Structural Analysis II
構造力学第二
(3)

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10. Method of Consistent Deformations (変形法)

10.1 Nature of Compatibility Methods

●The methods that are broadly classified as compatibility methods（変位適合法） are those in which the key relationships used in the solution are compatibility equations that are formulated through the superposition (重ね合わせ) of a set of partial solutions, each of which satisfies the requirements of equilibrium.

●Many methods can be classified as compatibility methods. In this Chapter, we focus on the method of consistent deformation (変形法).

10.2 Redundancies: External versus Internal (外部不静定 対 内部不静定)

●As described in Chapter 9, redundant forces are those that can be removed from the structure without impairing the stable integrity (**安定性**) of the structure.

●These redundant forces may be either external or internal. In the former case, the redundant forces (**不静定力**) are reaction forces (**反力**), whereas in the latter, the redundant forces are member forces (**断面力**).

10.3 Determination of Redundant Reactions

10.3.1 Single Redundant Reaction (**余剰反力が**1**つの場合**)

- The simple propped cantilevered beam (**先端で支持された片持ちばり**) solution that was discussed in Sections 9.1-9.3 was an application of the method of consistent deformation.

- The structure of that example is shown in more general sense in Fig. 10.1.

● The objective of the analysis is to determine the four independent reaction components, R_1-R_4, and the internal member forces (断面力) for member ab.

● The consideratioins of section 3.3 shows that the structure is statically indeterminate to the first order (1次不静定構造物).

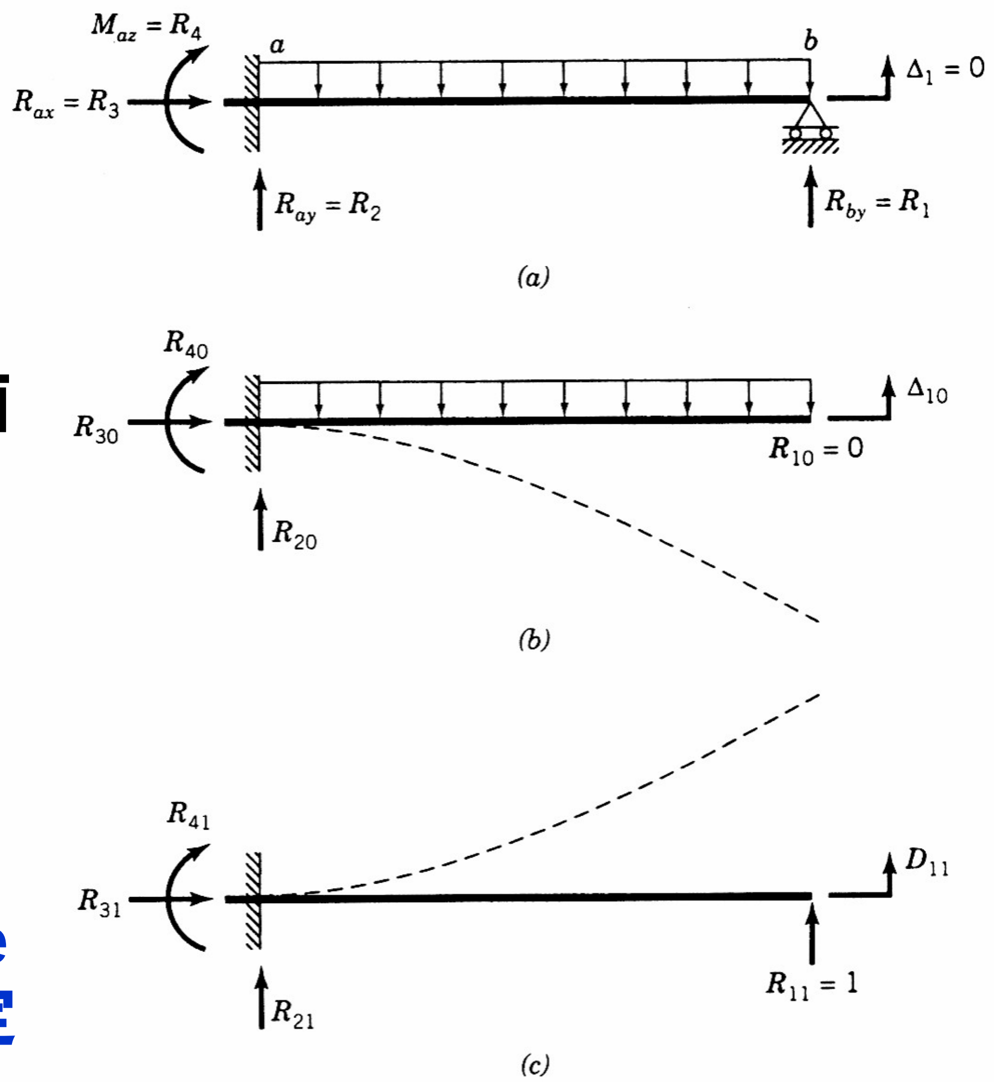


Figure 10.1 Consistent deformation analysis of a statically indeterminate beam structure. *(a)* Statically indeterminate structure. *(b)* Statically determinate primary structure. *(c)* Unit value of R_1.

- Upon removal of R_1, the statically determinate primary structure (主構造) of Fig. 10.1(b) remains.
- Since this structure is statically determinate, the reactions R20-R40can be determined.
- Corresponding to this arrangement, there is a displacement Δ_{10} at the point and in the direction of the released redundant.

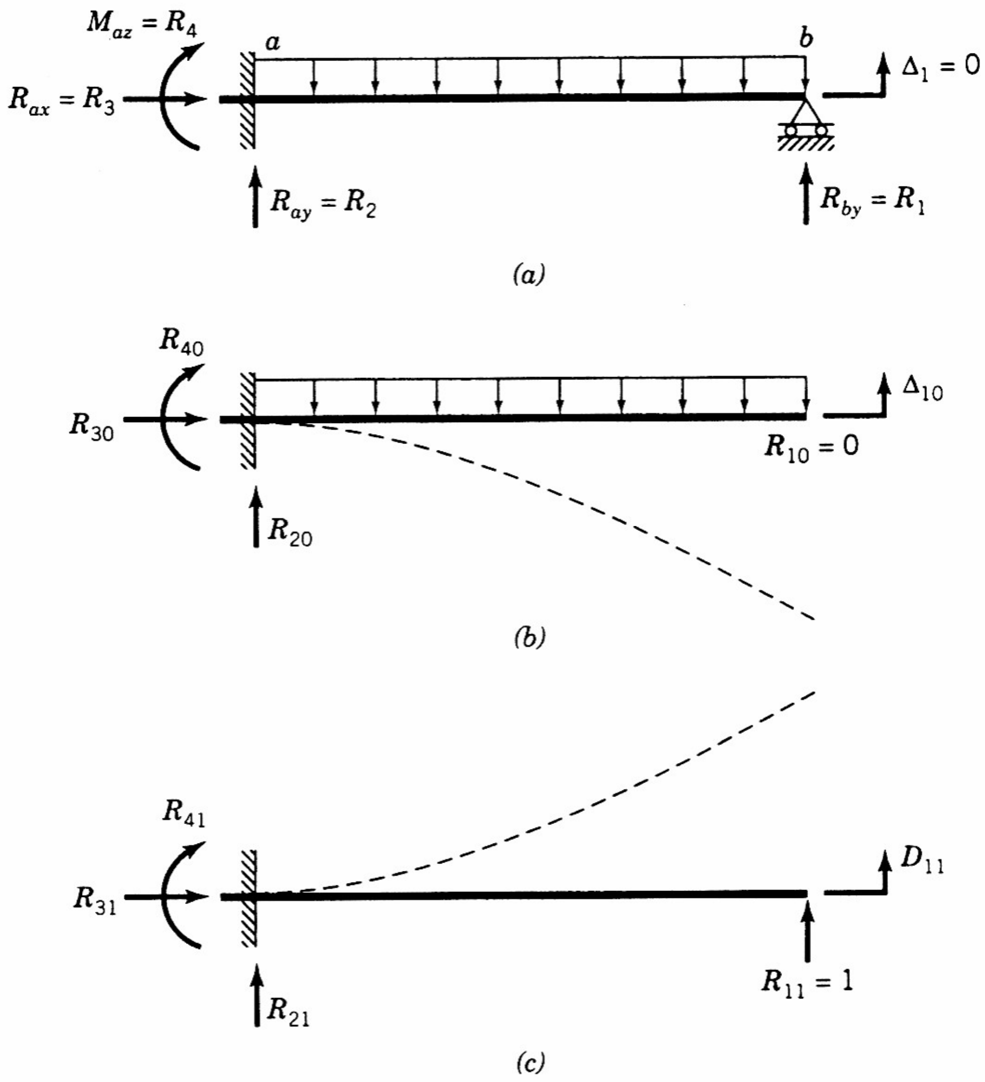


Fig. 10.1

- This displacement is, of course, in violation of the prescribed boundary condition for point b of the original structure, which requires that $\Delta_1 = 0$.
- Thus, the solution of the primary structure must be altered to meet the boundary condition.
- For this purpose, introduce a unit value of the redundant reaction (R_{11}=1) on the primary structure as shown in Fig. 10.1 (c).

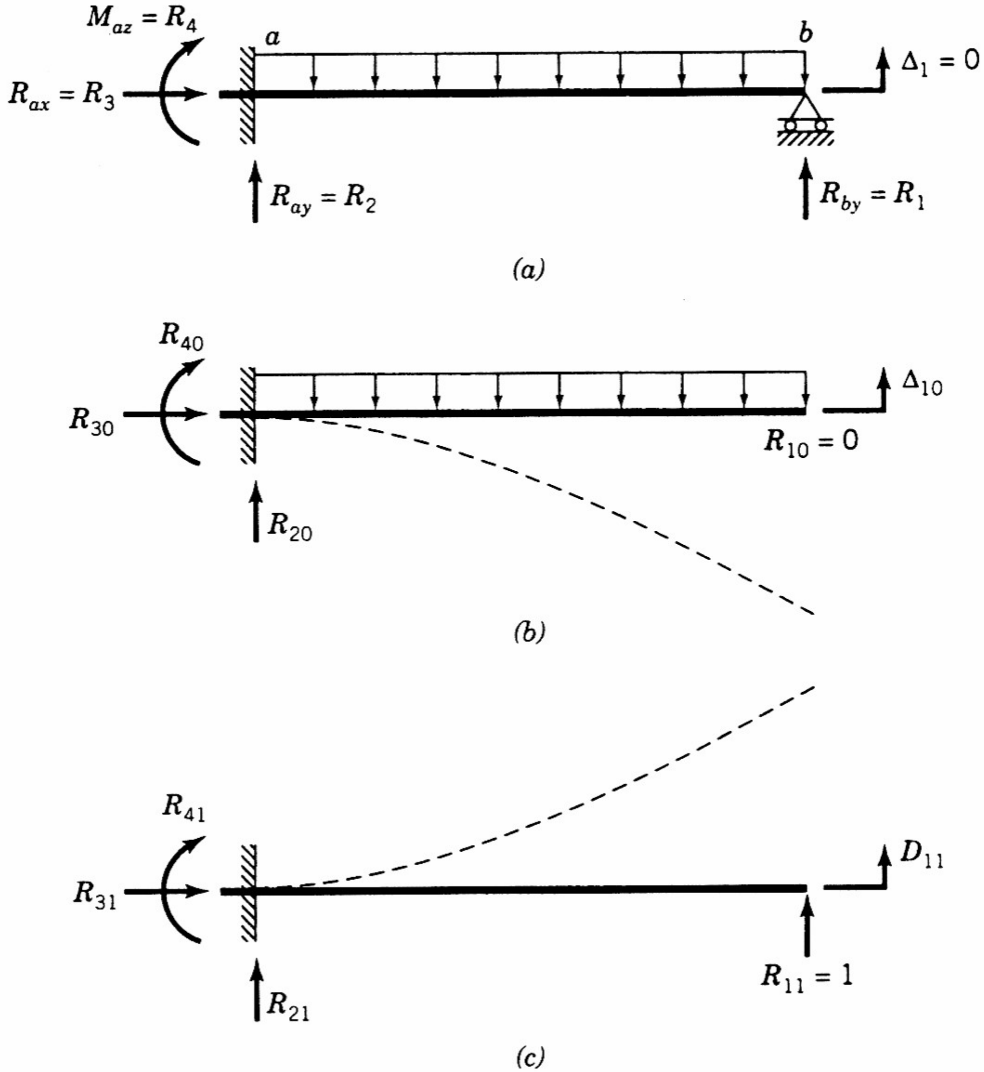


Fig. 10.1

● Here, the reactions R_{21} – R_{41} result from a static analysis of the primary structure.

●The displacement corresponding to the released redundant is identified as D_{11}, which is the flexibility coefficient (たわみ性係数、フレキシビリティー係数) that expresses the deflection at the point and in the direction of R_1 that is caused by a unit value of R_1.

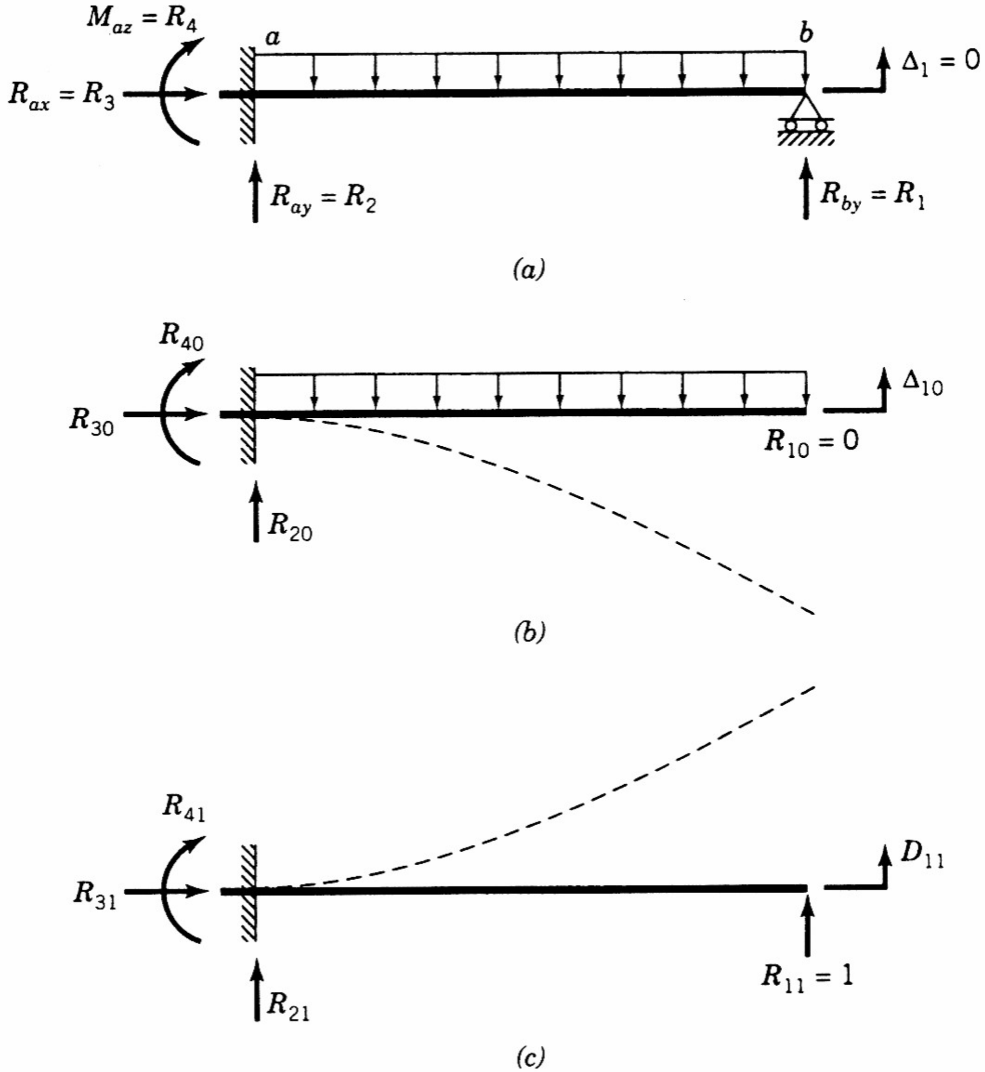


Fig. 10.1

● The deflection at the point in the direction of the released redundant caused by the redundant reaction R_1 is identified as Δ_{1R} and is given by

$$\Delta_{R1} = D_{11} R_1 \qquad (10.1)$$

● The displacement Δ_{10} and Δ_{1R} are combined to give the final displacement Δ_1 as

$$\Delta_{10} + D_{11} R_1 = \Delta_1 = 0$$

(10.2)

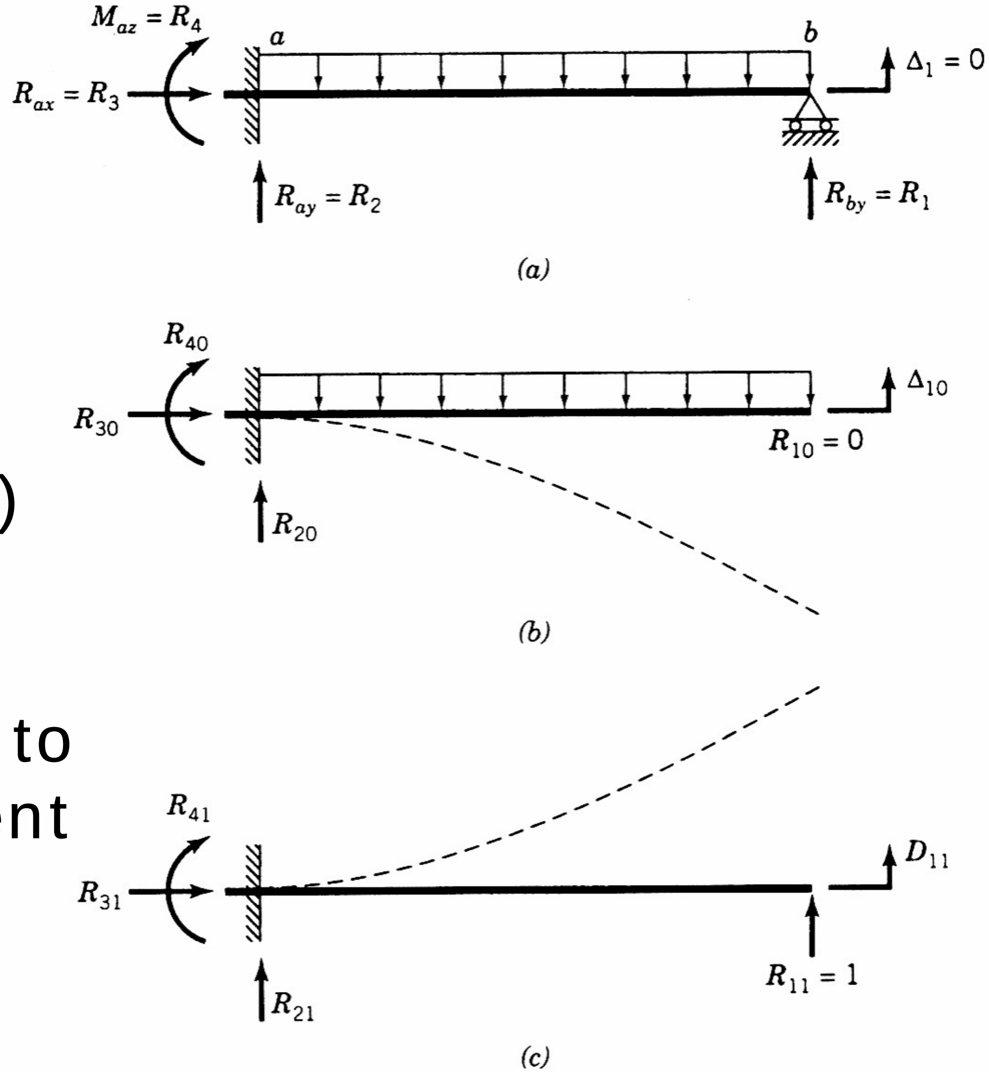


Fig. 10.1

● Solving Eq.(10.2) for R_1, we have

$$R_1 = -\frac{\Delta_{10}}{D_{11}} \qquad (10.3)$$

$$\Delta_{10} + D_{11}R_1 = \Delta_1 = 0 \qquad (10.2)$$

● In Fig. 10.1, all displacements are positive when upward. Thus, Δ_{10} is actually negative as shown in Fig. 10.1(b).

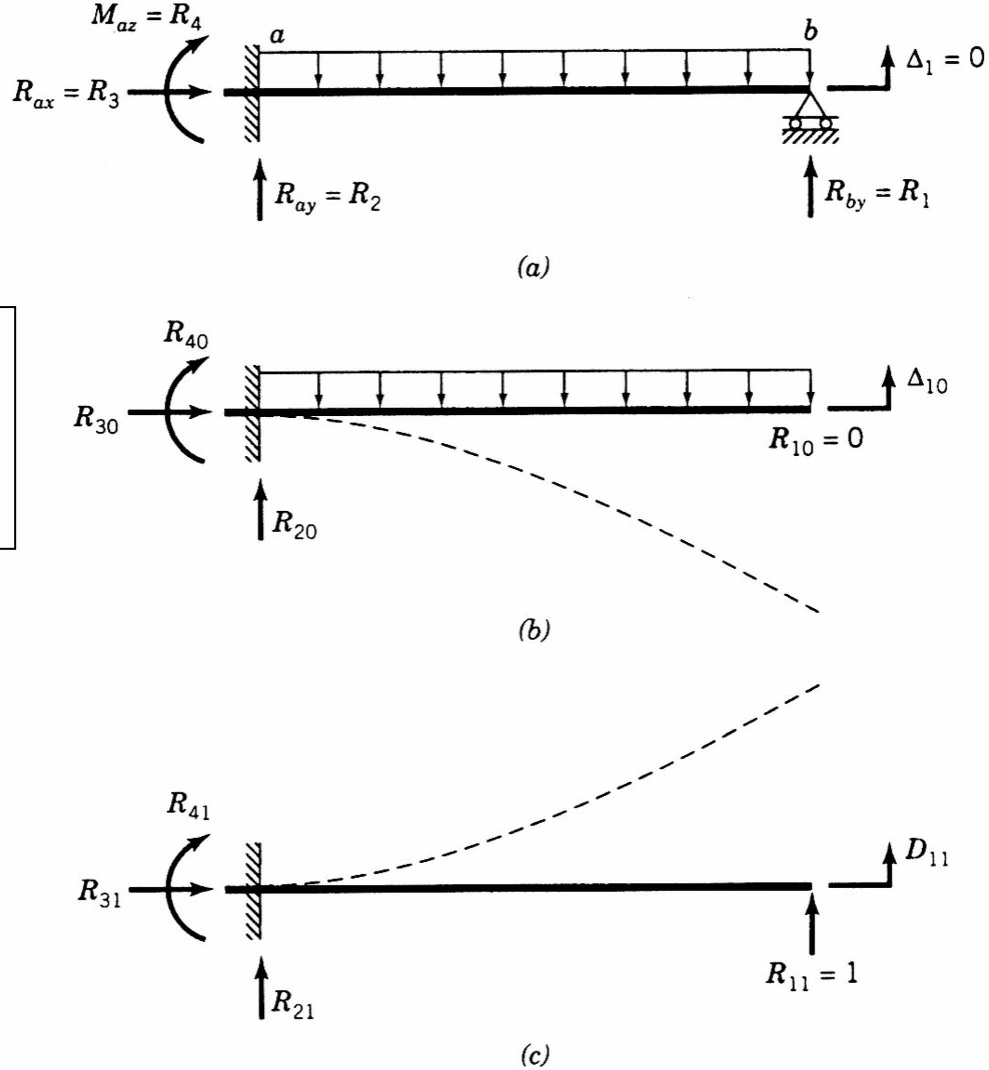


Fig. 10.1

$$\Delta_{10} + D_{11}R_1 = \Delta_1 = 0 \qquad (10.2)$$

- It should be noted that Eq. (10.2) is a compatible equation that has units of displacement.
- Since D11 has units of displacement, the quantity R1 is untitled.

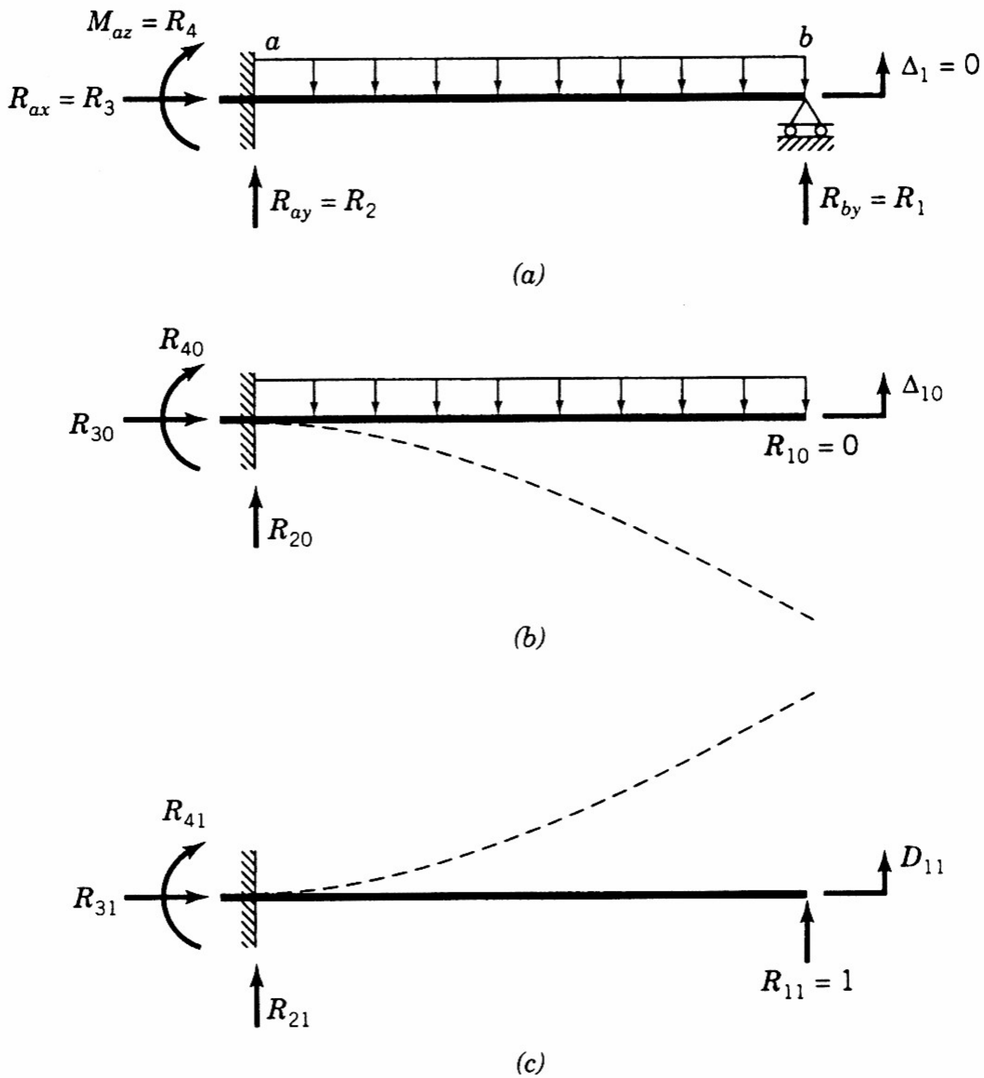


Fig. 10.1

- Once R1 has been determined, statics could be applied to determine the nonredundant reactions.
- Alternatively, there is a more general approach. The superposition pattern expressed in Eq. (10.2) for displacement holds for all other aspects of the solution.
- Thus, to determine one of the nonredundant reactions R_q, we have

$$R_q = R_{q0} + R_{q1} R_1 \qquad (10.4)$$

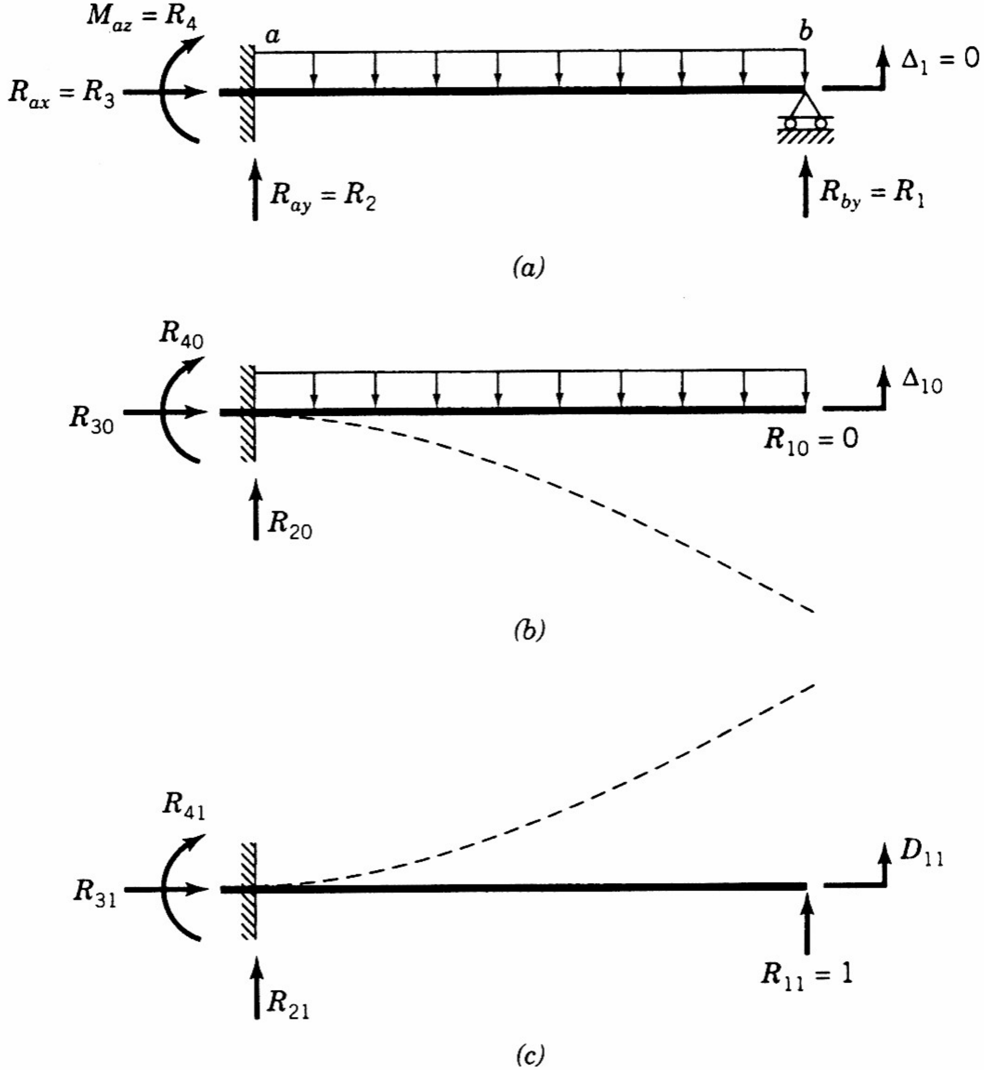


Fig. 10.1

- Or, in a more general form, if S is taken as any response quantity of interest, such as a reaction force or any internal force component in member as, then

$$S = S_0 + S_1 R_1 \qquad (10.5)$$

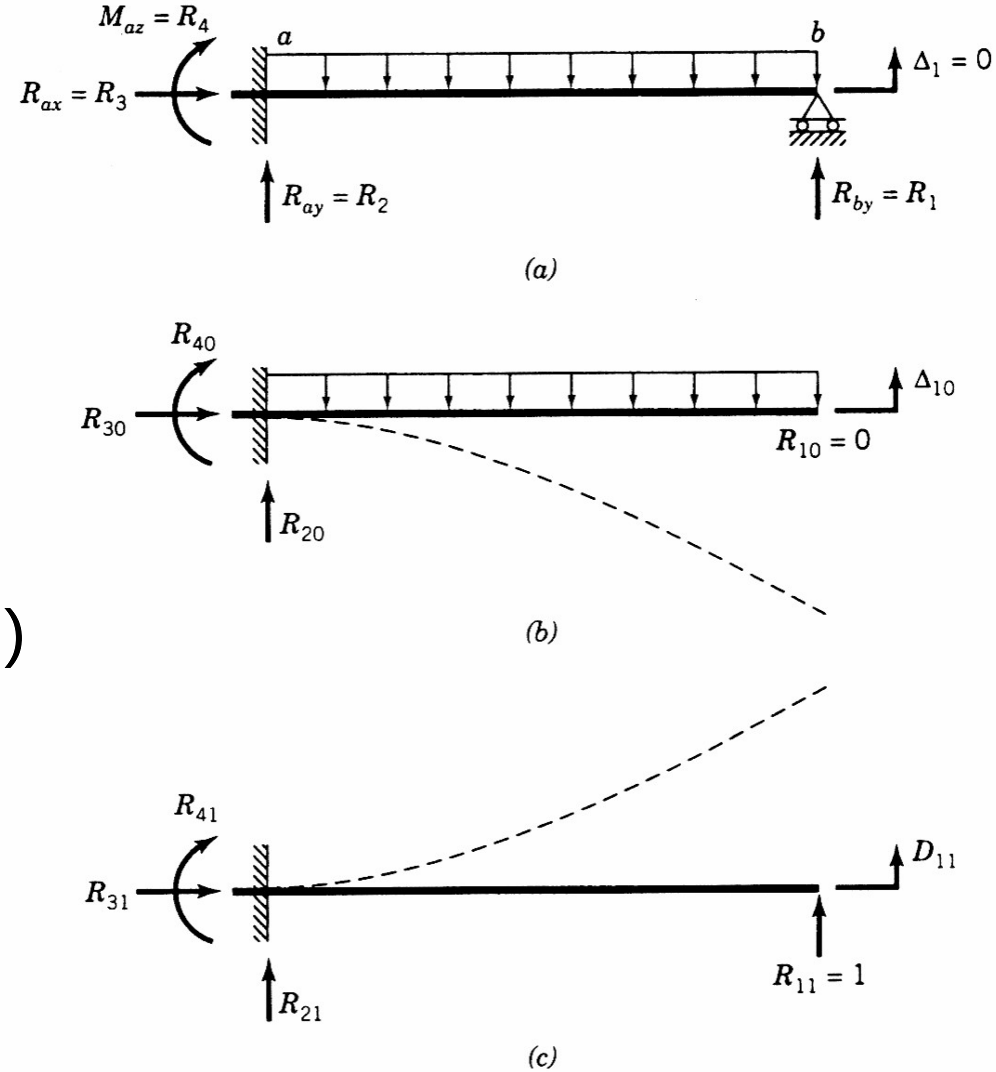


Fig. 10.1

10.3.2 Two Redundant Reactions (余剰反力が2つある場合)

● Consider next the continuous beam as shown in Fig. 10.2(a). This beam is twice statically indeterminate.

● One way to reduce the given structure to a statically determinate primary structure is to remove the two interior reactions as shown in Fig. 10.2(b).

● The primary structure can now be analyzed by the method of statics.

● The displacement Δ_{10} and Δ_{20} can be determined.

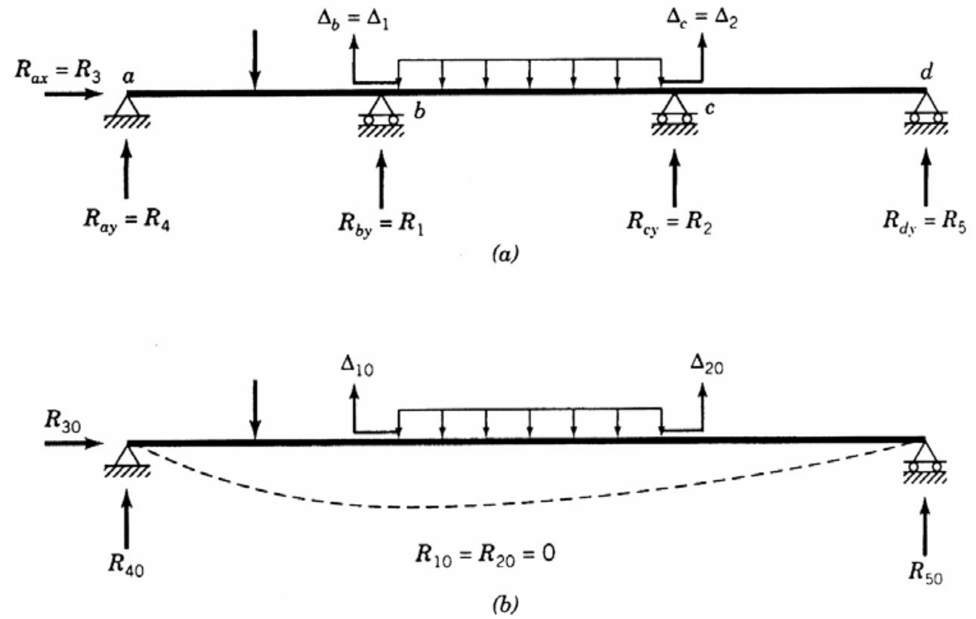


Fig. 10.2

● Because Δ_{10} and Δ_{20} are in violation of the boundary condition of the original structure, it is necessary to modify the solution of the primary structure until the displacements at these points are compatible with the prescribed boundary conditions.

● The required modification is accomplished by introducing unit values of the redundant reactions on the primary structure

and determining the effects that these individual loading cases have on the displacements where compatibility is to be restored.

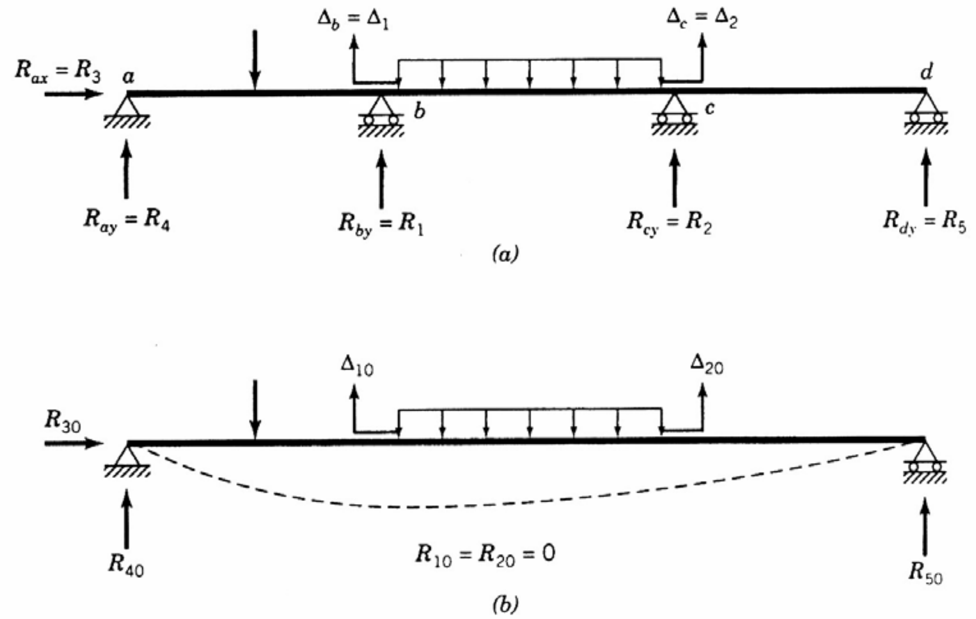


Fig. 10.2

● These unit load cases are shown in Fig. 10.2(c). They can be analyzed in accordance with static considerations.

●Each of these displacements is shown in Fig. 10.2(c) as D_{ij} .

● D_{ij} is the flexibility coefficient (**フレキシブル係数、柔性係数**) that expresses the displacement at the point in the direction of the redundant reaction

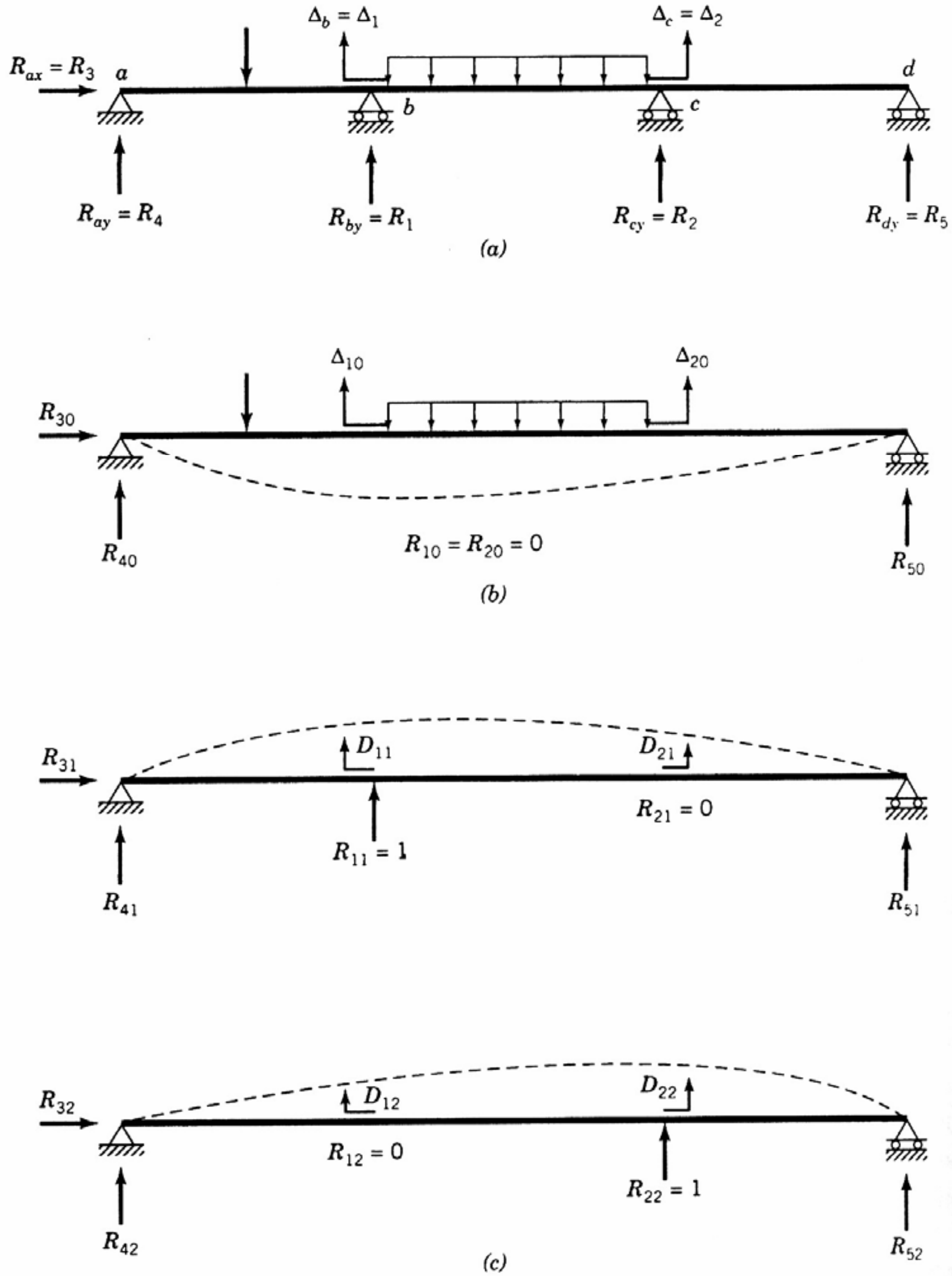


Figure 10.2 Statically indeterminate continuous beam. *(a)* Statically indeterminate beam. *(b)* Statically determinate primary structure. *(c)* Unit values of the redundant reactions.

● The total displacements at the points in the direction of redundant reactions are identified as Δ_{1R} and Δ_{2R} and are determined from the superposition (**重ね合わせ**).

$$\Delta_{1R} = D_{11}R_1 + D_{12}R_2$$
$$\Delta_{2R} = D_{21}R_1 + D_{22}R_2 \qquad (10.6)$$

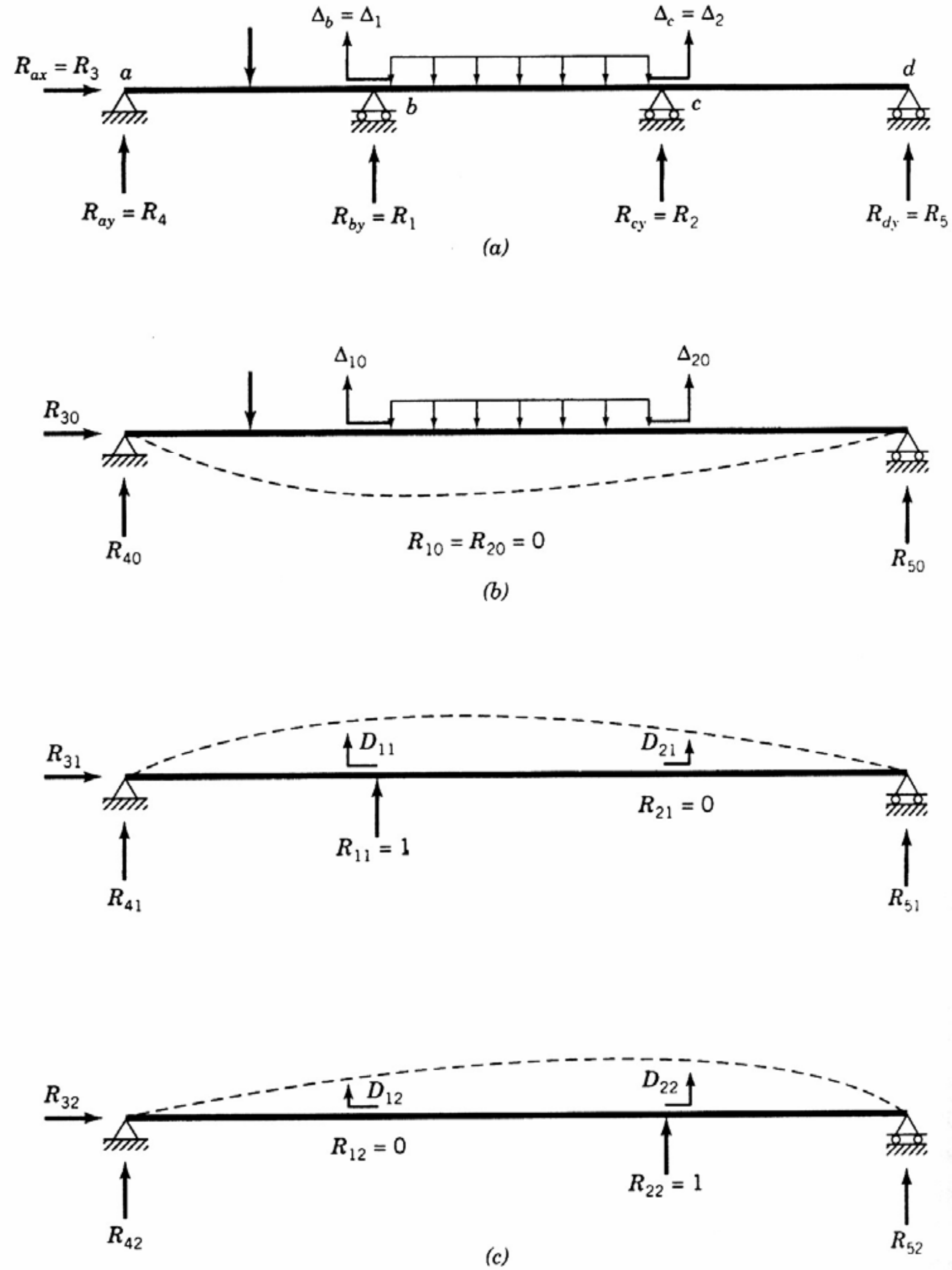


Figure 10.2 Statically indeterminate continuous beam. *(a)* Statically indeterminate beam. *(b)* Statically determinate primary structure. *(c)* Unit values of the redundant reactions.

● These displacements must be combined with Δ_{10} and Δ_{20} to yield the desired displacements of the original structures (refer to Fig. 10.2(a)).

$$\Delta_{10} + D_{11}R_1 + D_{12}R_2 = \Delta_1$$

$$\Delta_{20} + D_{21}R_1 + D_{22}R_2 = \Delta_2 \qquad (10.7)$$

● Eq. (10.7) represents compatibility equations (**変位適合条件**).

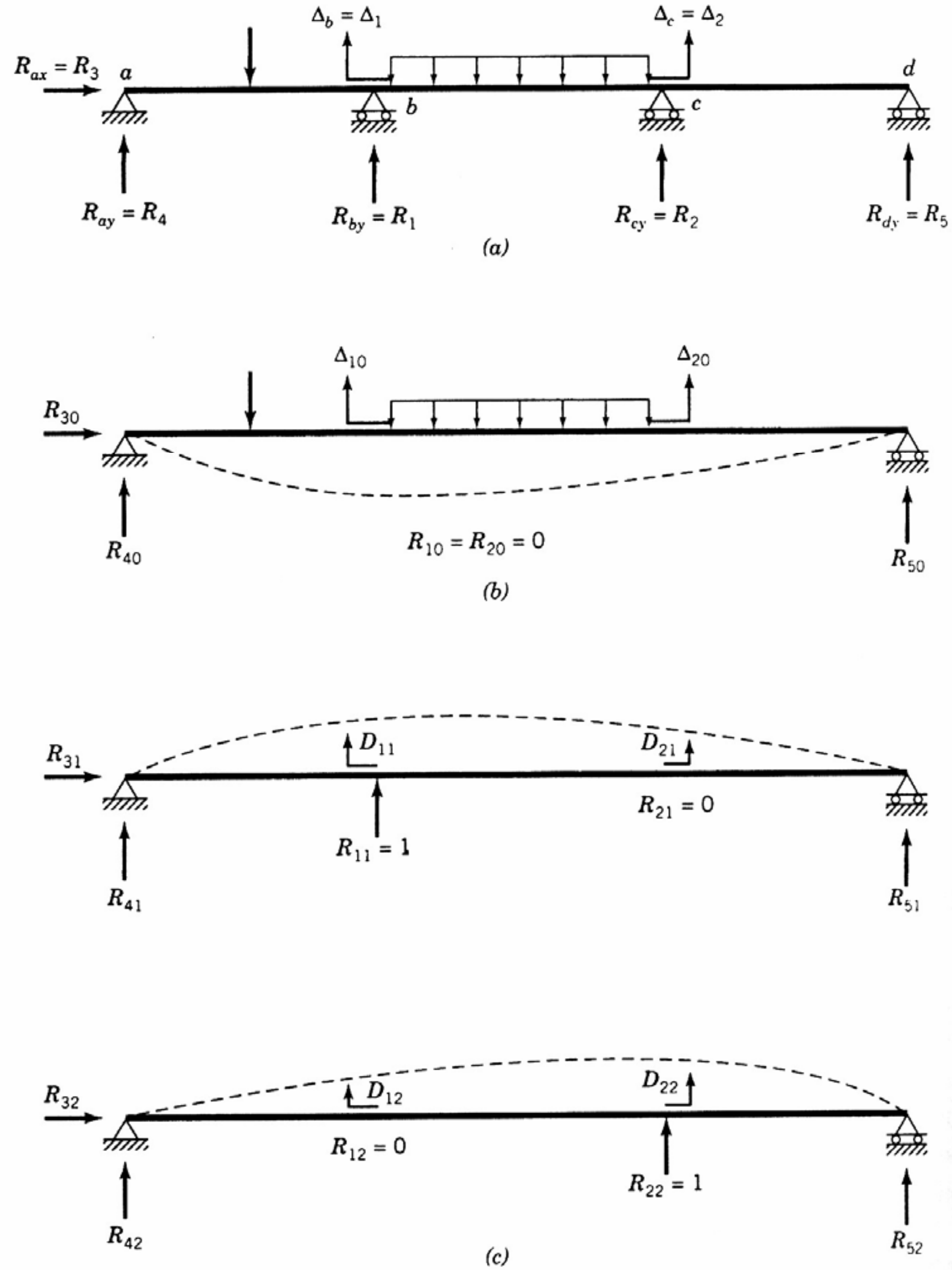


Figure 10.2 Statically indeterminate continuous beam. *(a)* Statically indeterminate beam. *(b)* Statically determinate primary structure. *(c)* Unit values of the redundant reactions.

● Eq. (10.7) can be written in matrix form (**マトリックス表示**) as

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 - \Delta_{10} \\ \Delta_2 - \Delta_{20} \end{Bmatrix} \tag{10.8}$$

$$\begin{aligned} \Delta_{10} + D_{11}R_1 + D_{12}R_2 &= \Delta_1 \\ \Delta_{20} + D_{21}R_1 + D_{22}R_2 &= \Delta_2 \end{aligned} \tag{10.7}$$

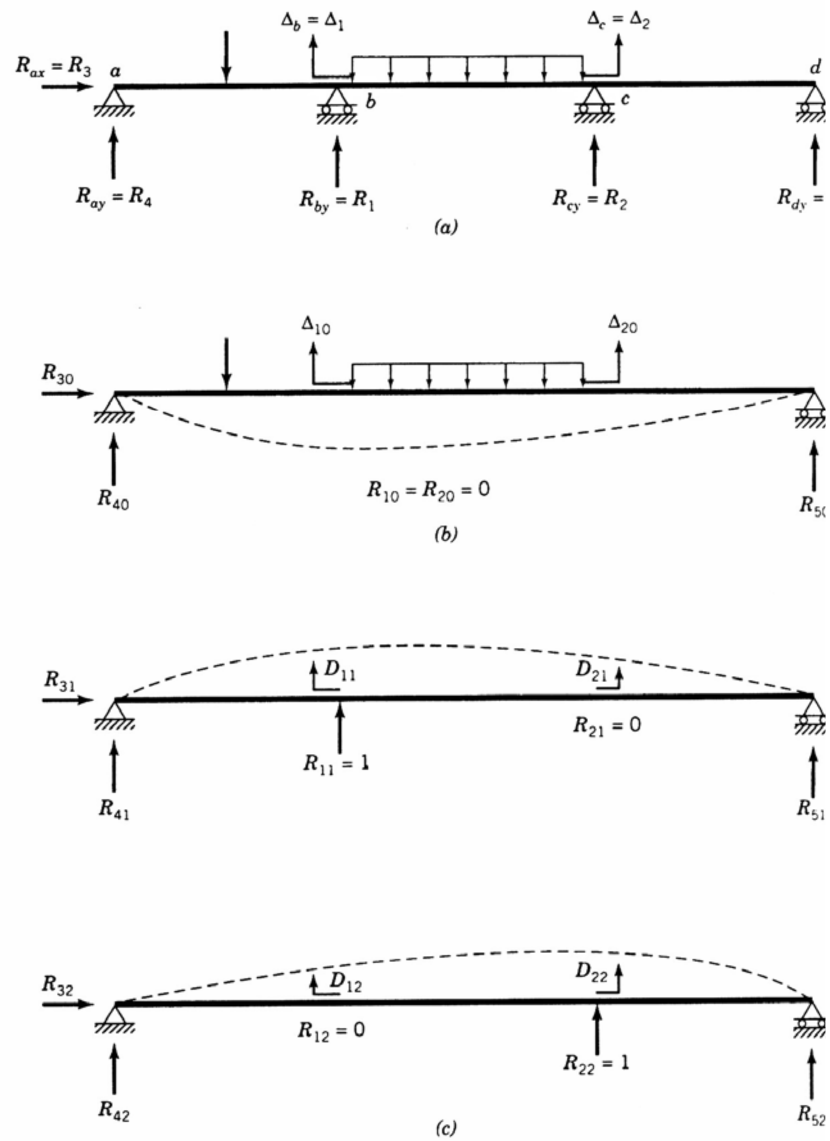


Figure 10.2 Statically indeterminate continuous beam. *(a)* Statically indeterminate bear *(b)* Statically determinate primary structure. *(c)* Unit values of the redundant reactions.

● In Eq.(10.8), the square matrix is the structural flexibility matrix (**フレキシビリティ行列、柔性行列**)

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 - \Delta_{10} \\ \Delta_2 - \Delta_{20} \end{Bmatrix} \qquad (10.8)$$

● The solution of Eq.(10.8) gives the magnitudes of redundant reactions. These reactions can be placed on the original structure, and the remaining reactions can be determined from statics.

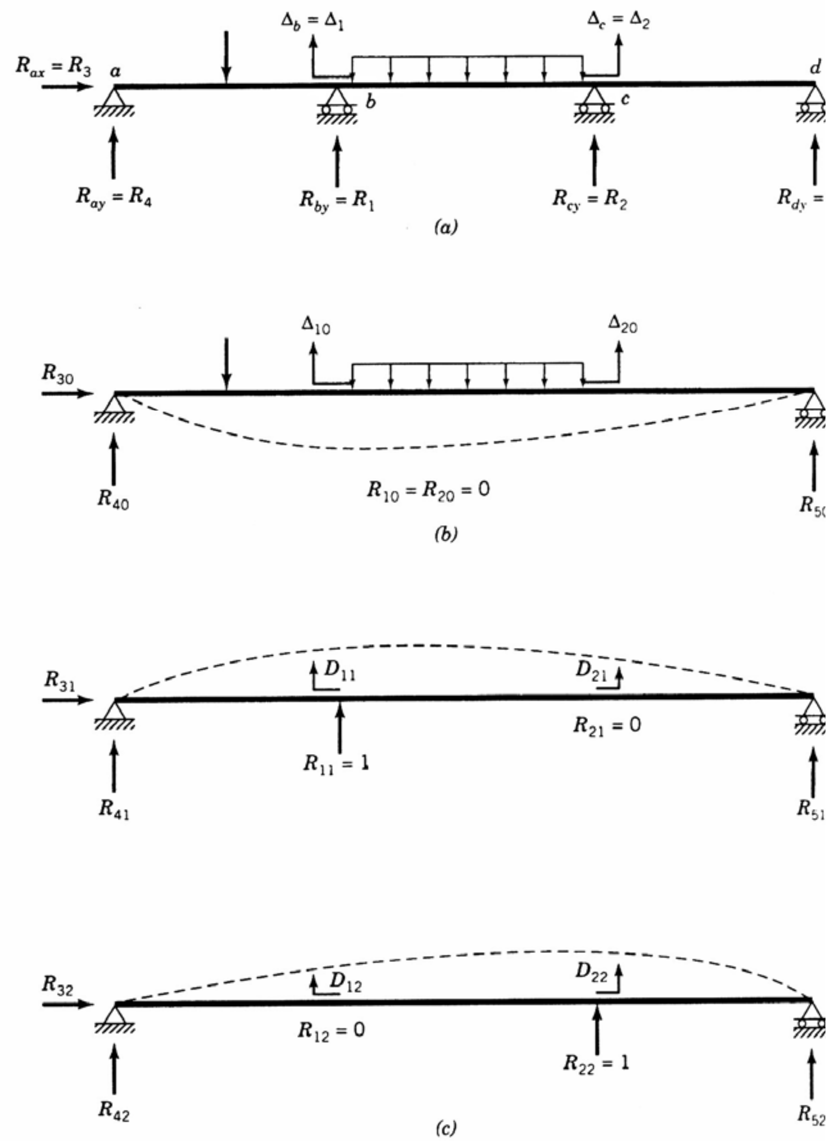


Figure 10.2 Statically indeterminate continuous beam. *(a)* Statically indeterminate bea *(b)* Statically determinate primary structure. *(c)* Unit values of the redundant reactions.

● Or, as a general procedure, the same superposition (**重ね合わせ法**) shown in Eq. (10.7) can be used for determining any other response quantities of interest（**関心のあるその他の諸量**), such as reaction, moment or shear.

●If S is taken as such a response quantity (**このような諸量の一つ**), then

$$S = S_0 + S_1 R_1 + S_2 R_2 \tag{10.9}$$

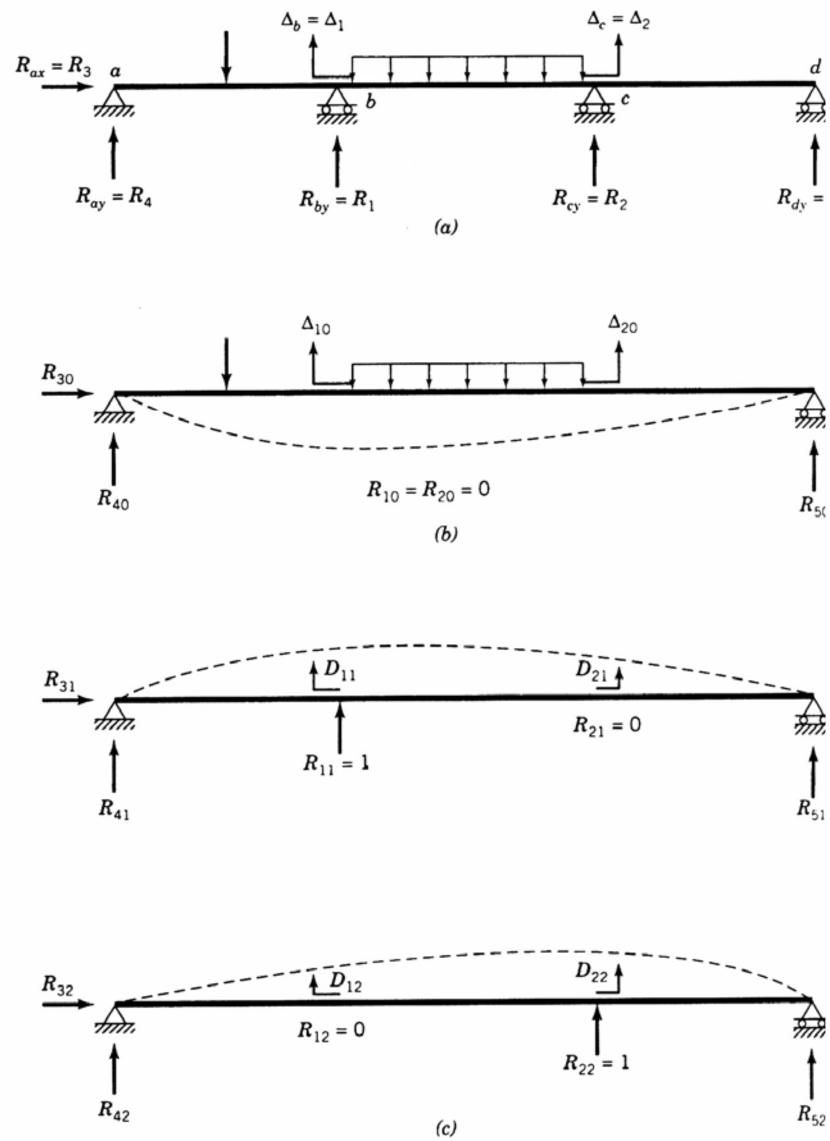


Figure 10.2 Statically indeterminate continuous beam. *(a)* Statically indeterminate bear *(b)* Statically determinate primary structure. *(c)* Unit values of the redundant reactions.

● In Eq. (10.9), S_0 is the value of S on the primary structure when the actual loading of the given structure is applied, and S_i is the value of S on the primary structure when a unit value of R_i is applied.

$$S = S_0 + S_1 R_1 + S_2 R_2 \qquad (10.9)$$

● Since superposition is valid only for linear elastic structures, the method of consistent deformations can be applied only to linear elastic structures.

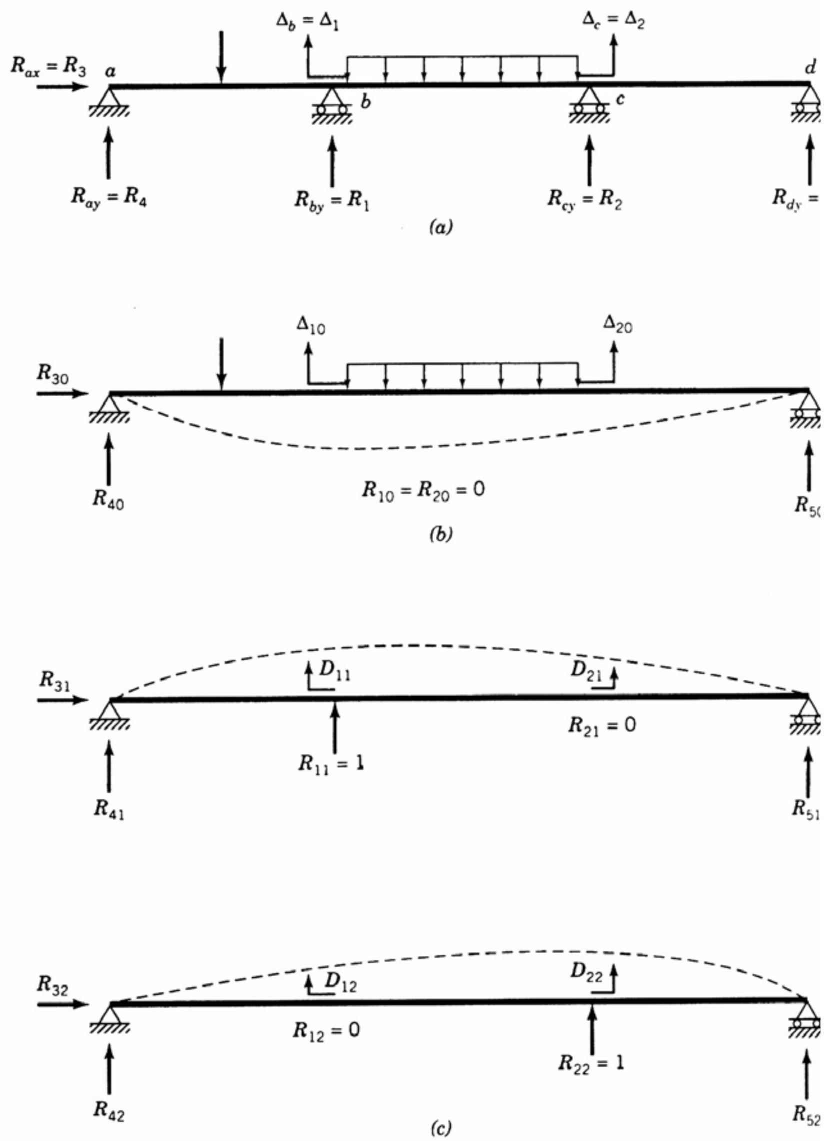


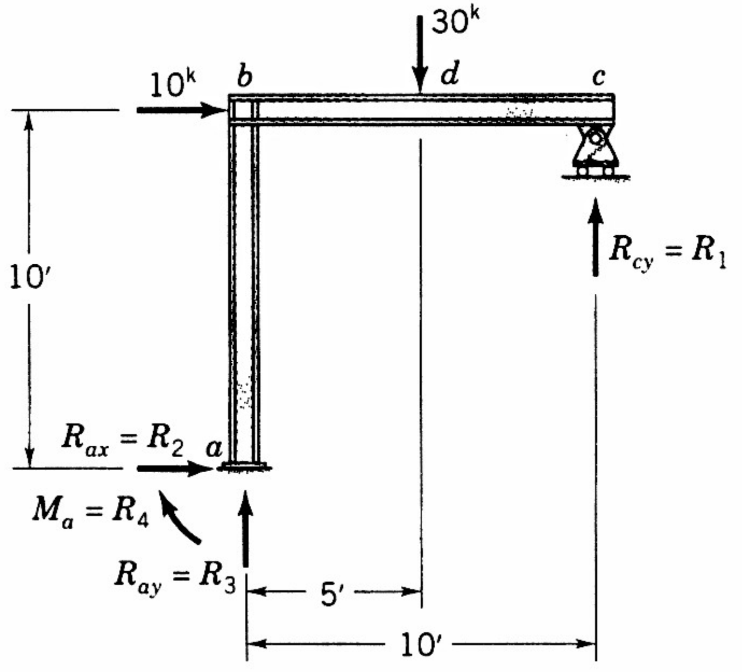
Figure 10.2 Statically indeterminate continuous beam. *(a)* Statically indeterminate bear *(b)* Statically determinate primary structure. *(c)* Unit values of the redundant reactions.

10.4 Application of the Method of Consistent Deformations

Example 10.1: Determine the reactions, and construct the moment diagram for the frame structure given. The quantity EI is the same for each member.

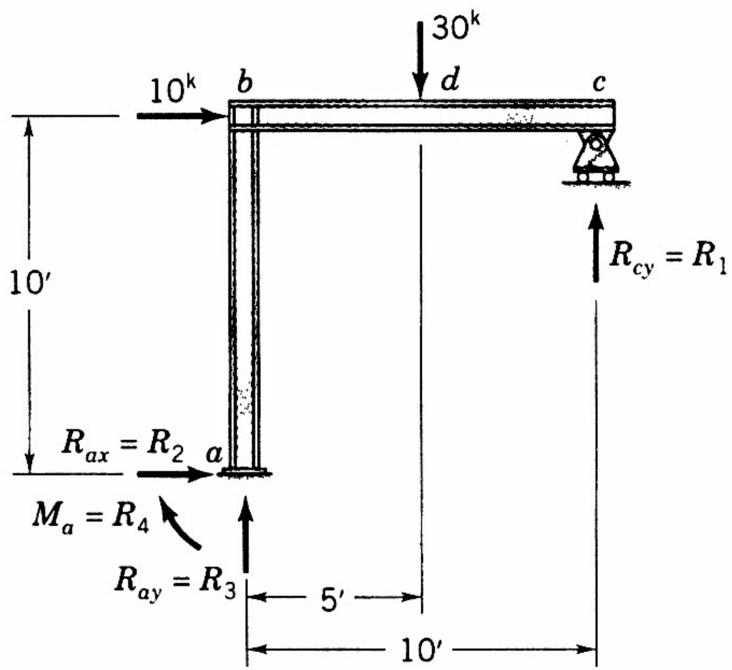
Structure classification

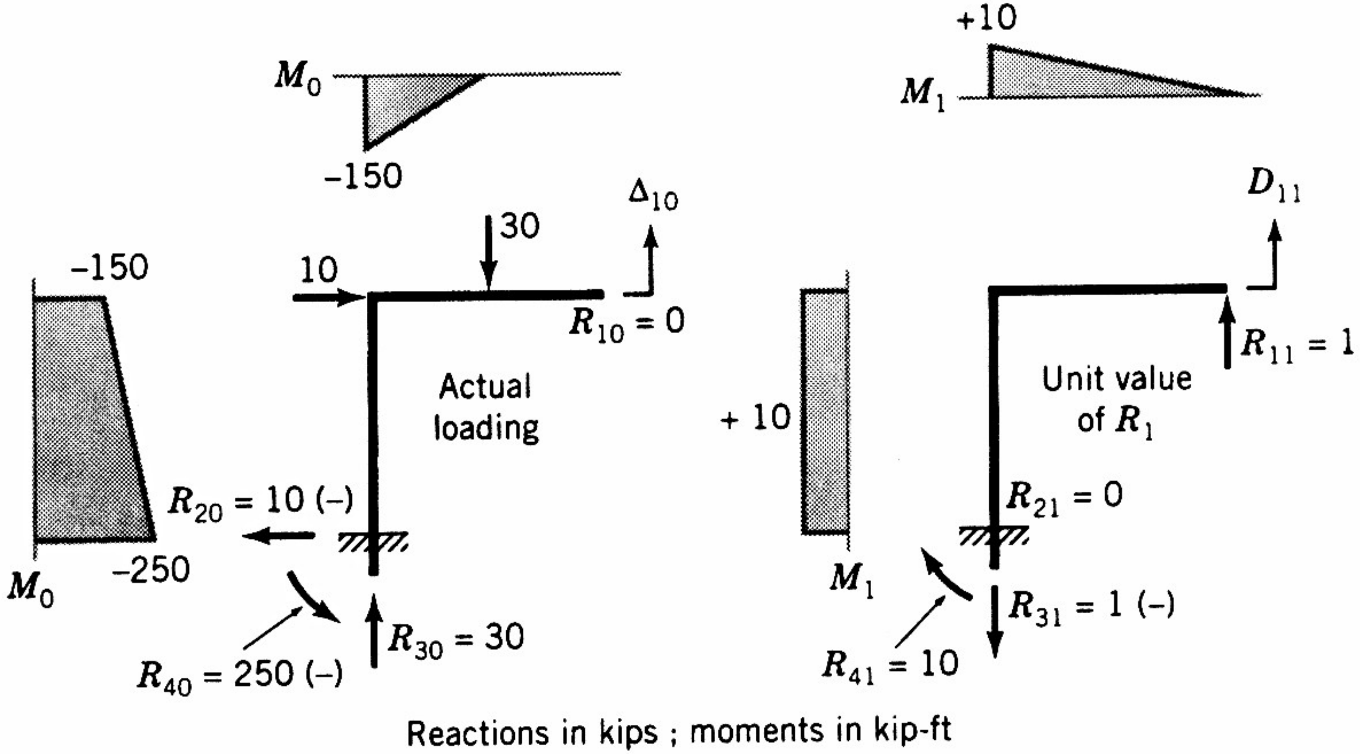
The structure is statically indeterminate to the first order.



Primary structure and loadings

- Select R_{cy} as the redundant reaction R_1, which produces a simple cantilever-type system as the primary structure.





Displacement calculation

- The moment-area method (**モーメント面積法**) is used because it is especially useful to a cantilevered-type structure.

Moment-area method (**モーメント面積法**, Refer to 8.4)

$$\theta_B^A = \int_A^B \frac{M}{EI} dx \qquad \Delta_B^A = \int_A^B \frac{M}{EI} \bar{x} dx$$

● The angle change between points A and B on the deflected structure, or the slope at B relative to A, is given by the area under the M/EI diagram between these points (First Moment-area method、**第1モーメント面積法**)

● The deflection at B on the deflected structure with respect to a line drawn tangent to point A on the structure is given by the static moment of the area under the M/EI diagram between A and B taken about an axis through point B (Second Moment-area method, **第2モーメント面積法**)

$$\theta_B^A = \int_A^B \frac{M}{EI} dx$$

(8.29)

$$\Delta_B^A = \int_A^B \frac{M}{EI} \bar{x} dx$$

(8.33)

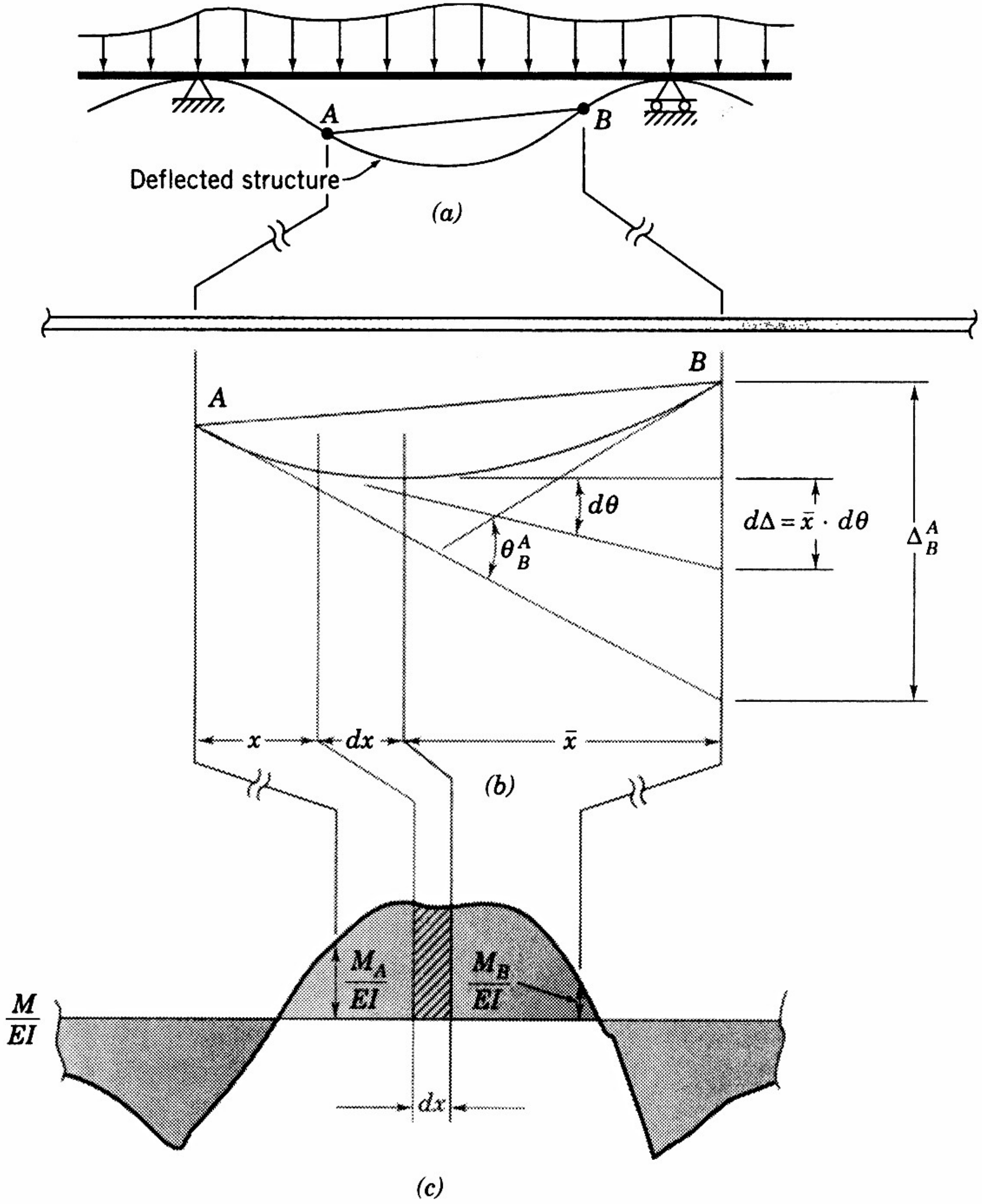


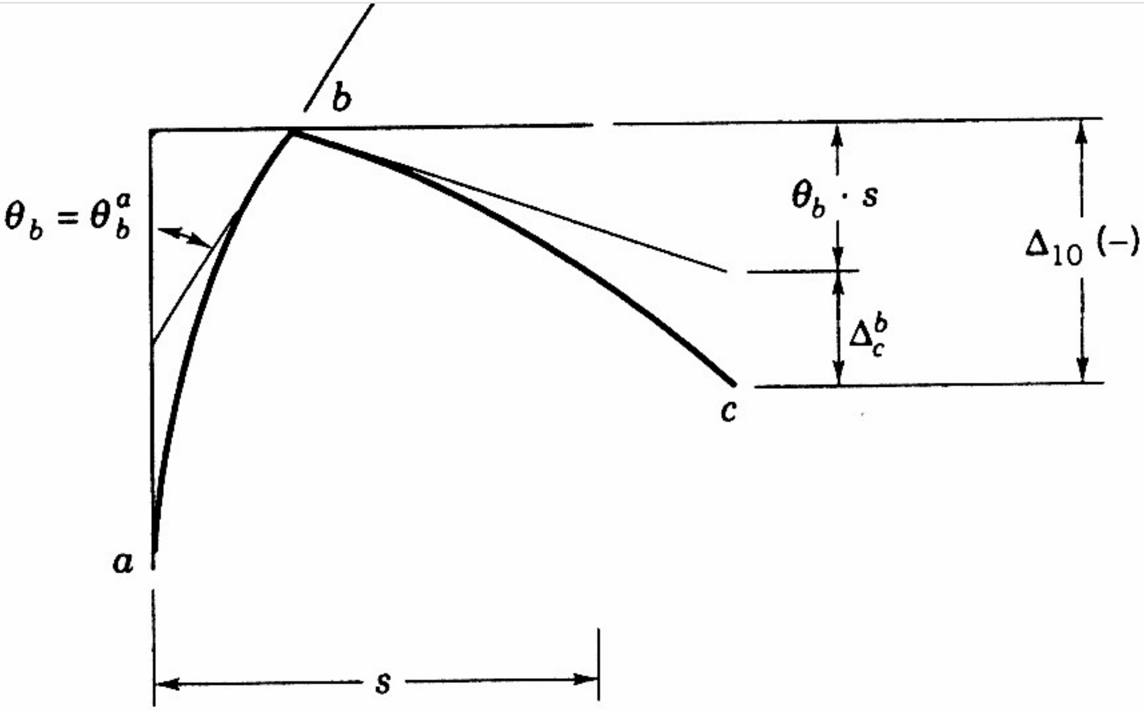
Figure 8.6 Development of moment–area theorems.

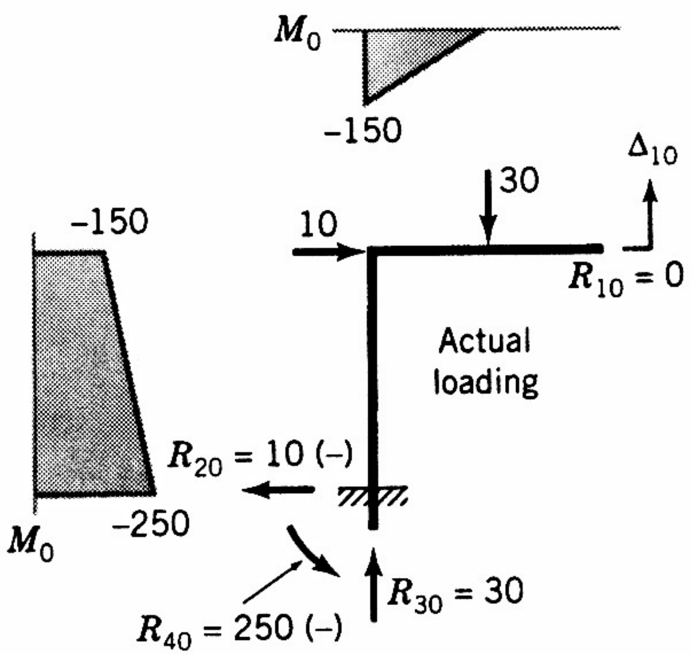
The displacement Δ_{10}

$$\theta_b = \theta_b^a = \left(\frac{250+150}{2EI}\right)\times 10 = \frac{2000}{EI}\ ft^2k$$

$$\Delta_c^b = \frac{150}{2EI}\times 5\times 8.33 = \frac{3124}{EI}\ ft^3k$$

$$\Delta_{10} = \frac{2000}{EI}\times 10 + \frac{3124}{EI} = \frac{23124}{EI}\ ft^3k$$



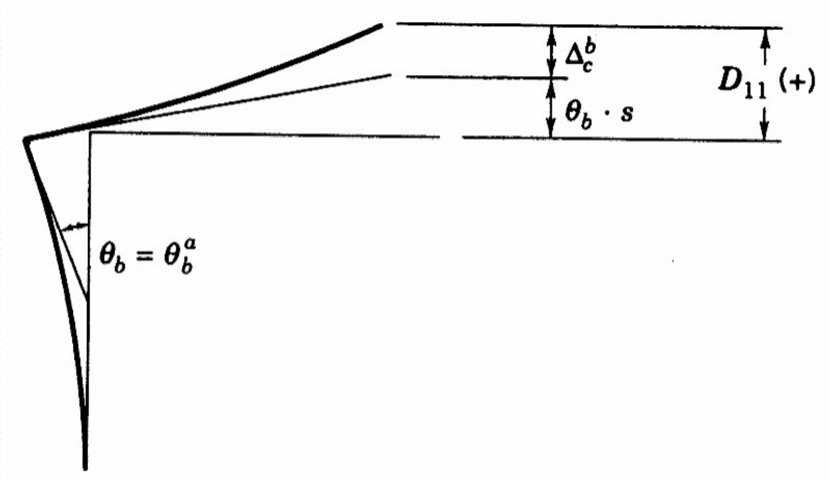


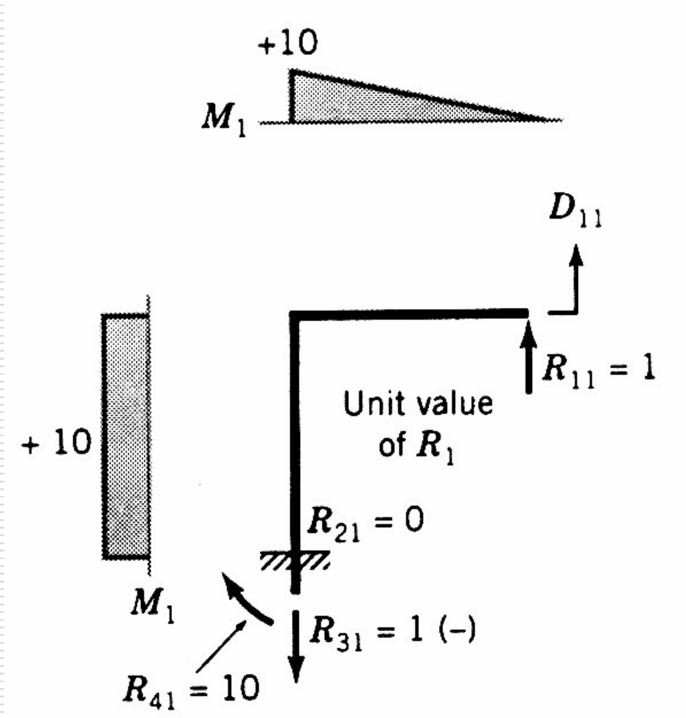
The Flexibility Coefficient D_{11}

$$\theta_b = \theta_b^a = \left(\frac{10}{EI}\right) \times 10 = \frac{100}{EI}\ ft^2k$$

$$\Delta_c^b = \frac{10}{2EI} \times 10 \times 6.67 = \frac{333.5}{EI}\ ft^3k$$

$$\Delta_{10} = \frac{100}{EI} \times 10 + \frac{333.5}{EI} = \frac{1333.5}{EI}\ ft^3k$$





Determination of Reactions

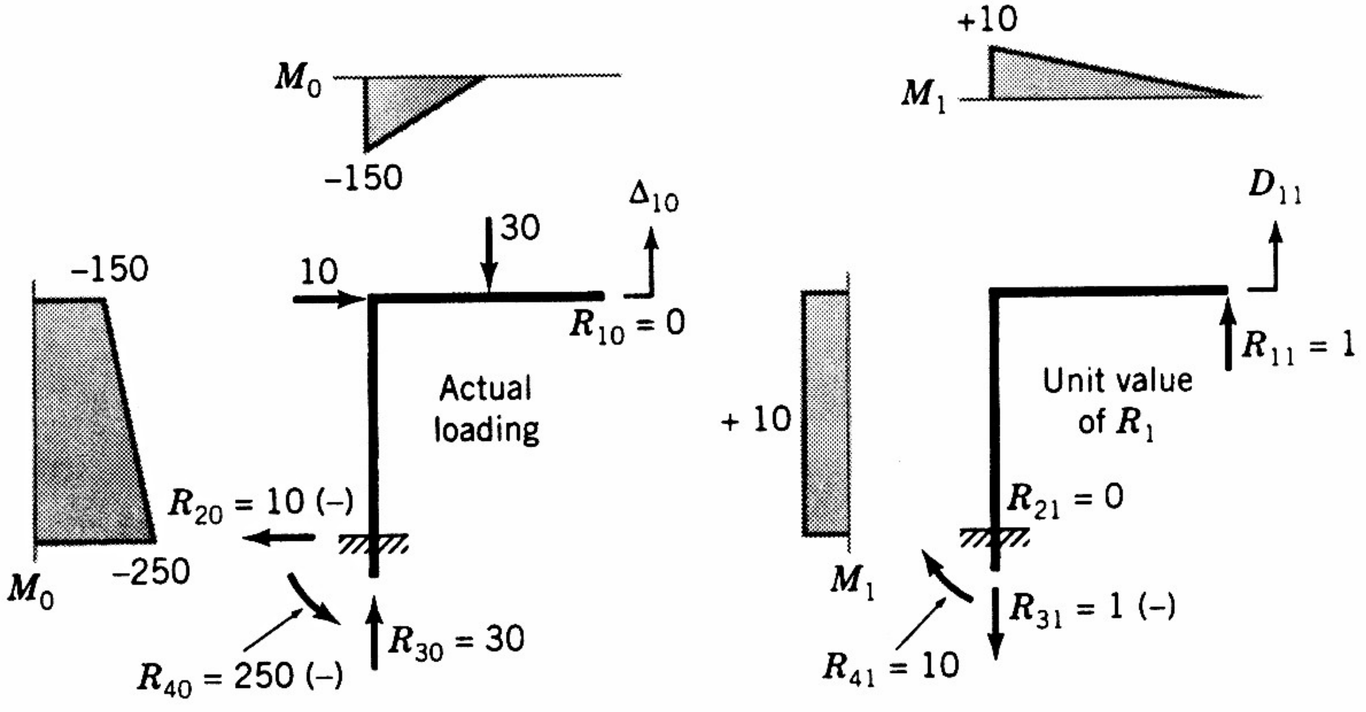
● The redundant reaction R1 is determined by imposing displacement compatibility at point c through the principle of superposition (重ね合わせの原理)

$$\Delta_{10} + D_{11} R_1 = \Delta_1 = 0 \qquad \text{(<-10.2)}$$

$$R_1 = -\frac{\Delta_{10}}{D_{11}}$$

$$= \frac{23{,}124}{1{,}333.5}$$

$$= 17.34$$



Reactions in kips ; moments in kip-ft

● The same superposition can be used to determine the remaining reactions

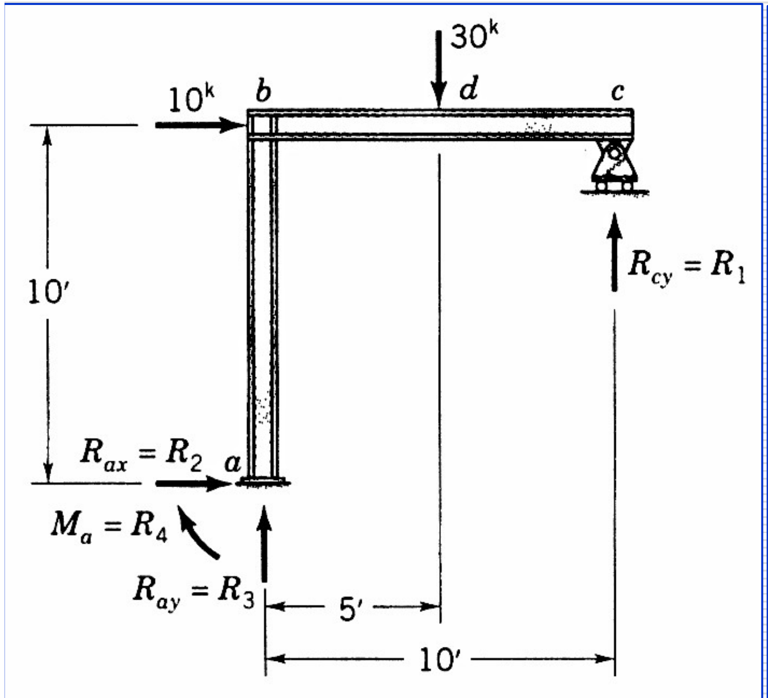
$$R_q = R_{q0} + R_{q1}R_1 \qquad (<-10.4)$$

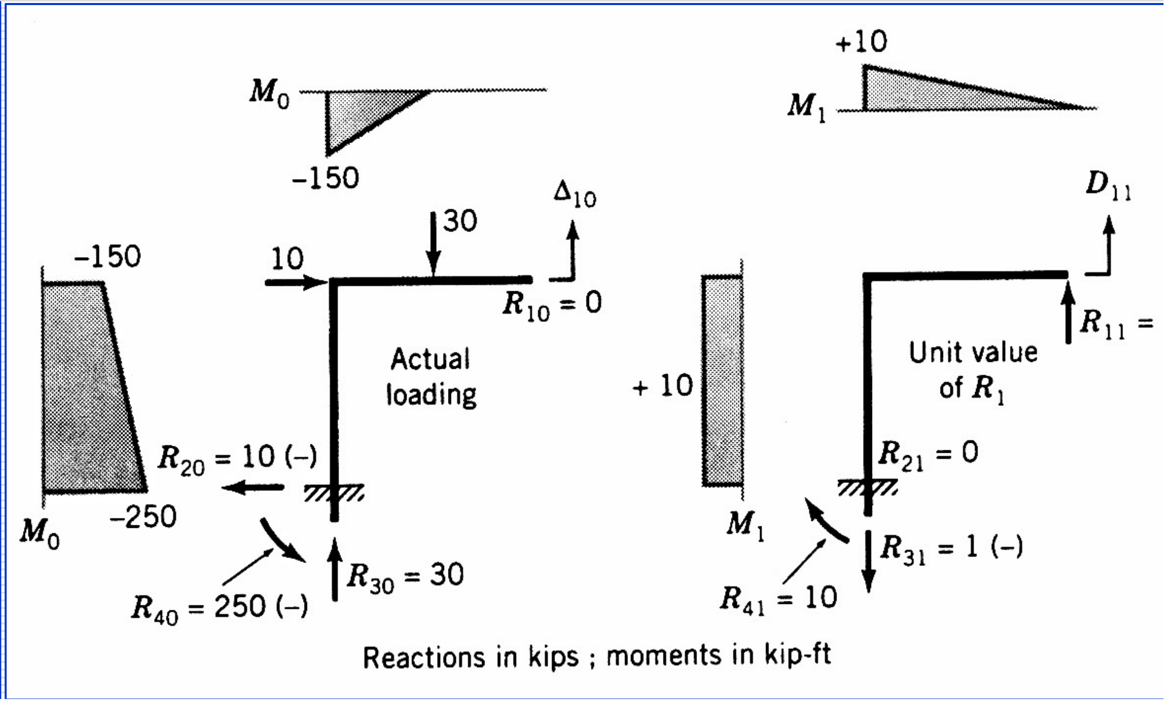
$$\Delta_{10} + D_{11}R_1 = \Delta_1 = 0 \qquad (<-10.2)$$

$$R_2 = -10 + (0 \times 17.34) = -10 kips$$

$$R_3 = 30 + (-1 \times 17.34) = 12.66 kips$$

$$R_4 = -250 + (10 \times 17.34) = -76.6 kips$$





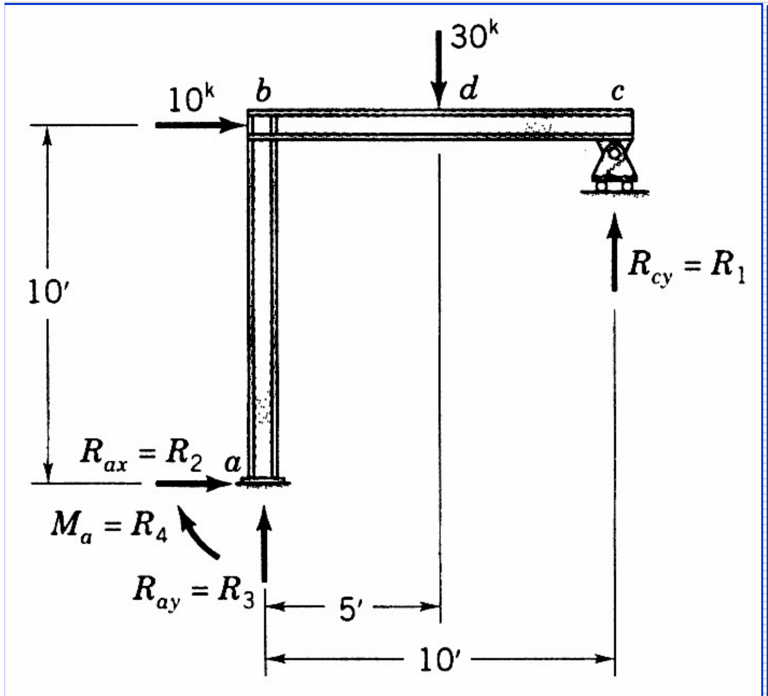
Final Moment

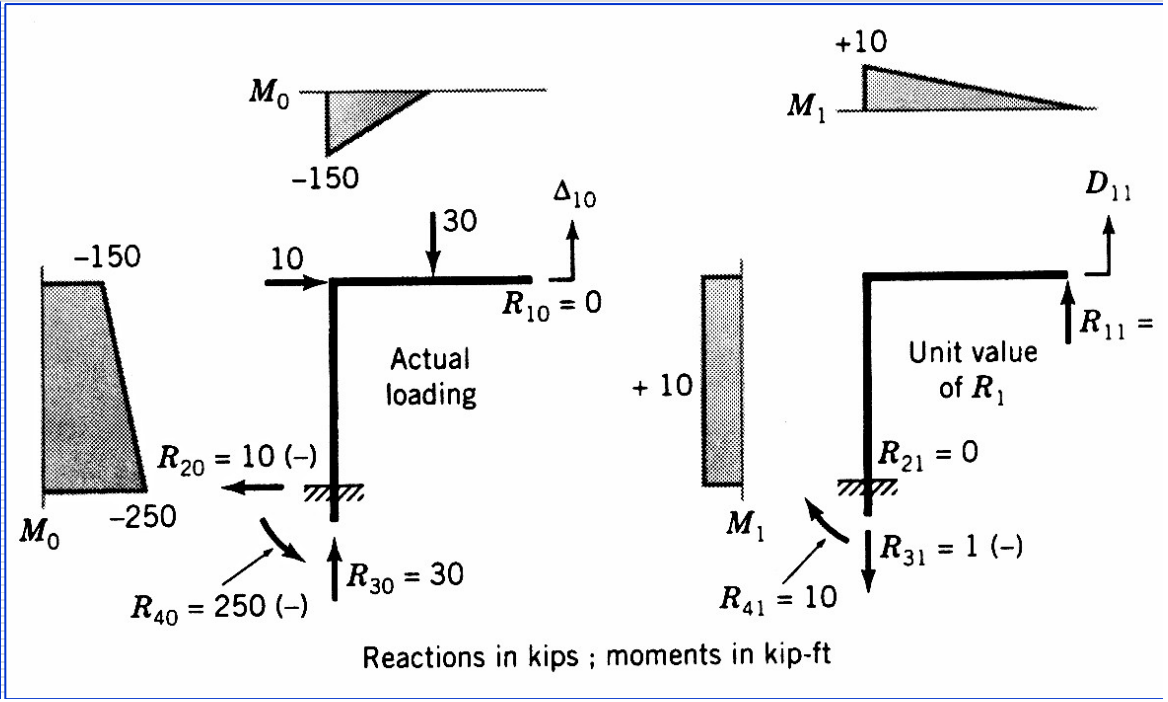
- Again, superposition provides the final moments at any point on the structure

$$M = M_0 + M_1 R_1$$

Moment for $R_1 = 1$ on primary structure

Moment for actual loading on primary structure





$$\begin{Bmatrix} M_a \\ M_b \\ M_c \\ M_d \end{Bmatrix} = \begin{Bmatrix} -250 \\ -150 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 10 \\ 10 \\ 0 \\ 5 \end{Bmatrix} \times 17.34 = \begin{Bmatrix} -76.6 \\ 23.4 \\ 0 \\ 86.7 \end{Bmatrix}$$