result of the impulse I, the mass of the system acquires an instantaneous velocity

$$
\begin{equation*}
\dot{v}_{0}=\frac{F_{0} \Delta t}{m}=\frac{I}{m} \tag{4.2}
\end{equation*}
$$


$\Delta t$ Fig. 4.1 Impulse force and impulse response

- Since $F(t)=0$ for $t>\tau$, there is no further excitation and hence no particular integral in the solution of the basic equation of motion.
- The system therefore executes free vibration
- If we measure time from the end of the impulse and assume no appreciable change in displacement during the short time $\Delta t$, the initial conditions are $v(0)=v_{0}=0$ and $\dot{v}(0)=\dot{v}_{0}$.

$\Delta t$ Fig. 4.1 Impulse force and impulse response
- Eq. $(2.50)$ represents the damped free vibration of the system with $v(0)=0$ and $\dot{v}(0)$.

- Denoting from Eq.(2.52), $\theta=-\pi / 2$, Eq. (2.50) becomes

$$
v(t)=\frac{\dot{v}_{0}}{\omega_{D}} \cos \left(\omega_{D} t-\frac{\pi}{2}\right) e^{-\omega \xi t}
$$

or

$$
\begin{equation*}
v(t)=\frac{I}{m \omega_{D}} e^{-\omega \xi t} \sin \omega_{D} t \tag{4.3}
\end{equation*}
$$

$$
\begin{align*}
& v(t)=\rho \cos \left(\omega_{D} t+\theta\right) e^{-\xi \omega t}  \tag{2.50}\\
& \rho=\sqrt{v(0)^{2}+\left(\frac{\dot{v}(0)+v(0) \xi \omega}{\omega_{D}}\right)^{2}}  \tag{2.51}\\
& \theta=-\tan ^{-1}\left(\frac{\dot{v}(0)+v(0) \xi \omega}{\omega_{D} v(0)}\right) \tag{2.52}
\end{align*}
$$

－Eq．（4．3）shows a damped sine－wave of circular frequency $\omega_{D}$ and decay of amplitude by $\exp (-\xi \omega \mathrm{t})$

$$
\begin{equation*}
v(t)=\frac{I}{m \omega_{D}} e^{-\omega \xi t} \sin \omega_{D} t \tag{4.3}
\end{equation*}
$$

－If we define the notation $h(t)$ to represent the response to a unit impulse，then Eq．（4．3）becomes

$$
\begin{equation*}
v(t)=I \cdot h(t) \tag{4.4}
\end{equation*}
$$

where，$h(t)$ is referred to as the impulse－response function（イン $F^{\prime}(t) \downarrow$ レス応答関数）．


Fig．4．1 Impulse force and impulse response

## 4．2 Response to the General Transient（過渡応答）

－Transient response of a SDOF system subjected to an arbitrary external force $F(t)$ may be considered that it is developed as a sequence of infinitesimal impulses each of which contributes to the total motion according to Eq．（4．3）．

$$
v(t)=\frac{I}{m \omega_{D}} e^{-\omega \xi t} \sin \omega_{D} t
$$



Fig．4．2 Transient response 9 as a sequence of impulses

- Assuming $\mathrm{v}(0)=\dot{v}(0)=0$ at $\mathrm{t}=0$ (this restriction shall later be removed), the response due to any one impulse in the absence of all others may be written from Eq. (4.3) as

$$
\begin{equation*}
d v(t)=\frac{F(\tau) d \tau}{m \omega_{D}} e^{-\xi \omega(t-\tau)} \sin \omega_{D}(t-\tau) \tag{4.5}
\end{equation*}
$$

- In Eq. (4.5) we have to carefully distinguish between $t$ (the time at which the response is desired) and $\tau$ (the time at which the impulse applies)

$$
v(t)=\frac{I}{m \omega_{D}} e^{-\omega \xi t} \sin \omega_{D} t
$$



Fig. 4.2 Transient response 10 as a sequence of impulses
－The effect of all impulses in the interval $t=0$ and $t=t$ contribute to the total response at time $t$ ．
－Because we now consider the linear system，the total response due to all impulses may be evaluated by superposition（重ね合わせ）of Eq．（4．5）as

$$
\begin{equation*}
v(t)=\int_{0}^{t} \frac{F(\tau)}{m \omega_{D}} e^{-\xi \omega(t-\tau)} \sin \omega_{D}(t-\tau) d \tau \tag{4.6}
\end{equation*}
$$

where $\tau$ is a dummy variable which disappears on integration．
－Eq．（4．6）is known as Duhamel integral（デュアメル積分），superposition integral（重 ね合わせ積分）or convolution integral（畳み込み積分）．


Fig． 4.2

- If the system has initial condition $v(0)=v_{0}$, and $\dot{v}(0)=\dot{v}_{0}$, the complete solution of the response becomes from Eq. (2.49) and Eq. (4.6) as

$$
\begin{align*}
v(t)=\{ & \left.v(0) \cos \omega_{D} t+\left(\frac{\dot{v}(0)+v(0) \xi \omega}{\omega_{D}}\right) \sin \omega_{D}\right\} e^{-\xi \omega t} \\
& +\int_{0}^{t} \frac{F(\tau)}{m \omega_{D}} e^{-\xi \omega(t-\tau)} \sin \omega_{D}(t-\tau) d \tau \tag{4.7}
\end{align*}
$$

in which, the first term represents the free vibration of the system.

$$
\begin{gather*}
v(t)=\int_{0}^{t} \frac{F(\tau)}{m \omega_{D}} e^{-\xi \omega(t-\tau)} \sin \omega_{D}(t-\tau) d \tau  \tag{4.6}\\
v(t)=\left\{v(0) \cos \omega_{D} t+\left(\frac{\dot{v}(0)+v(0) \xi \omega}{\omega_{D}}\right) \sin \omega_{D}\right\} e^{-\xi \omega t} \tag{2.49}
\end{gather*}
$$

－Eq．（4．6）can be rewritten using the impulse response function（インパルス応答関数）given by Eq． （4．4）as

$$
\begin{equation*}
v(t)=\int_{0}^{t} F(\tau) h(t-\tau) d \tau \tag{4.8}
\end{equation*}
$$

where

$$
\begin{equation*}
h(t-\tau)=\frac{1}{m \omega_{D}}\left\{e^{-\xi \omega(t-\tau)} \sin \omega_{D}(t-\tau)\right\} \tag{4.9}
\end{equation*}
$$

－Depending on the complexity of the system，it may or may not be easy to find $h(t-\tau)$ analytically．It is sometimes possible to obtain good approximations to $h(t-\tau)$ experimentally．

$$
\begin{gather*}
v(t)=I \cdot h(t) \\
v(t)=\int_{0}^{t} \frac{F(\tau)}{m \omega_{D}} e^{-\xi \omega(t-\tau)} \sin \omega_{D}(t-\tau) d \tau \tag{4.6}
\end{gather*}
$$

## 4．3 Response of SDOF System to Earthquake

 Ground Motions（地震動を受ける1自由度系の応答）－Based on Eq．（2．17），the external force $F(t)$ which applies to a SDOF system is

$$
\begin{equation*}
F(t)=-m \ddot{u}_{g}(t) \tag{4.10}
\end{equation*}
$$

－Hence substitution of the above equation to Eq． （4．6）leads to

$$
\begin{equation*}
v(t)=\frac{1}{\omega_{D}} \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t-\tau)} \sin \omega_{D}(t-\tau) d \tau \tag{4.11}
\end{equation*}
$$

$$
\begin{align*}
& v(t)=-\int_{0}^{t} \frac{F(\tau)}{m \omega_{D}} e^{-\xi \omega_{n}(t-\tau)} \sin \omega_{D}(t-\tau) d \tau \\
& m \ddot{v}(t)+c \dot{v}(t)+k v(t)=-m \ddot{v}_{g}(t) \equiv p_{\text {eff }}(t) \tag{4.17}
\end{align*}
$$

## - Denoting

$$
\frac{d}{d t}\left\{\int_{0}^{t} f(\tau, t) d \tau\right\}=\int_{0}^{t} \frac{\partial f(\tau, t)}{\partial t} d \tau+f(\tau, t)_{\tau=t}
$$

differentiation of Eq. (4.11) leads to

$$
\begin{align*}
\dot{v}(t) & =-\int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t-\tau)} \cos \omega_{D}(t-\tau) d \tau \\
& +\frac{\xi}{\sqrt{1-\xi^{2}}} \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t-\tau)} \sin \omega_{D}(t-\tau) d \tau \tag{4.12}
\end{align*}
$$

$$
\begin{equation*}
v(t)=-\frac{1}{\omega_{D}} \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t-\tau)} \sin \omega_{D}(t-\tau) d \tau \tag{4.11}
\end{equation*}
$$

- A further differentiation of Eq. (4.11) yields $\ddot{v}(t)$.
- Absolute acceleration $\ddot{v}_{a}$ (note that $\ddot{v}_{a}$ was represented as $\ddot{v}^{t}$ in Eq. (2.15), but it is
represented as $\ddot{v}_{a}$ here for convenience of the notation), which is important for evaluation of the inertia force, is

$$
\begin{align*}
& \ddot{v}_{a}(t)=\ddot{v}(t)+\ddot{u}_{g}(t) \\
& =\frac{\omega_{n}\left(1-2 \xi^{2}\right)}{\sqrt{1-\xi^{2}}} \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t-\tau)} \sin \omega_{D}(t-\tau) d \tau \\
& \quad+2 \omega_{n} \xi \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t-\tau)} \cos \omega_{D}(t-\tau) d \tau \\
& =\omega_{D} \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t-\tau)}\left\{\left(1-\frac{\xi^{2}}{1-\xi^{2}} \sin \omega_{D}(t-\tau)\right)\right. \\
& \left.\quad+\frac{2 \xi}{\sqrt{1-\xi^{2}}} \cos \omega_{D}(t-\tau) d \tau\right\} \tag{4.13}
\end{align*}
$$

