Structural Dynamics 構造動力学 (4)

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Chapter 4 Response to General Dynamic Loading: Superposition Methods 一般的な荷重に対する応答 (重ね合わせ法に基づく)

This chapter is described based on "Dynamics of Structures" by Shelton Cherry.

4.1 Impulsive Excitation: The Impulse Function インパルス応答(インパルス関数)

• Consider an idealized SDOF viscous damped system (under critically damped system) at rest at t=0 at which time a force F_0 is applied to the mass for a very short interval Δt .

• Therefore at time t=0, the system is subjected to a single impulse $I = F_0 \Delta t$ ($\hbar a$) as shown in Fig. 4.1



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• From impulse-momentum principle, we have

$$F_0 \Delta t = m \dot{v}_0 \tag{4.1}$$

 Eq. (4.1) implies that as a result of the impulse *I*, the mass of the system acquires an instantaneous velocity

$$\dot{v}_0 = \frac{F_0 \Delta t}{m} = \frac{I}{m} \tag{4.2}$$



• Since F(t)=0 for $t>\tau$, there is no further excitation and hence no particular integral in the solution of the basic equation of motion.

• The system therefore executes free vibration

• If we measure time from the end of the impulse and assume no appreciable change in displacement during the short time Δt , the initial conditions are $v(0) = v_0 = 0$ and $\dot{v}(0) = \dot{v}_0$.



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• Eq. (2.50) represents the damped free vibration of the system with v(0) = 0 and $\dot{v}(0)$.

 F_0

$$v(t) = \rho \cos(\omega_D t + \theta) e^{-\xi \omega t} \qquad (2.50)$$

$$\rho = \sqrt{v(0)^2 + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D}\right)^2} \qquad (2.51)$$

$$\theta = -\tan^{-1}\left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D v(0)}\right) \qquad (2.52)$$

$$F(t)$$

$$f(t)$$

$$\Delta t \quad \text{Fig. 4.1 Impulse force and impulse response}$$

• Denoting from Eq. (2.52), $\theta = -\pi/2$, Eq. (2.50) becomes $v(t) = \frac{\dot{v}_0}{\omega_D} \cos(\omega_D t - \frac{\pi}{2}) e^{-\omega \xi t}$ or $v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin \omega_D t \qquad (4.3)$

$$v(t) = \rho \cos(\omega_D t + \theta) e^{-\xi \omega t} \qquad (2.50)$$

$$\rho = \sqrt{v(0)^2 + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D}\right)^2} \qquad (2.51)$$

$$\theta = -\tan^{-1}\left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D v(0)}\right) \qquad (2.52)$$

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• Eq.(4.3) shows a damped sine-wave of circular frequency ω_D and decay of amplitude by exp(- $\xi\omega t$)

$$v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin \omega_D t \qquad (4.3)$$

• If we define the notation h(t) to represent the response to a unit impulse, then Eq. (4.3) becomes

 $v(t) = I \cdot h(t) \tag{4.4}$

where, h(t) is referred to as the impulse-response function (インパルス応答関数).



4.2 Response to the General Transient (過渡応答)

• Transient response of a SDOF system subjected to an arbitrary external force F(t) may be considered that it is developed as a sequence of infinitesimal impulses each of which contributes to the total motion according to Eq. (4.3).

 $\frac{I}{1}e^{-\omega\xi t}\sin\omega_D t$ (4.3)



as a sequence of impulses

• Assuming $v(0) = \dot{v}(0) = 0$ at t=0 (this restriction shall later be removed), the response due to any one impulse in the absence of all others may be written from Eq. (4.3) as

 $dv(t) = \frac{F(\tau)d\tau}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin\omega_D(t-\tau)$ (4.5)

• In Eq. (4.5) we have to carefully distinguish between t (the time at which the response is desired) and τ (the time at which the impulse applies)

$$v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin \omega_D t$$
(4.3)



The effect of all impulses in the interval t=0 and t=t contribute to the total response at time t.
Because we now consider the linear system, the total response due to all impulses may be evaluated by superposition (重ね合わせ) of Eq. (4.5) as

$$v(t) = \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin\omega_D(t-\tau)d\tau \qquad (4.6)$$

where τ is a dummy variable which disappears on integration.

● Eq. (4.6) is known as Duhamel integral (デュアメル積 分), superposition integral (重 ね合わせ積分) or convolution integral (畳み込み積分).



• If the system has initial condition $v(0) = v_0$, and $\dot{v}(0) = \dot{v}_0$, the complete solution of the response becomes from Eq. (2.49) and Eq. (4.6) as

$$v(t) = \left\{ v(0)\cos\omega_D t + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D}\right)\sin\omega_D \right\} e^{-\xi\omega t} + \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)}\sin\omega_D (t-\tau)d\tau \qquad (4.7)$$

in which, the first term represents the free vibration of the system.

$$v(t) = \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin\omega_D(t-\tau)d\tau \qquad (4.6)$$
$$v(t) = \left\{ v(0)\cos\omega_D t + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D}\right)\sin\omega_D \right\} e^{-\xi\omega t} \quad (2.49)$$

Eq. (4.6) can be rewritten using the impulse response function (インパルス応答関数) given by Eq. (4.4) as

$$v(t) = \int_0^t F(\tau) h(t - \tau) d\tau$$
 (4.8)

where

$$h(t-\tau) = \frac{1}{m\omega_D} \left\{ e^{-\xi\omega(t-\tau)} \sin\omega_D(t-\tau) \right\} \quad (4.9)$$

•Depending on the complexity of the system, it may or may not be easy to find $h(t-\tau)$ analytically. It is sometimes possible to obtain good approximations to $h(t-\tau)$ experimentally.

$$v(t) = I \cdot h(t) \qquad (4.4)$$

$$v(t) = \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin \omega_D(t-\tau) d\tau \qquad (4.6)$$
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4.3 Response of SDOF System to Earthquake Ground Motions (地震動を受ける1自由度系の応答)

• Based on Eq. (2.17), the external force F(t) which applies to a SDOF system is

$$F(t) = -m\ddot{u}_g(t) \tag{4.10}$$

• Hence substitution of the above equation to Eq. (4.6) leads to $1 + \frac{\xi}{2} = \frac{\xi}{2} (t + \tau)$

$$v(t) = \frac{1}{\omega_D} \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \sin \omega_D (t-\tau) d\tau \quad (4.11)$$

$$v(t) = -\int_{0}^{t} \frac{F(\tau)}{m\omega_{D}} e^{-\xi\omega_{n}(t-\tau)} \sin\omega_{D}(t-\tau) d\tau \quad (4.6)$$

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = -m\ddot{v}_{g}(t) \equiv p_{eff}(t) \quad (4.17)_{14}$$



$$\frac{d}{dt}\left\{\!\int_0^t f(\tau,t)d\tau\right\}\!\!=\!\int_0^t \frac{\partial f(\tau,t)}{\partial t}d\tau + f(\tau,t)_{\tau=t}$$

differentiation of Eq. (4.11) leads to



(4.12)

 $v(t) = -\frac{1}{\omega_D} \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \sin \omega_D (t-\tau) d\tau \quad (4.11)$ 15

•A further differentiation of Eq. (4.11) yields $\ddot{v}(t)$.

•Absolute acceleration \ddot{v}_a (note that \ddot{v}_a was represented as \ddot{v}^t in Eq. (2.15), but it is represented as \ddot{v}_a here for convenience of the notation), which is important for evaluation of the inertia force, is

$$\begin{split} \ddot{v}_{a}(t) &= \ddot{v}(t) + \ddot{u}_{g}(t) \\ &= \frac{\omega_{n}(1 - 2\xi^{2})}{\sqrt{1 - \xi^{2}}} \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t - \tau)} \sin \omega_{D}(t - \tau) d\tau \\ &\quad + 2\omega_{n} \xi \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t - \tau)} \cos \omega_{D}(t - \tau) d\tau \\ &= \omega_{D} \int_{0}^{t} \ddot{u}_{g}(\tau) e^{-\xi \omega_{n}(t - \tau)} \left\{ \left(1 - \frac{\xi^{2}}{1 - \xi^{2}} \sin \omega_{D}(t - \tau) \right) \\ &\quad + \frac{2\xi}{\sqrt{1 - \xi^{2}}} \cos \omega_{D}(t - \tau) d\tau \right\} \end{split}$$
(4.13)