

# Extensive Form Trembling-hand Perfect Eq.

Agent normal form of an extensive form game :

For each player, each of his information sets is a player.

Their payoffs are the same as the player's payoff.

EFTHPE in an extensive form

= NFTHPE in its agent normal form

NFTHPE in a normal form game  $\Gamma = (N = \{0, 1, \dots, I\}, \{\Delta(S_i)\}, \{u_i\})$  is a Nash eq.  $\sigma$  satisfying the following:

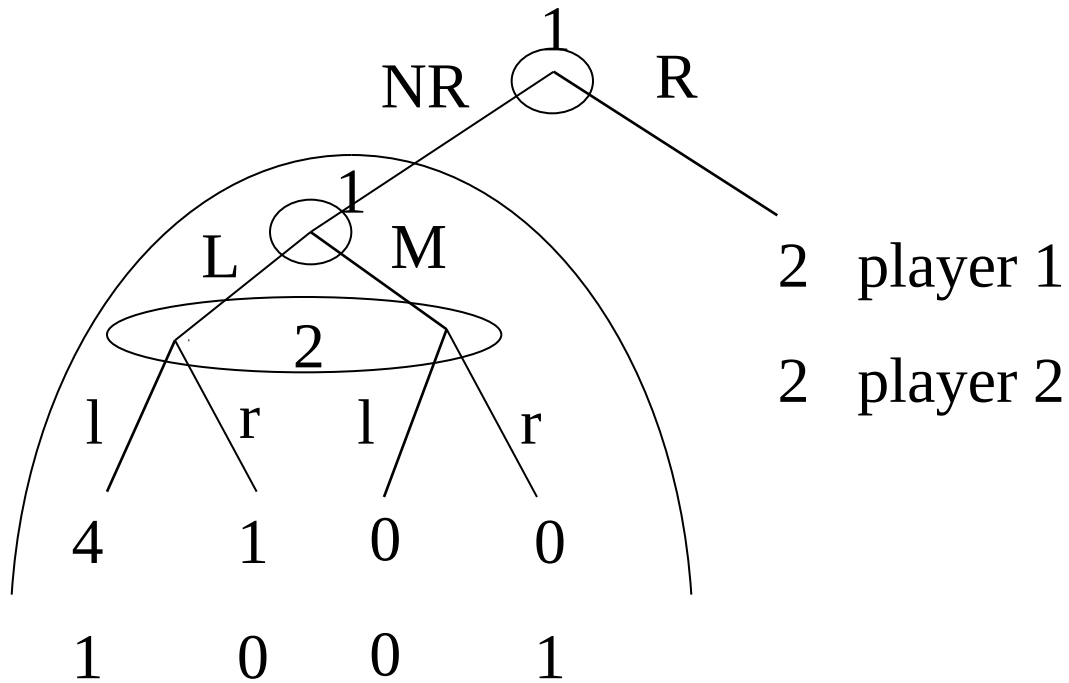
$\exists$  a sequence of perturbed games  $\{\Gamma_{\varepsilon_k}\}_{k=1}^{\infty}$  that converges to  $\Gamma$ , i.e.,  $\lim_{k \rightarrow \infty} \varepsilon_i^k(s_i) = 0 \quad \forall s_i \in S_i$ , for which

$\exists$  a sequence of Nash eq.  $\{\sigma^k\}_{k=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} \sigma^k = \sigma$ .

$$\Gamma_{\varepsilon_k} = [N = \{0, 1, \dots, I\}, \{\Delta_{\varepsilon_k}(S_i)\}, \{u_i\}]$$

$$\Delta_{\varepsilon_k}(S_i) = \{\sigma_i \mid \sigma_i(s_i) \geq \varepsilon_i(s_i) \quad \forall s_i \in S_i \text{ and } \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$$

# Extensive Form Trembling-hand Perfect Eq.



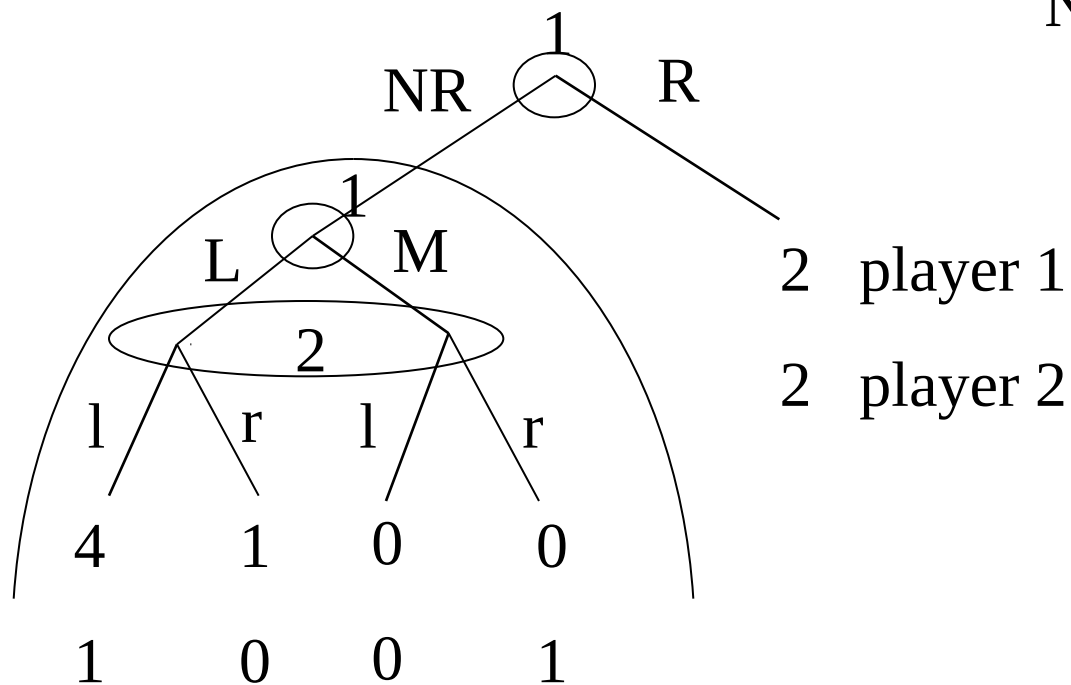
Subgame

| 2<br>1 | l                   | r            |
|--------|---------------------|--------------|
| L      | <u>4</u> , <u>1</u> | <u>1</u> , 0 |
| M      | 0, 0                | 0, <u>1</u>  |

Nash eq. (L, l) (unique)

SPNE ((NR-L), 1)

# Extensive Form Trembling-hand Perfect Eq.



Normal form game

| 2 \ 1 | l                   | r                   |
|-------|---------------------|---------------------|
| NR-L  | <u>4</u> , <u>1</u> | 1, 0                |
| NR-M  | 0, 0                | 0, <u>1</u>         |
| R-L   | 2, <u>2</u>         | <u>2</u> , <u>2</u> |
| R-M   | 2, <u>2</u>         | <u>2</u> , <u>2</u> |

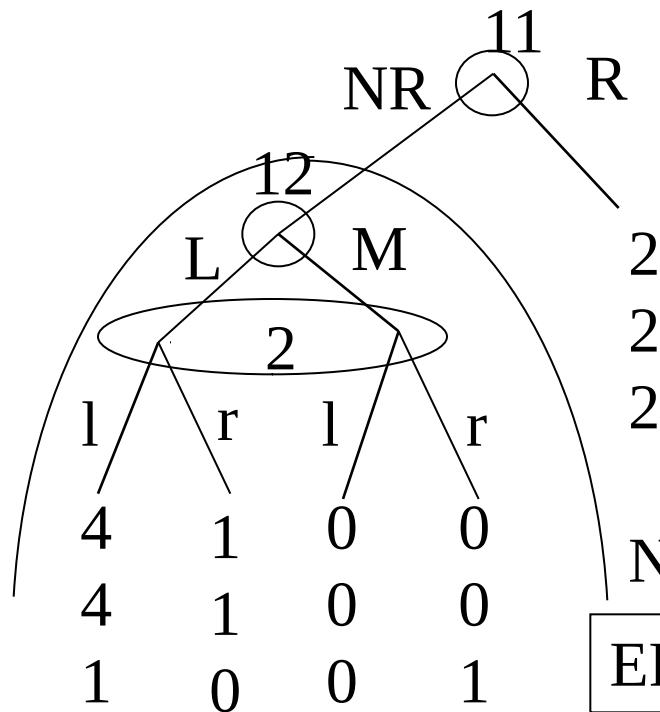
Nash eq. (NR-L, l) (R-L, r), (R-M, r)

NFTHPE (NR-L, l) (R-L, r), (R-M, r)

$$\varepsilon^k = (1/k^2, 1/k, 1/k, 1/k), \quad \sigma^k = (1/k^2, 1/k, 1-(1/k^2+2/k), 1/k)$$

$\lim_{k \rightarrow \infty} \sigma^k = \text{R-L}$  and  $r$  is b.r. to  $\sigma^k \rightarrow (\text{R-L}, r)$  NETHPE  
similar for  $(\text{R-M}, r)$

# Extensive Form Trembling-hand Perfect Eq.



Agent normal form

11 plays NR

11 plays R

| 12 \ 2 | l            | r              |
|--------|--------------|----------------|
| 12     |              |                |
| L      | <u>4,4,1</u> | 1, <u>1</u> ,0 |
| M      | 0,0,0        | 0,0, <u>1</u>  |

| 12 \ 2 | l             | r            |
|--------|---------------|--------------|
| 12     |               |              |
| L      | 2, <u>2,2</u> | <u>2,2,2</u> |
| M      | <u>2,2,2</u>  | <u>2,2,2</u> |

Nash eq. (NR,L, l), (R,L, r), (R,M, l), (R,M, r)

EFTHPE (NR,L, l)

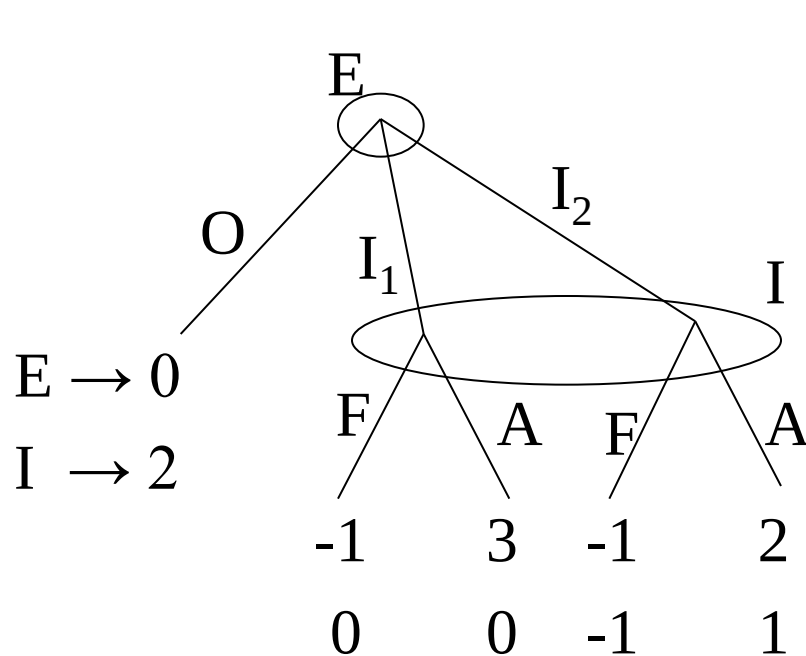
(R, L, r) : 2's payoff

$$(\varepsilon, 1-\varepsilon), (1-\varepsilon', \varepsilon'), (\varepsilon'', 1-\varepsilon'') \rightarrow \varepsilon((1-\varepsilon')\varepsilon'' + \varepsilon'(1-\varepsilon'')) + 2(1-\varepsilon)$$

$$(1-\varepsilon'', \varepsilon'') \rightarrow \varepsilon((1-\varepsilon')(1-\varepsilon'') + \varepsilon'\varepsilon'') + 2(1-\varepsilon) \uparrow$$

similar for (R, M, l) and (R, M, r)

# Extensive Form Trembling-hand Perfect Eq.



|   |                | I                   |                     |
|---|----------------|---------------------|---------------------|
|   |                | F                   | A                   |
| E | O              | <u>0</u> , <u>2</u> | 0, <u>2</u>         |
|   | I <sub>1</sub> | -1, <u>0</u>        | <u>3</u> , <u>0</u> |
|   | I <sub>2</sub> | -1, -1              | 2, <u>1</u>         |

Nash eq (SPNE)  $\rightarrow (O, F), (I_1, A)$

A weakly dominates F

$((O, F), (1, 0))$  SE

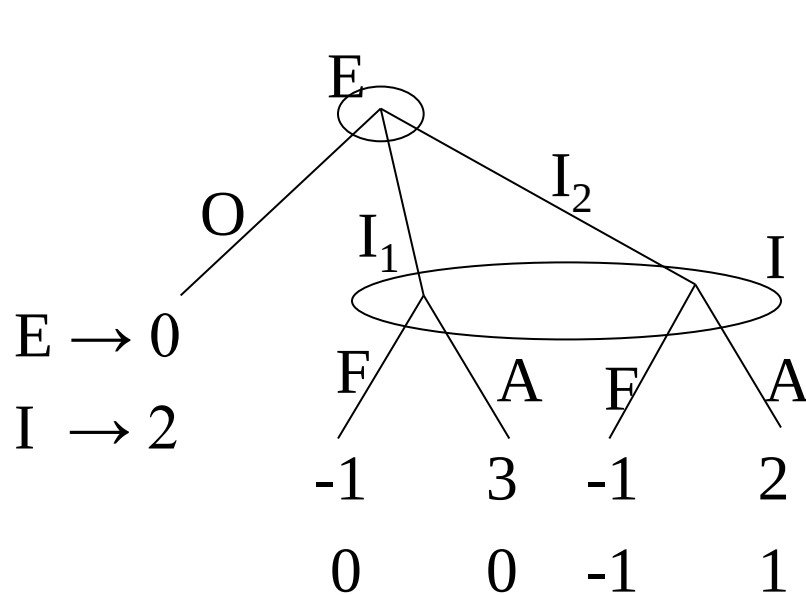
F is sequentially rational given (1, 0)

Let  $\sigma_E^k = (1 - (1/k + 1/k^2), 1/k, 1/k^2)$ . Then  $\lim_{k \rightarrow \infty} \sigma_E^k = (1, 0, 0) = O$

Furthermore belief is  $((1/k)/(1/k + 1/k^2), (1/k^2)/(1/k + 1/k^2))$

$= (k/(k+1), 1/(k+1)) \rightarrow (1, 0) \text{ (as } k \rightarrow \infty)$

# Extensive Form Trembling-hand Perfect Eq.



|   |                | I                   |                     |
|---|----------------|---------------------|---------------------|
|   |                | F                   | A                   |
| E | O              | <u>0</u> , <u>2</u> | 0, <u>2</u>         |
|   | I <sub>1</sub> | -1, <u>0</u>        | <u>3</u> , <u>0</u> |
|   | I <sub>2</sub> | -1, -1              | 2, <u>1</u>         |

Nash eq (SPNE)  $\rightarrow (O, F), (I_1, A)$

A weakly dominates F

$((O, F), (1, 0))$  SE but not EFTHPE

Let E's strategy  $(1-\varepsilon-\varepsilon', \varepsilon, \varepsilon')$ . Then

I's payoffs  $\rightarrow F: 2(1-\varepsilon-\varepsilon') + 0\varepsilon + (-1)\varepsilon' = 2 - 2\varepsilon - 3\varepsilon'$

$A: 2(1-\varepsilon-\varepsilon') + 0\varepsilon + 1\varepsilon' = 2 - 2\varepsilon - 1\varepsilon' > \text{payoff under F}$

EFTHPE  $\rightarrow$  SE

# Bilateral Bargaining

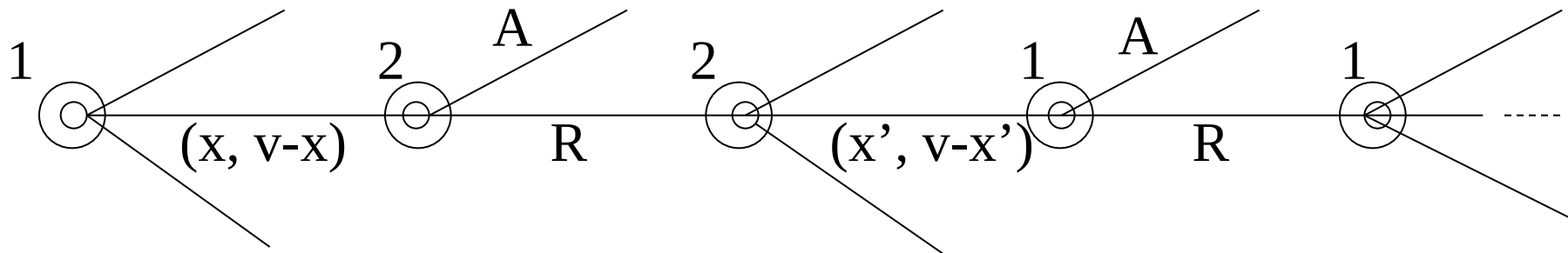
Players 1, 2 determine the split of  $v$

Player 1 makes an offer of a split  $(x, v-x)$  ( $0 \leq x \leq v$ )

Player 2 “accepts”  $\rightarrow$  1 gets  $x$ ; 2 gets  $v-x$

or “rejects”  $\rightarrow$  2 makes an offer of a split

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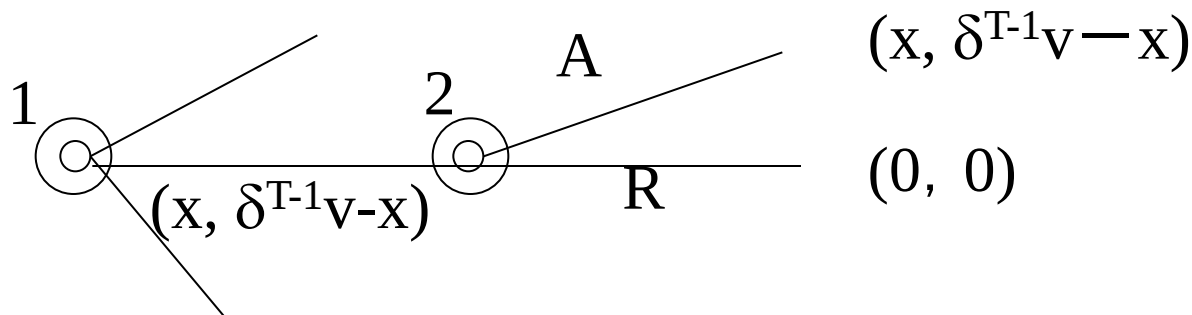


SPE ?

A : accept, R : reject

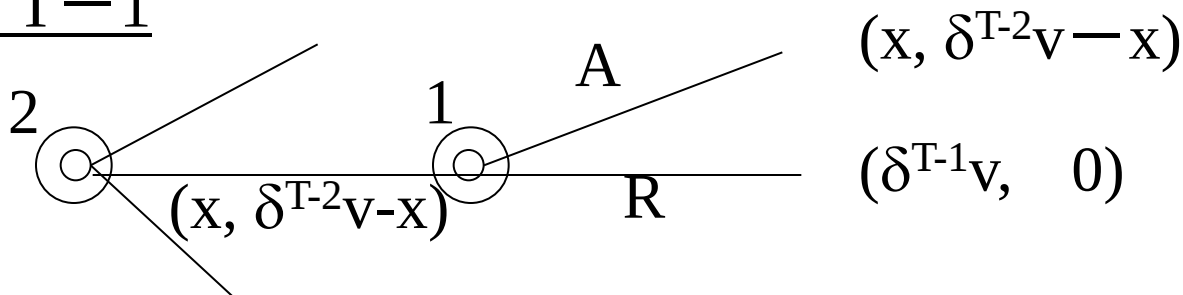
# Finite Horizon (T (odd) ) periods

## Period T



Unique SPE  $\rightarrow ((\delta^{T-1}v, 0), A)$  discounted payoffs  $(\delta^{T-1}v, 0)$

## Period T-1



Unique SPE  $\rightarrow ((\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v), A)$

discounted payoffs  $(\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v)$

# Finite Horizon (T (odd) ) periods

## Period T

Unique SPE  $\rightarrow ((\delta^{T-1}v, 0), A)$  discounted payoffs  $(\delta^{T-1}v, 0)$

## Period T-1

Unique SPE  $\rightarrow ((\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v), A)$

discounted payoffs  $(\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v)$

## Period T-2

Unique SPE  $\rightarrow ((\delta^{T-3}v - \delta^{T-2}v + \delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v), A)$

discounted payoffs  $((\delta^{T-3}v - \delta^{T-2}v + \delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v)$

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## Period 1

Unique SPE  $\rightarrow (v - \delta v + \delta^2 v - \dots + \delta^{T-1}v, \delta v - \delta^2 v + \dots - \delta^{T-1}v), A)$

Discounted payoffs  $(v - \delta v + \delta^2 v - \dots + \delta^{T-1}v, \delta v - \delta^2 v + \dots - \delta^{T-1}v)$

# Finite Horizon (T odd) periods

## Period 1

Unique SPE  $\rightarrow (v - \delta v + \delta^2 v - \dots + \delta^{T-1} v, \delta v - \delta^2 v + \dots - \delta^{T-1} v), A)$

Discounted payoffs  $(v - \delta v + \delta^2 v - \dots + \delta^{T-1} v, \delta v - \delta^2 v + \dots - \delta^{T-1} v)$

$$\begin{aligned} 1\text{'s payoff} &= v(1 - \delta + \delta^2 - \dots + \delta^{T-1}) = v((1 - (-\delta)^T) / (1 + \delta)) \\ &= v(1 + \delta^T) / (1 + \delta) = v^*_1(T) \\ &\rightarrow v / (1 + \delta) \quad (\text{as } T \rightarrow \infty) \end{aligned}$$

$$\begin{aligned} 2\text{'s payoff} &= v(1 - (1 + \delta^T) / (1 + \delta)) = v(\delta - \delta^T) / (1 + \delta) \\ &= v^*_2(T) = v - v^*_1(T) \\ &\rightarrow v \delta / (1 + \delta) \quad (\text{as } T \rightarrow \infty) \end{aligned}$$

# Finite Horizon (T (even) ) periods

## Period 1

Unique SPE  $\rightarrow (v - \delta v_1^*(T-1), \delta v_1^*(T-1)), A)$

Discounted payoffs  $(v - \delta v_1^*(T-1), \delta v_1^*(T-1))$

1's payoff

$$\begin{aligned} v - \delta v(1 + \delta^{T-1}) / (1 + \delta) &= v(1 - \delta^T) / (1 + \delta) \\ &\rightarrow v / (1 + \delta) \quad (\text{as } T \rightarrow \infty) \end{aligned}$$

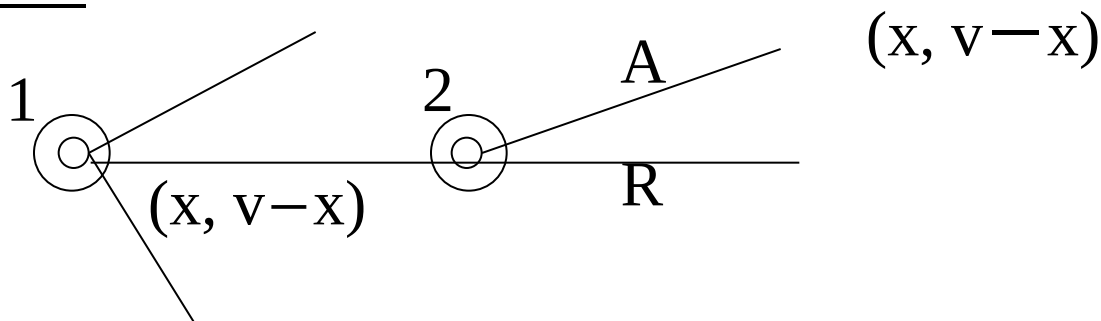
2's payoff

$$\begin{aligned} \delta v(1 + \delta^{T-1}) / (1 + \delta) &= v(\delta + \delta^T) / (1 + \delta) \\ &\rightarrow \delta v / (1 + \delta) \quad (\text{as } T \rightarrow \infty) \end{aligned}$$

# Infinite Horizon

## Stationary SPNE

### Period 1



$v_1^+ = \max$  payoff to 1 in any SPNE

2 can gain at most  $\delta v_1^+$  if he rejects

2 will accept if he gets ( more than or equal to)  $\delta v_1^+$

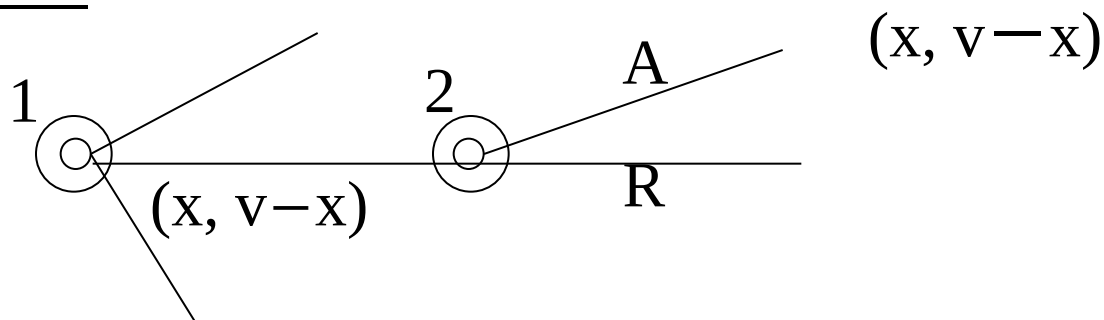
1 gets at least  $v - \delta v_1^+$

$v_1^- = \min$  payoff to 1 in any SPNE

$$\rightarrow v_1^- \geq v - \delta v_1^+$$

# Infinite Horizon

## Period 1



$$v_1^- \geq v - \delta v_1^+$$

Show  $v_1^+ \leq v - \delta v_1^-$

2 can gain at least  $\delta v_1^-$  if he rejects

2 will reject if he gets less than  $\delta v_1^-$

1 gets at most  $v - \delta v_1^-$  when 2 accepts his offer

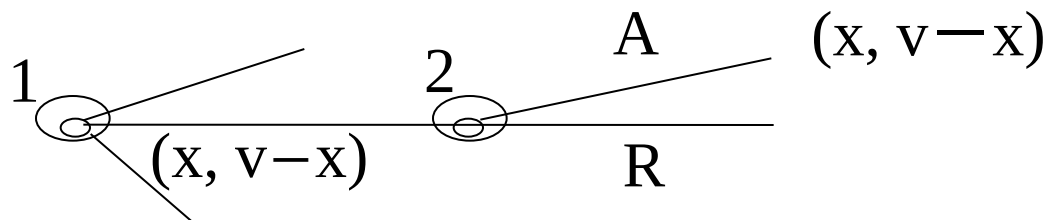
When 2 rejects, 2 gains at least  $\delta v_1^-$  in period 2.

→ 1 can gain at most  $\delta v - \delta v_1^-$  ( $< v - \delta v_1^-$ )

Thus  $v_1^+ \leq v - \delta v_1^-$

# Infinite Horizon

## Period 1



$$v_1^- \geq v - \delta v_1^+ \quad v_1^+ \leq v - \delta v_1^-$$

$$v_1^+ \leq v - \delta v_1^- \leq v_1^- + \delta v_1^+ - \delta v_1^-$$

$$(1-\delta)v_1^+ \leq (1-\delta)v_1^- \rightarrow v_1^+ \leq v_1^- \rightarrow v_1^+ = v_1^- = v_1^0$$

$$v_1^0 = v - \delta v_1^0 \rightarrow v_1^0 = v / (1+\delta)$$

$$v_2^0 = v - v_1^0 = v - v / (1+\delta) = \delta v / (1+\delta)$$

SPNE  $\rightarrow$  a player making an offer offers  $\delta v / (1+\delta)$

a player accepts an offer iff the offer  $\geq \delta v / (1+\delta)$

(payoffs in finite horizon when  $T \rightarrow \infty$ )

# Assignments

Problem Set 11 (due June 30)

Exercise 9.B.7 (p.302)