Structural Analysis II
構造力学第二
(2)

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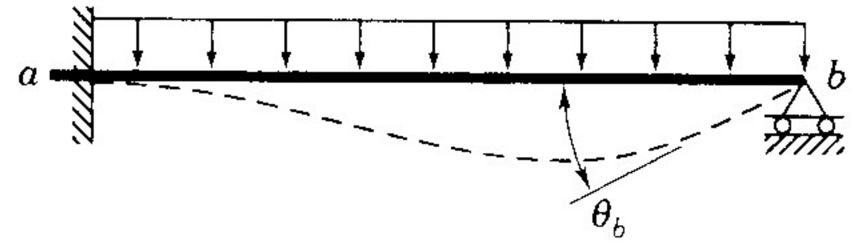
9.4 KINEMATIC INDETERMINACIES (幾何学的不静定)；REDUNDANCIES

1) What is kinematic indeterminacies?

- static indeterminacy refer to number of force quantities that must be determined in order to render the equilibrium solution complex.
- On the other hand, kinematic indeterminacy refers to the number of displacement quantities (kinematic degree of freedom) that are necessary to define the deformation response of the structure.

2) Example of kinematic indeterminacy



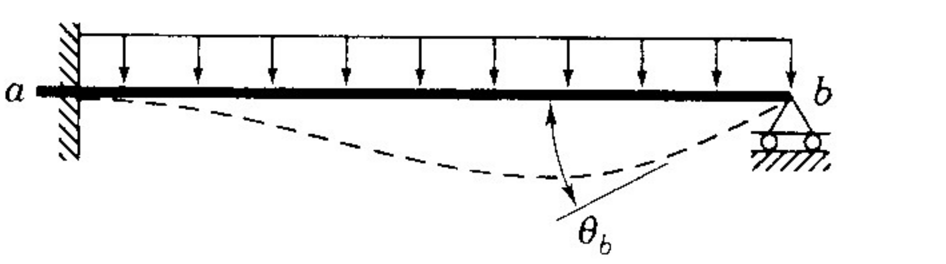
(a) Kinematically indeterminate structure with redundant displacement

Fig. 9.5 Selection of redundant displacement

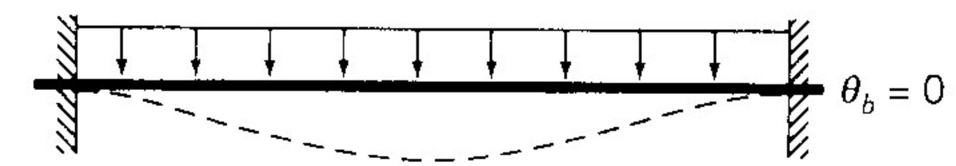
- Since the structure is fixed at point a and vertically restrained at point b and the axial deformation is zero, there is only one kinematic degree of freedom, θ_b
- We identify the rotation θ_b as a kinematic redundant.

● If the rotation θ_b is removed from the structure ($\theta_b = 0$), the resulting primary structure is called kinematically determinate (**幾何学的静定**).

● Fig. 9.5(b) shows the kinematically determinate primary structure.



(a) Kinematically indeterminate structure with redundant displacement



(b) Kinematically determinate primary structure

Fig. 9.5 Selection of redundant displacement

9.5 ALTERNATIVE FORM OF ANALYSIS

- Consider again the structure of Fig. 9.1 as shown in Fig. 9.6(a).

- An end moment M_{bz} has been added which must be eventually be set equal to zero.

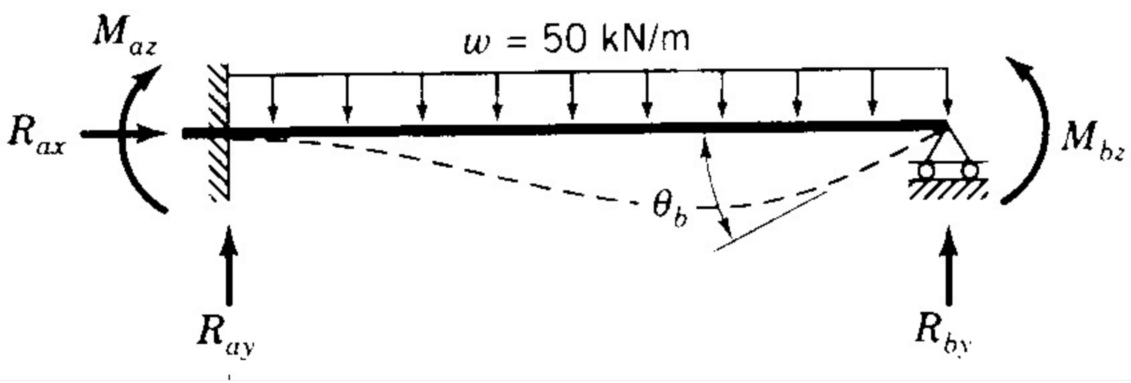


Fig. 9.6(a) Loading & deformation of a statically indeterminate beam

●Next, the statically determinate primary structure of Fig. 6.3(c) is separately loaded with w, R_{by} and M_{bz} as shown in Fig. 9.6(b)

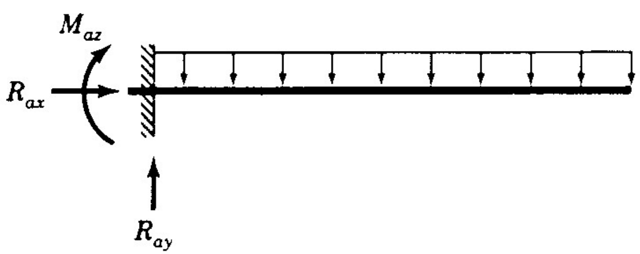


Fig. 9.3(c) Stable and statically determinate primary structure

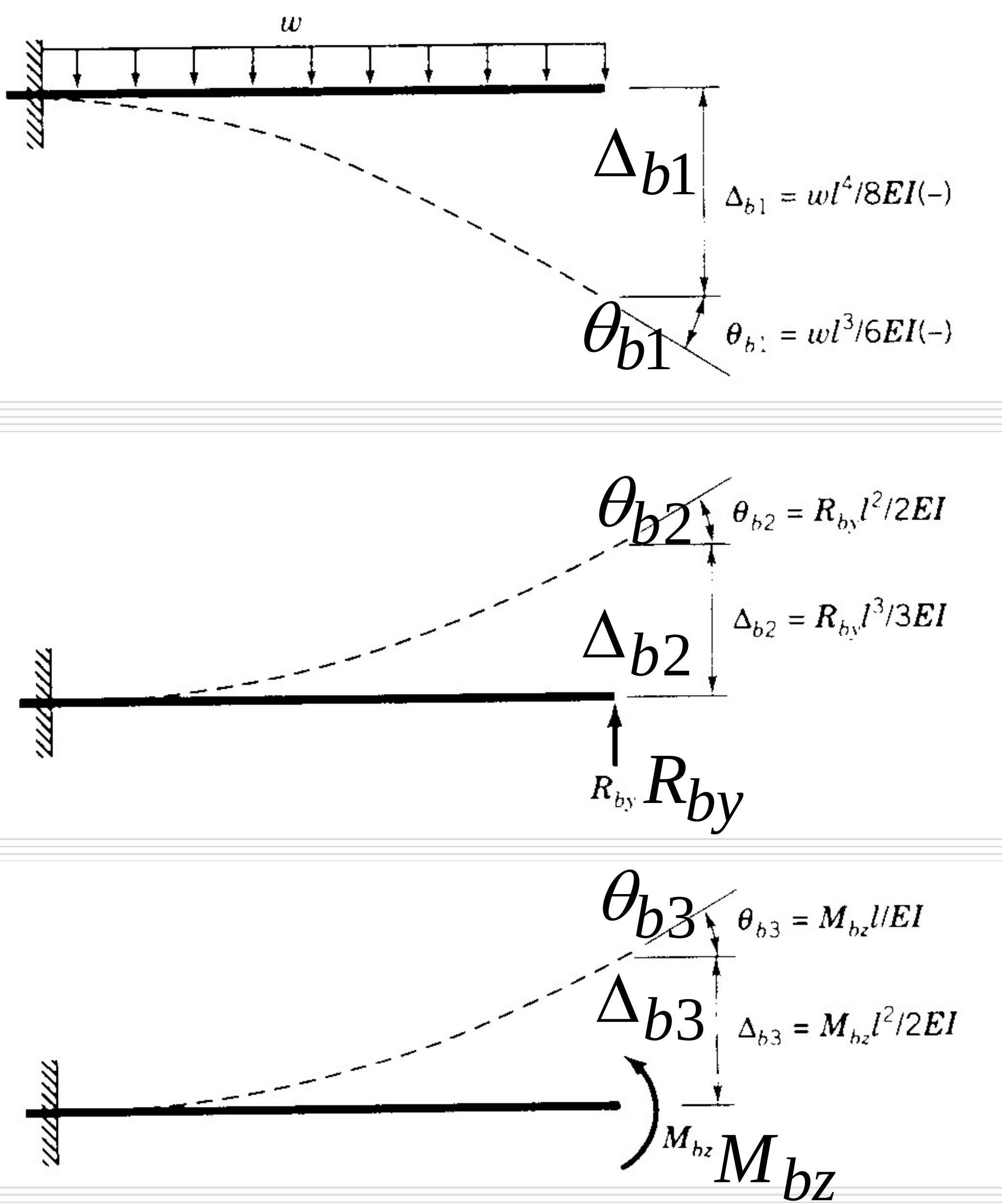


Fig. 9.6 (b)

●The separate solutions of Fig. 9.6(b) must be superimposed to obtain the correct boundary conditions for the given structure of Fig. 9.6(a) as

$$\Delta_{b1} + \Delta_{b2} + \Delta_{b3} = \Delta_b = 0$$

$$\theta_{b1} + \theta_{b2} + \theta_{b3} = \theta_b \qquad (9.6)$$

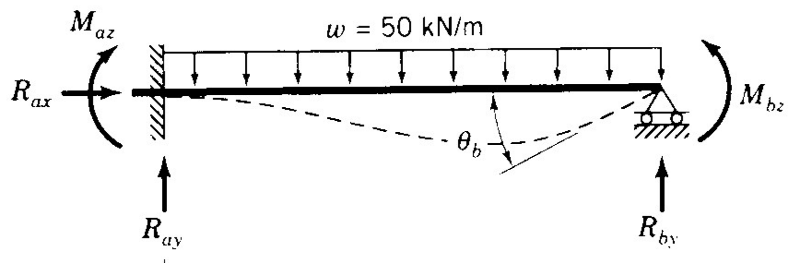


Fig. 9.6(a)

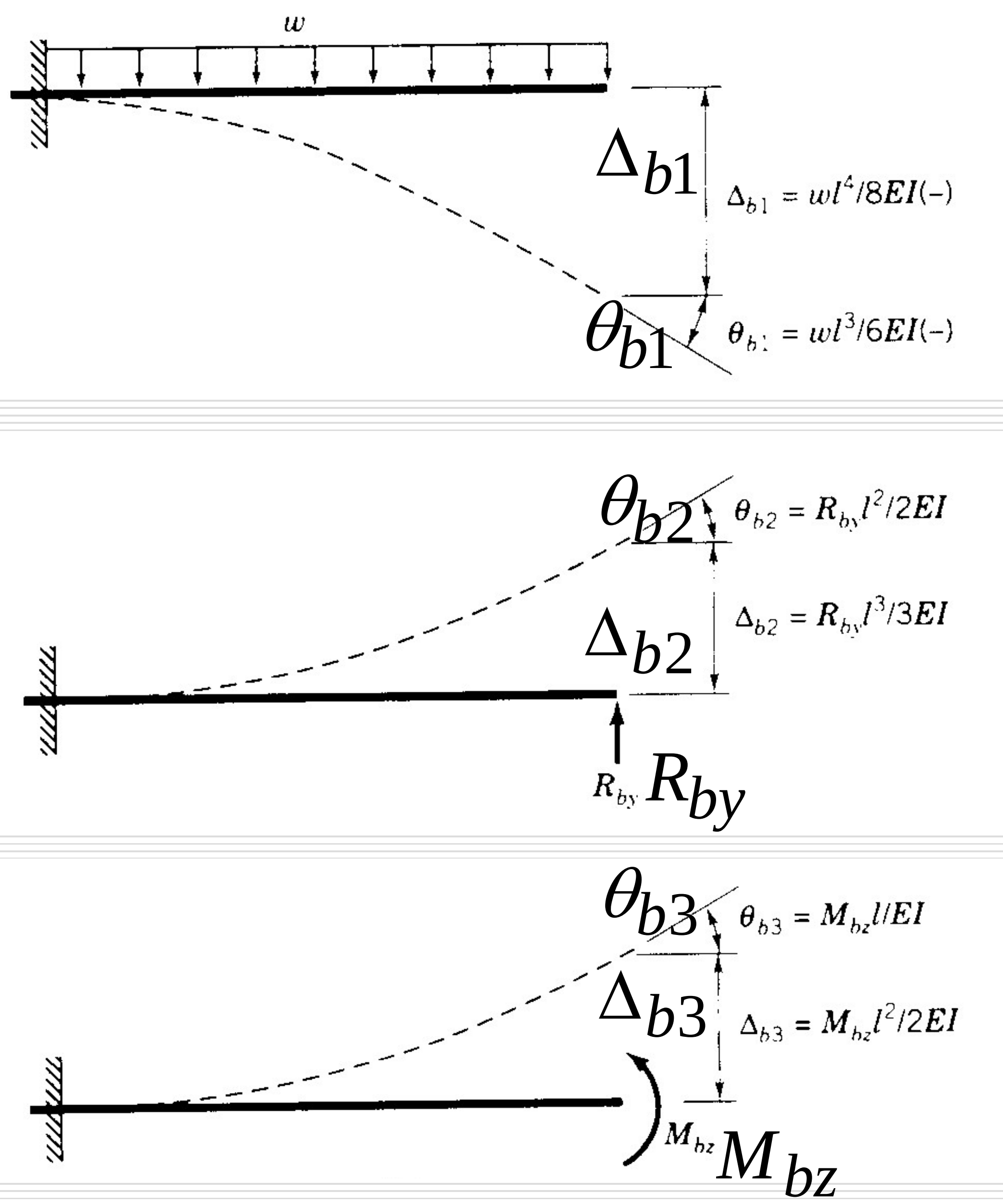


Fig. 9.6 (b)

- Substitution of the displacement quantities into Eq. (9.6) yields

$$\frac{R_{by}l^3}{3EI}+\frac{M_{bz}l^2}{2EI}=\frac{wl^4}{8EI}$$

$$\frac{R_{by}l^2}{2EI}+\frac{M_{bz}l}{EI}=\frac{wl^3}{6EI}+\theta_b$$

(9.7)

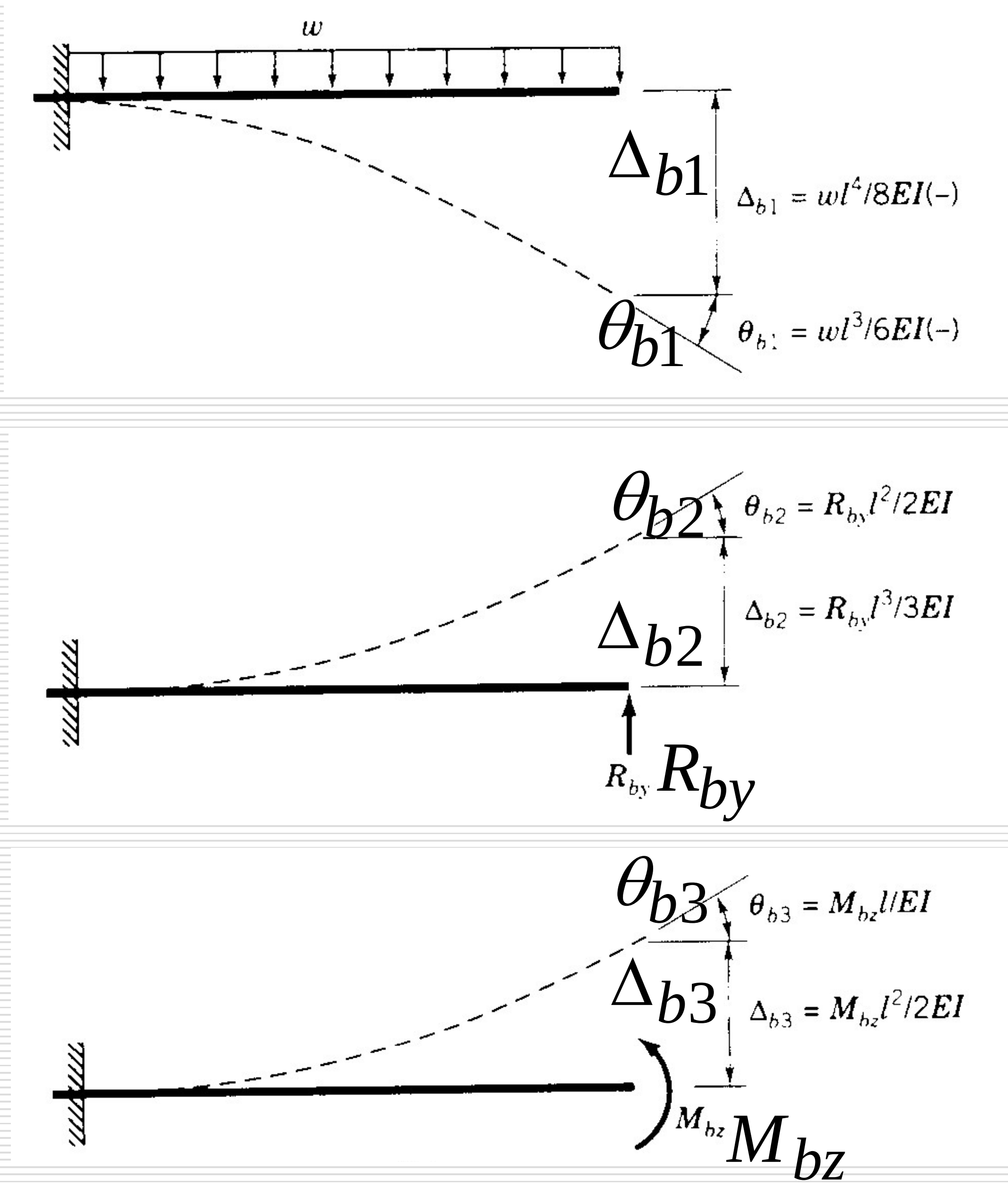


Fig. 9.6 (b)

●Static equilibrium can be written

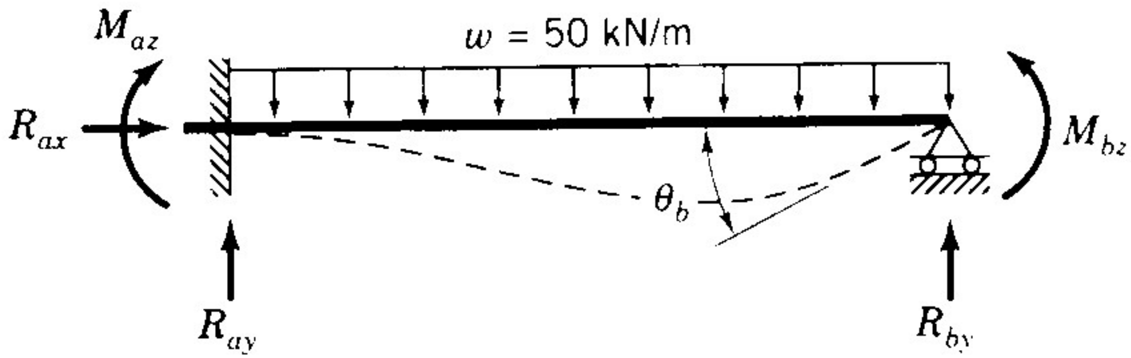


Fig. 9.6(a) Loading & deformation of a statically indeterminate beam

$$R_{ay} + R_{by} = wl$$

$$M_{az} + R_{ay}l - wl\frac{l}{2} - M_{bz} = 0$$

$$\boxed{\begin{aligned} &\frac{R_{by}l^3}{3EI} + \frac{M_{bz}l^2}{2EI} = \frac{wl^4}{8EI} \\ &\frac{R_{by}l^2}{2EI} + \frac{M_{bz}l}{EI} = \frac{wl^3}{6EI} + \theta_b \qquad (9.7) \end{aligned}}$$

●The static equilibrium and Eq. (9.7) can be solved as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ l & 1 & 0 & -1 \\ 0 & 0 & l^3/3EI & l^2/2EI \\ 0 & 0 & l^2/2EI & l/EI \end{bmatrix} \begin{Bmatrix} R_{ay} \\ M_{az} \\ R_{by} \\ M_{bz} \end{Bmatrix} = \begin{Bmatrix} wl \\ wl^2/2 \\ wl^4/8EI \\ wl^3/6EI + \theta_b \end{Bmatrix}$$

$$\boxed{\begin{aligned} & R_{ay} + R_{by} = wl \\ & M_{az} + R_{ay} l - wl\frac{l}{2} - M_{bz} = 0 \\ & \frac{R_{by} l^3}{3EI} + \frac{M_{bz} l^2}{2EI} = \frac{wl^4}{8EI} \\ & \frac{R_{by} l^2}{2EI} + \frac{M_{bz} l}{EI} = \frac{wl^3}{6EI} + \theta_b \qquad (9.7) \end{aligned}}$$

●Solving the equation shown in the previous page, we have

$$R_{ay} = \frac{wl}{2} + \frac{3k_b}{2l}\theta_b \qquad R_{by} = \frac{wl}{2} - \frac{3k_b}{2l}\theta_b$$

$$M_{az} = -\frac{wl^2}{12} - \frac{k_b}{2}\theta_b \qquad M_{bz} = -\frac{wl^2}{12} + k_b\theta_b \tag{9.8}$$

where,

$$k_b = \frac{4EI}{l}$$

●Eq. (9.8) expresses all of the response quantities in terms of the single kinematic degree of freedom θ_b

●Solutions of the kinemetically determinate primary structure can be obtained from Eq. (9.8) by substituting $\theta_b = 0$

$$R_{ay1} = \frac{wl}{2} \quad R_{by1} = \frac{wl}{2} \quad M_{az1} = -\frac{wl^2}{12} \quad M_{bz1} = -\frac{wl^2}{12} \tag{9.8}$$

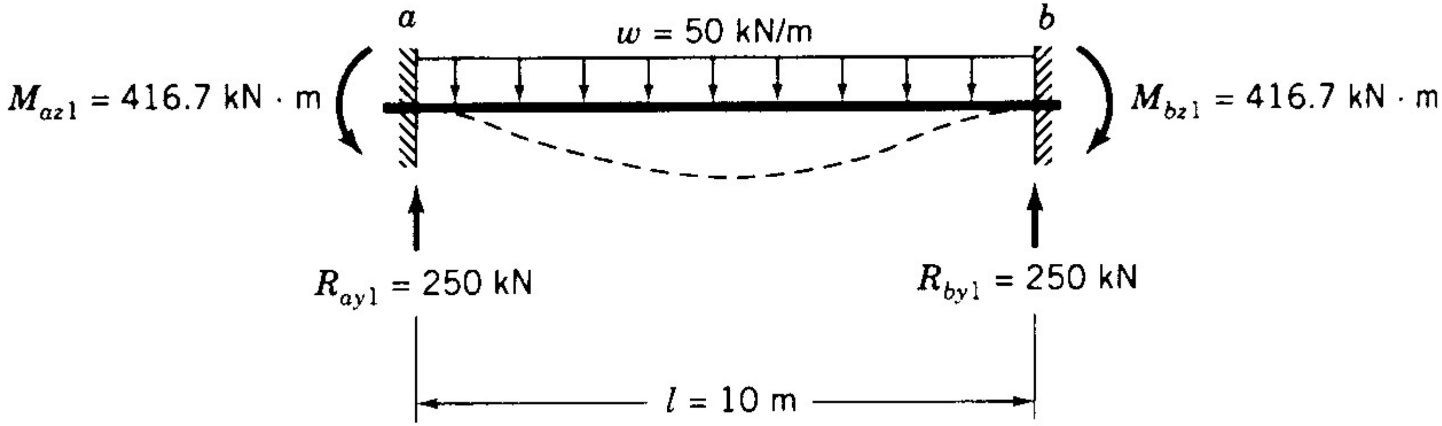
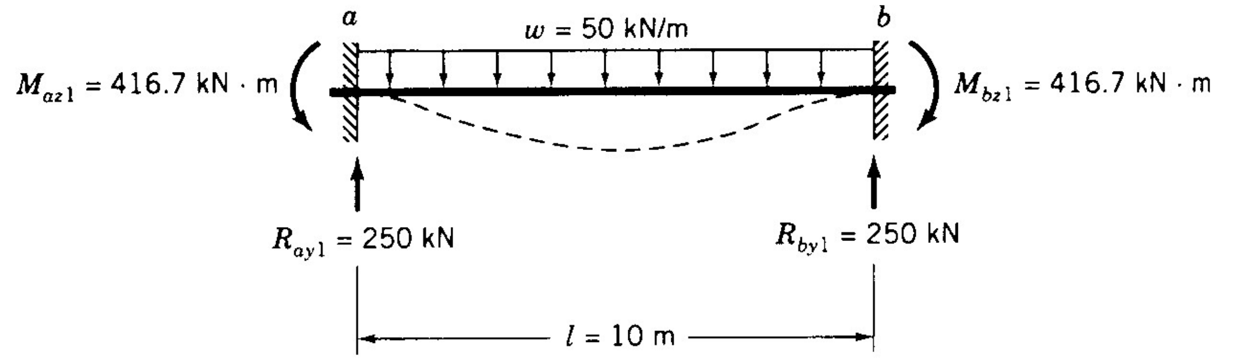


Fig. 9.8 (a) Primary structure subjected to given loading

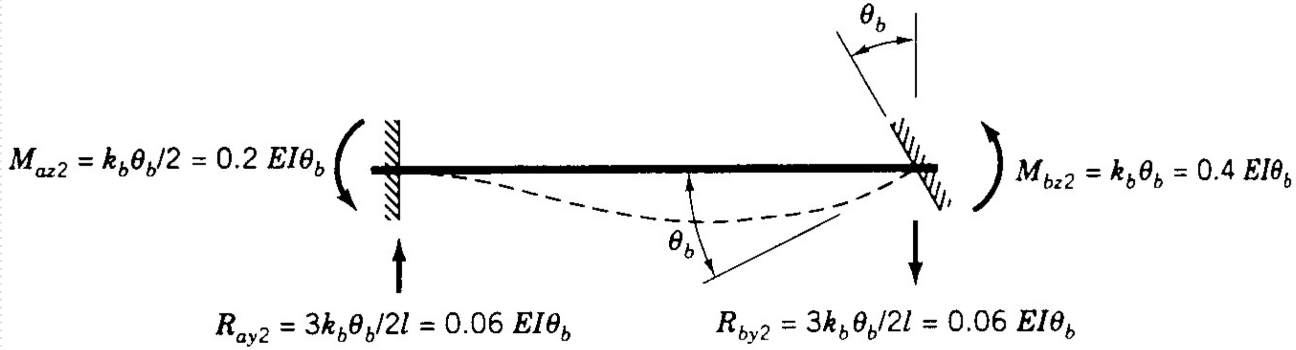
$$R_{ay} = \frac{wl}{2} + \frac{3k_b}{2l}\theta_b \qquad R_{by} = \frac{wl}{2} - \frac{3k_b}{2l}\theta_b$$

$$M_{az} = -\frac{wl^2}{12} - \frac{k_b}{2}\theta_b \qquad M_{bz} = -\frac{wl^2}{12} + k_b\theta_b \tag{9.8}$$

- Although this solution is a valid equilibrium solution, it violates the required force boundary condition on the moment at point b.
- To remedy this problem, the primary structure is subjected to the redundant displacement θ_b



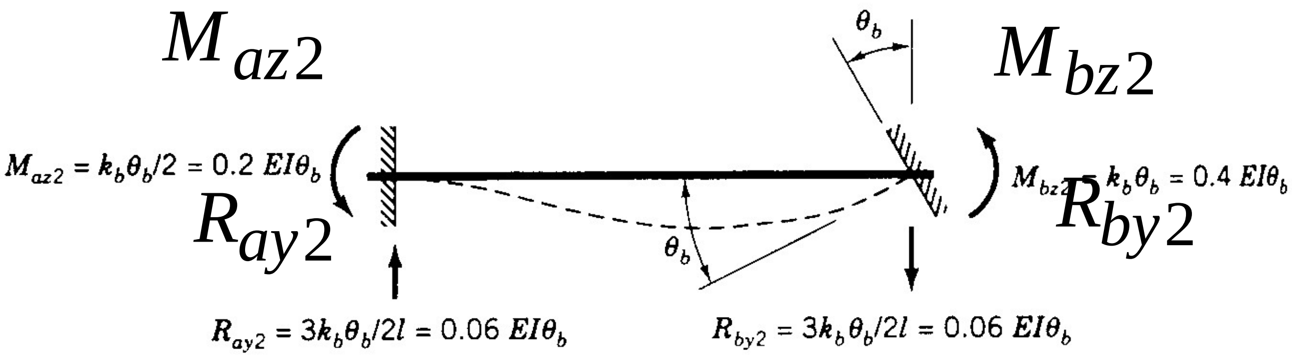
(a) Primary structure subjected to given load



(b) Primary structure subjected to redundant rotation

Fig. 9.8 Kinematic primary structure

●Assuming that w=0 in Eq. (9.8), and based on the associated end moments and reactions take on the values as



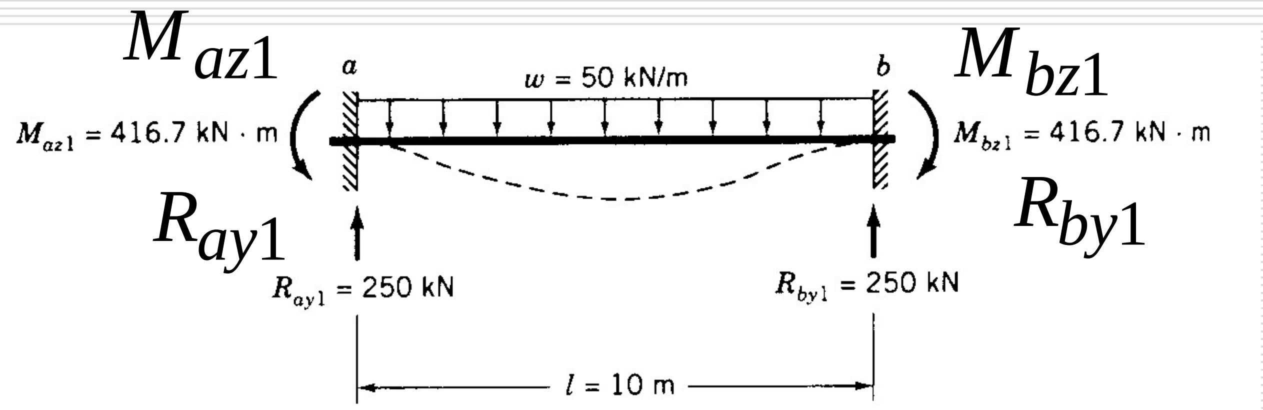
(b) Primary structure subjected to redundant rotation

$$R_{ay2} = \frac{3k_b}{2l}\theta_b = 0.06EI\theta_b \qquad R_{by2} = \frac{3k_b}{2l}\theta_b = 0.06EI\theta_b$$

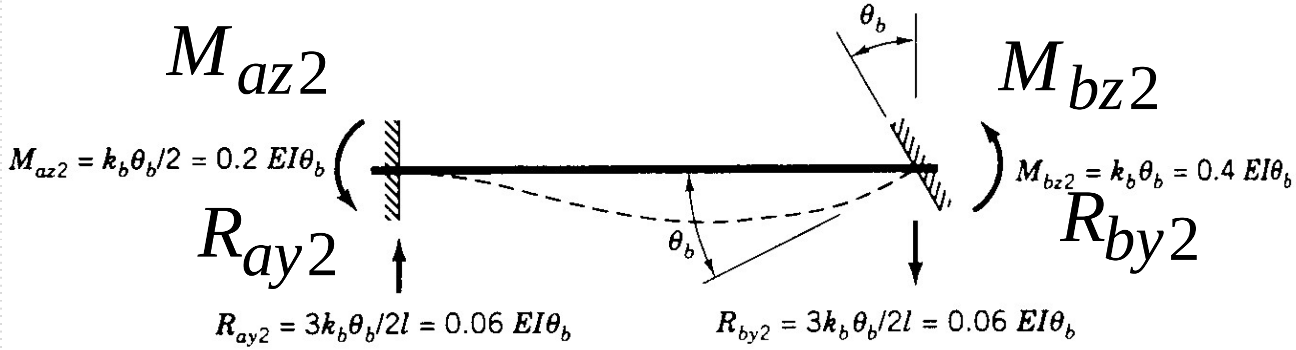
$$M_{az2} = -\frac{k_b}{2}\theta_b = -0.2EI\theta_b \qquad M_{bz2} = k_b\theta_b = 0.4EI\theta_b$$

$$\boxed{\begin{aligned} R_{ay} &= \frac{wl}{2} + \frac{3k_b}{2l}\theta_b \qquad & R_{by} &= \frac{wl}{2} - \frac{3k_b}{2l}\theta_b \\ M_{az} &= -\frac{wl^2}{12} - \frac{k_b}{2}\theta_b \qquad & M_{bz} &= -\frac{wl^2}{12} + k_b\theta_b \end{aligned} \qquad (9.8)}$$

- For a solution that reflects the proper loading and satisfies the designated force boundary conditions at point b, the solution shown in Figs. 9.8(a) and 9.8(b) must be superimposed so that the final moment at point b is zero.



(a) Primary structure subjected to given load



(b) Primary structure subjected to redundant rotation

Fig. 9.8 Kinematic primary structure

- Thus,

$$M_{bz1} + M_{bz2} = 0$$

- Upon substitution,

$$M_{bz1} + M_{bz2} = -\frac{wl^2}{12} + \frac{4EI}{l}\theta_b = 0$$

- from which

$$\theta_b = \frac{wl^3}{48EI}$$

- Retuning to Eq. (9.8), we have

$$R_{ay} = \frac{5wl}{8} = +312.5kN \qquad R_{by} = \frac{3wl}{8} = 187.5kN$$

$$M_{az} = -\frac{wl^2}{8} = -625kNm \qquad M_{bz} = 0$$

$$R_{ay} = \frac{wl}{2} + \frac{3k_b}{2l}\theta_b \qquad R_{by} = \frac{wl}{2} - \frac{3k_b}{2l}\theta_b$$

$$M_{az} = -\frac{wl^2}{12} - \frac{k_b}{2}\theta_b \qquad M_{bz} = -\frac{wl^2}{12} + k_b\theta_b \qquad (9.8)$$

9.6 STATIC VERSUS KINEMATIC INDETERMINACY

1) Which is better between static and kinematic determinacy?

- The above example problem had only one static redundancy, R_{by}, and only one kinematic redundancy, θ_b. So whether the static indeterminacy or kinematic determinacy is better does not matter.

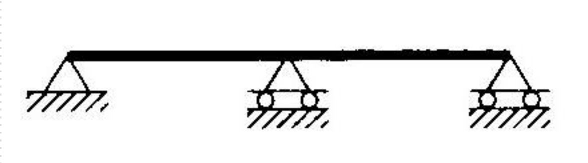
- However, as the structure becomes more complex, static vs. kinematic indeterminacy becomes important

2) Example of static and kinematic indeterminacy

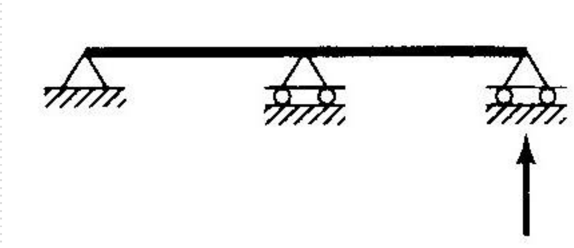
Structure 1

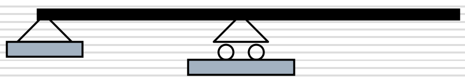
Primary structure

Degree of Indeterminacy



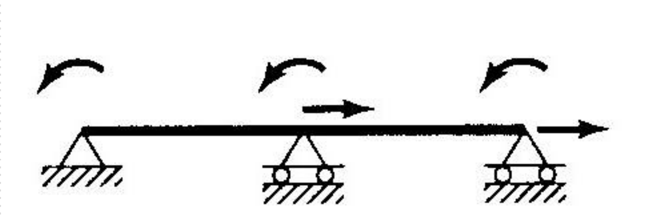
Static classification

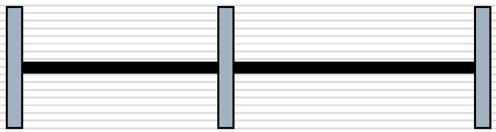




1

Kinematic classification





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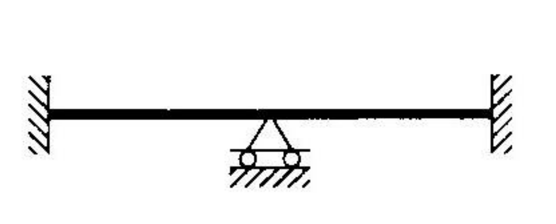
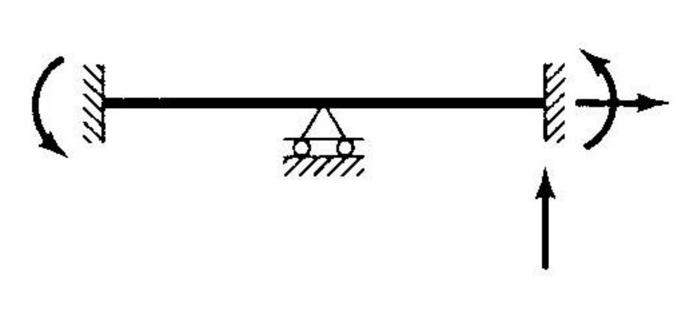
Structure 2

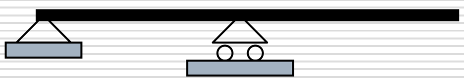
Primary structure

Degree of Indeterminacy



Static classification





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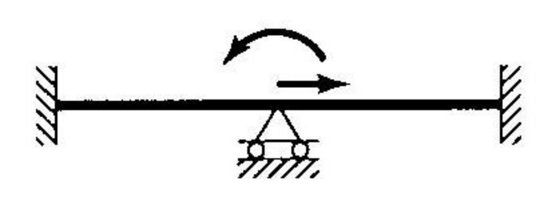
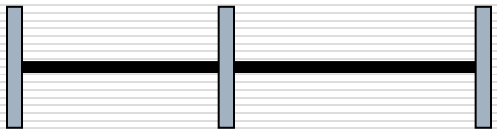
Kinematic classification





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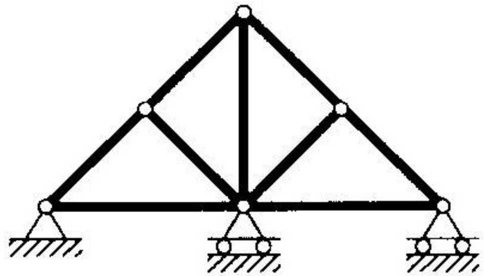
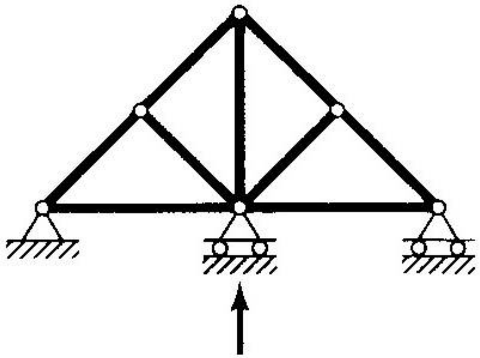
Structure 3

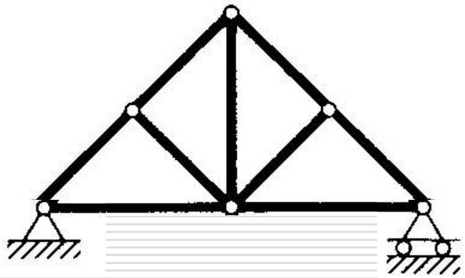


Primary structure

Degree of Indeterminacy

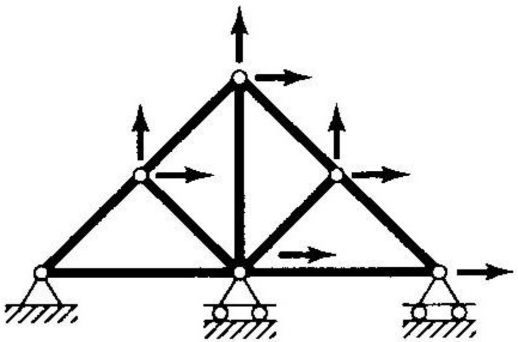
Static classification

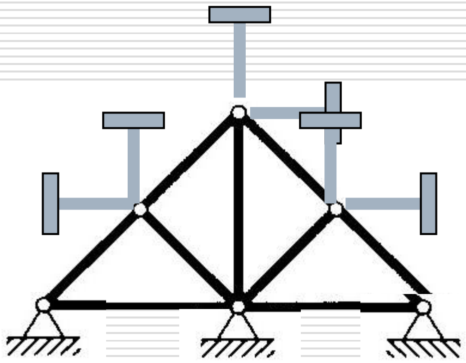




1

Kinematic classification





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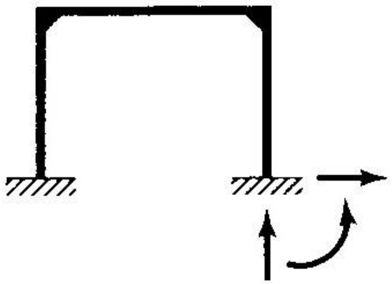
Structure 4

Primary structure

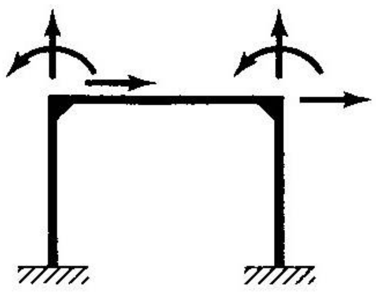
Degree of Indeterminacy

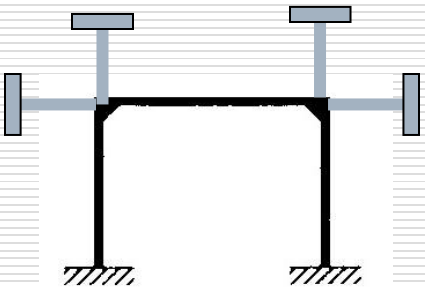
Static classification



3

Kinematic classification





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3) Important observation for static versus kinematic indeterminacy

- In establishing the statically determinate primary structure, there are a number of alternative selection patters for redundant force quantities.
- However, in developing the kinematically determinate primary structure, no selection process is necessary. Instead, all of the displacement quantities that are necessary to describe the structure's response automatically become redundant quantities.

9.7 COMPATIBILITY METHOD OF ANALYSIS (変位適合解析法)

1) What is the compatibility method?

- The compatibility method is based on the solution of a set of equations that express compatibility relationship throughout the structure.

- The compatibility method is also referred as the force method（荷重法） or the flexibility method (フレキシビリティー法), since the unknowns in the governing equations are forces and the coefficients of those unknowns are flexibility (displacement quantities)(たわみ性、フレキシビリティー) .

2) Flexibility and Stiffness (フレキシビリティーと剛性)

$F = ku$ (A1)

where,

F: force (**力**)
u: displacement (**変位**)
K: stiffness (**剛性**)

Equilibrium method, or displacement method, or stiffness method

$u = fF$ (A2)

where,

F: force (**力**)
u: displacement (**変位**)
f: flexibility (**たわみ性、フレキシビリティー**)

Compatibility method, or Force method, or Flexibility method

3) Example of the compatibility method

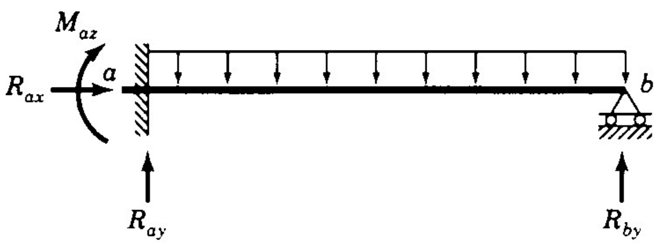
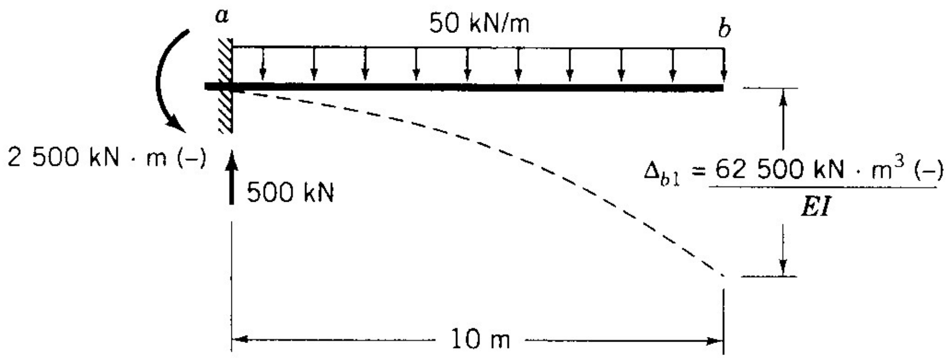


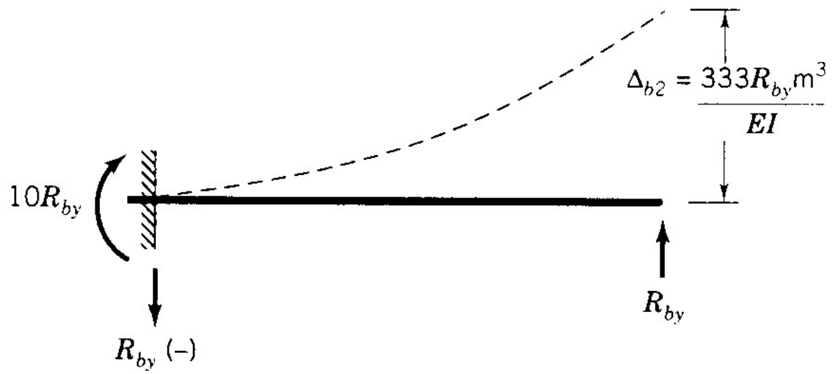
Fig. 9.3 (a) statically indeterminate structure



(a) Primary structure subjected to given loading

●As we studied in 9.3, we have the compatibility condition for a statically determinate structure as

$$\Delta_b = \Delta_{b1} + \Delta_{b2} \tag{9.3}$$



(b) Primary structure subjected to redundant reaction

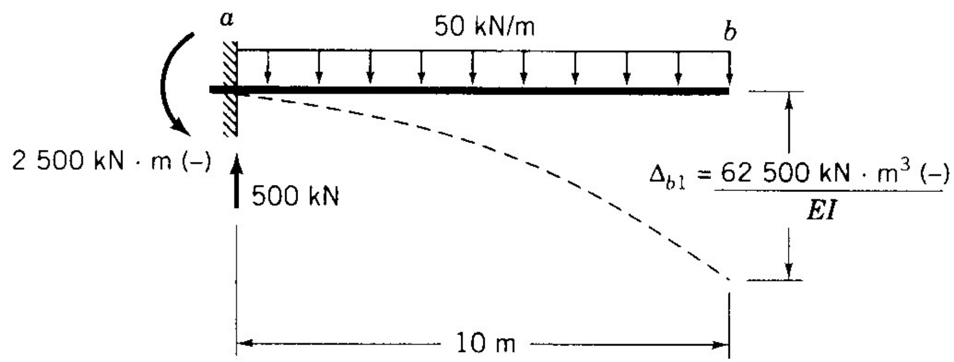
Fig. 9.4 Primary structure of Fig. 9.3

● Because

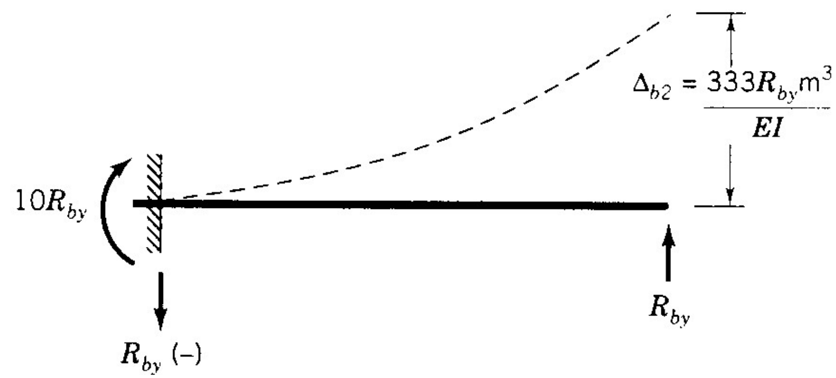
$$\Delta_{b1} = -\frac{wl^4}{8EI} \qquad \Delta_{b2} = \frac{R_{by}l^3}{3EI}$$

●Thus, R_{by} can be obtained as

$$R_{by} = \frac{3}{8}wl = 187.5kN$$



(a) Primary structure subjected to given loading



(b) Primary structure subjected to redundant reaction

Fig. 9.4 Primary structure of Fig. 9.3

4) Are requirements of equilibrium approved in the compatibility method?

- The final solution must satisfy both the conditions of compatibility and the requirements of equilibrium.
- When the method is properly formulated, the compatibility equations represent a superposition of a set of partial solutions, each of which satisfies the requirements of equilibrium.

9.8 EQUILIBRIUM METHOD OF ANALYSIS (荷重つり合い法)

1) What is the equilibrium method?

- The equilibrium method is based on the solution of a set of equations that express the equilibrium requirements for the structure.

- The equilibrium method is also called the displacement method (**変位法**) or the stiffness method (**剛性法**).

2) Example of the equilibrium method (Refer to Eq. (9.9))

$$M_{bz1} + M_{bz2} = 0$$

- Upon substitution,

$$-\frac{wl^2}{12} + \frac{4EI}{l}\theta_b = 0$$

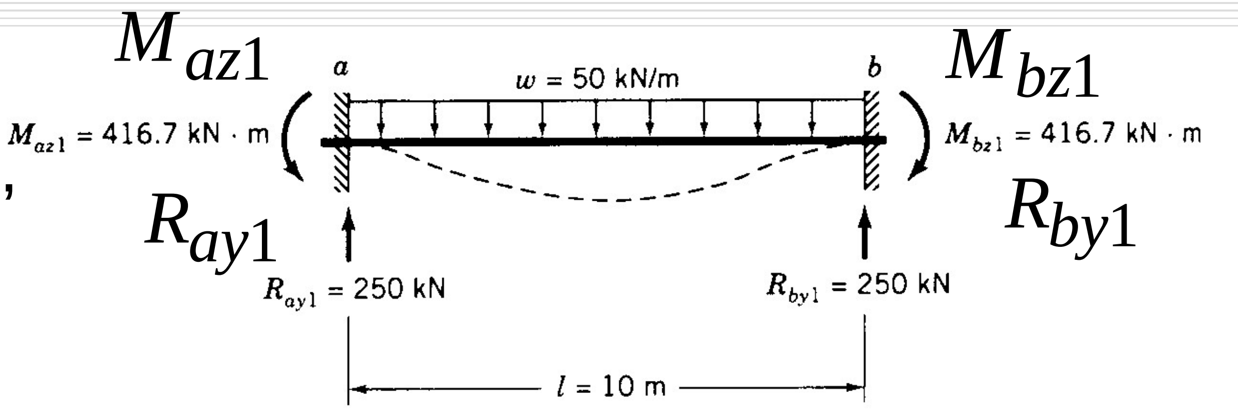
- from which

$$\theta_b = \frac{wl^3}{48EI}$$

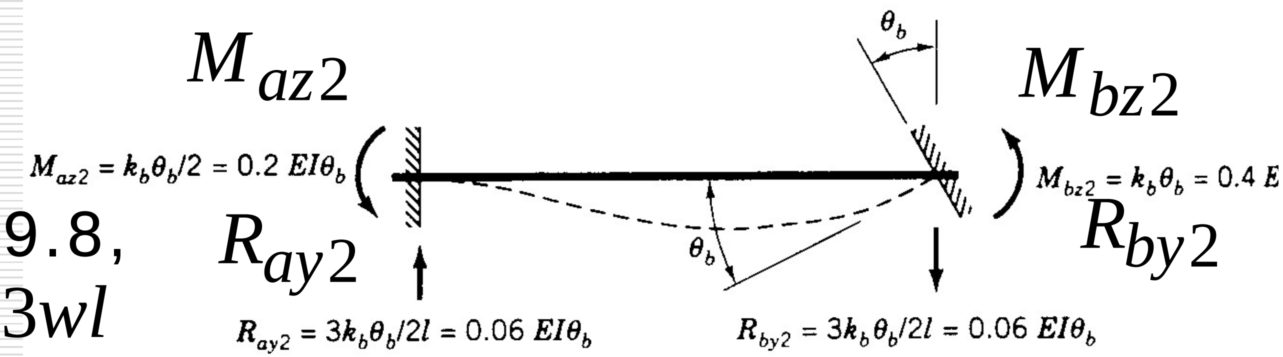
- Retuning to Fig. 9.8,

$$R_{ay} = \frac{5wl}{8} \qquad R_{by} = \frac{3wl}{8}$$

$$M_{az} = -\frac{wl^2}{8} \qquad M_{bz} = 0$$



(a) Primary structure subjected to given loa



(b) Primary structure subjected to redundant rotation

Fig. 9.8 Kinematic primary structure

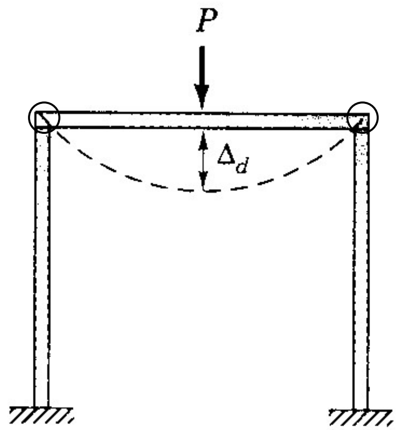
3) Advantage of equilibrium method

- For larger structures, there are many redundant displacement quantities. There is one redundant displacement quantity corresponding to each kinematic degree of freedom.
- Therefore, a number of equilibrium equations must be solved simultaneously for the displacement quantities.
- However unlike the compatibility method, there is no selectivity in choosing the redundants. All the kinematic degree of freedom are taken as redundant displacements.
- This is a major advantage for the automation that is needed in computer programming.

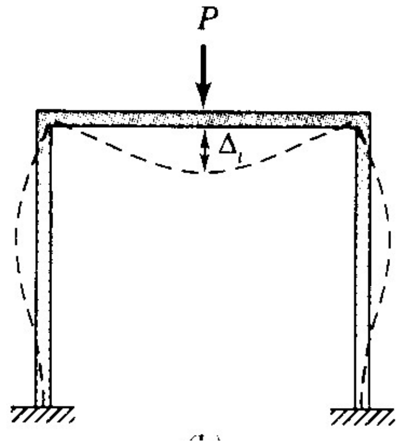
9.9 BEHAVIORAL CHARACTERISTICS OF STATICALLY INDETERMINATE STRUCTURES

1) Why are statically indeterminate structures used?

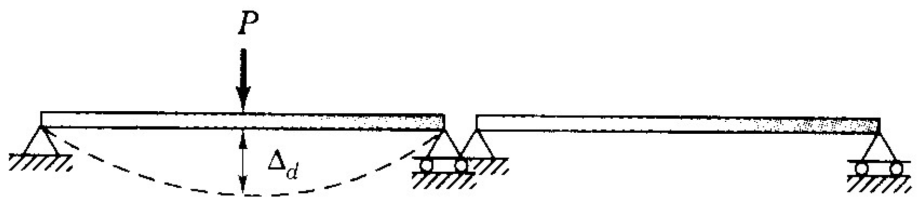
(1) A statically indeterminate structure displays greater stiffness in resisting load than does a comparable statically determinate structure.



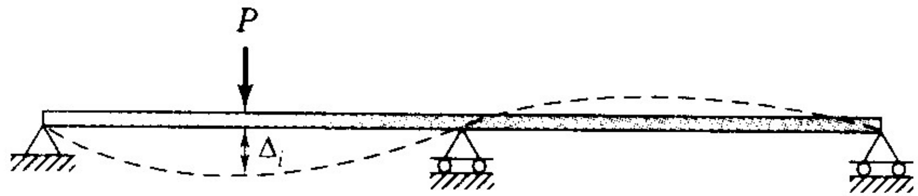
(a) Post and lintel



(b) Portal frame



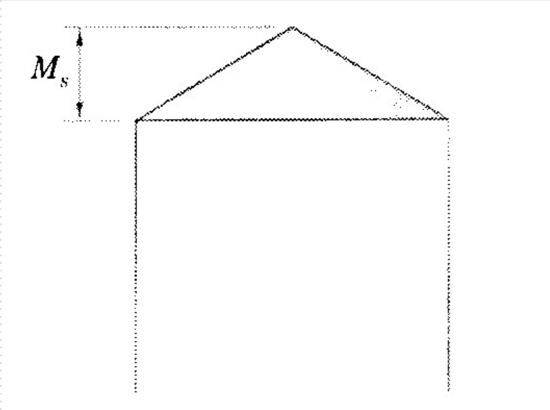
(c) Two simply supported beams



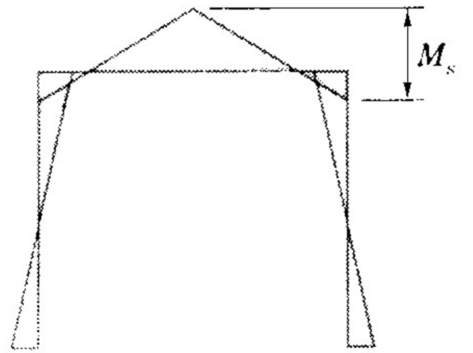
(d) Two-span continuous beam

Fig. 9.9

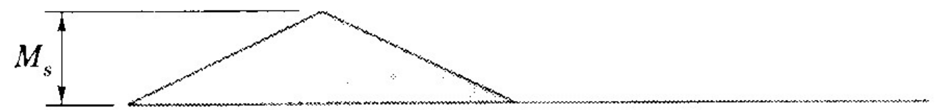
(2) A statically indeterminate structure has lower stress intensities than would a comparative statically determinate structure.



(a) Post and lintel



(b) Portal frame



(c) Two simply supported beams

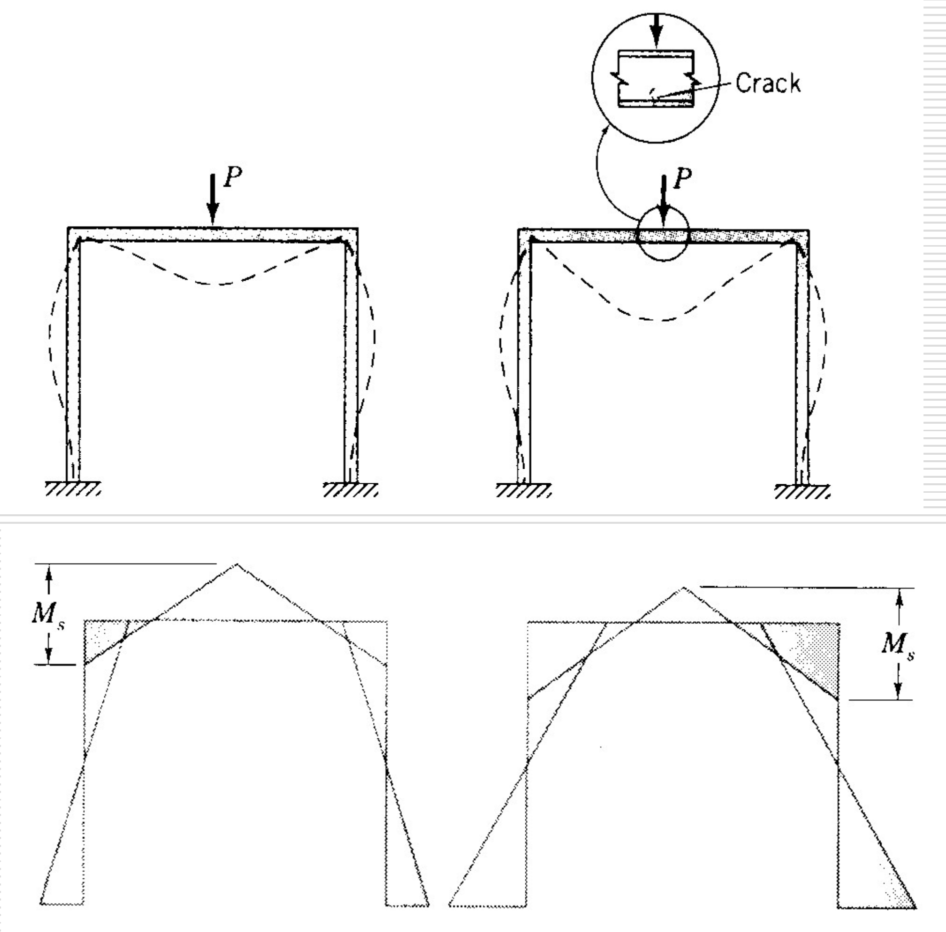
(d) Two-span continuous beam

Fig. 9.10 Comparative moment diagrams

(3) A statically indeterminate structure is safer than would a comparative statically determinate structure.

● There are either internal force or external reactions that are not needed for stability in a statically indeterminate structure.

● No matter how these force components are removed, the structure will not become unstable.



(a) Statically indeterminate frame

(b) Partially collapsed frame

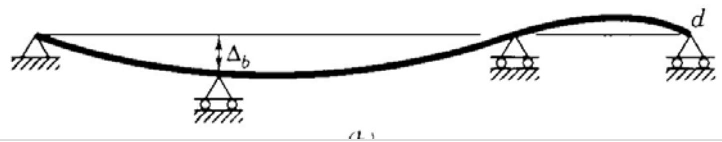
Fig. 9.11 Redistribution of moment

3) What is disadvantage of a statically indeterminate structure?

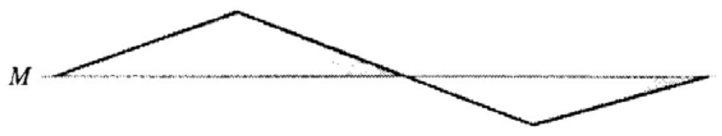
- Continuity which is the advantage of a statically indeterminate structure can be the primary disadvantage.

- If point b of a three-span continuous beam settled as shown in Fig. 9.12 (b), bending moment as shown in Fig. 9.12 (c) would be developed. Thus, without any loading on the structure, sizable moments would be introduced at the interior support points.

(a) Three-span continuous beam



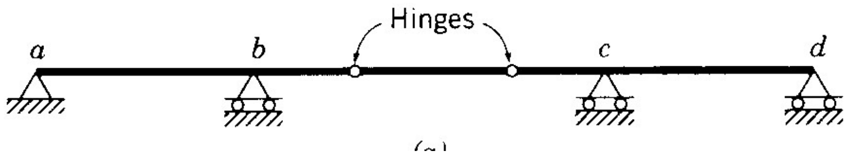
(b) Deformations caused by relative settlement



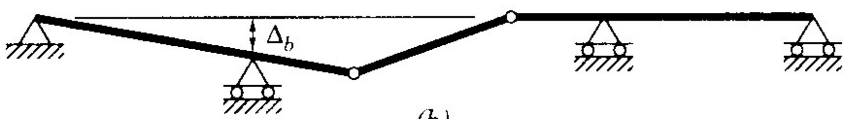
(c) Moment induced by settlement

Fig. 9.12

●In situations where such settlements can occur, a statically determinate structure should be employed as shown in Fig. 9.13.



(a) Cantilever structure



(b) Relative settlement

Fig. 9.13