Example 9.B.1

E

 $E \rightarrow 0$ $I \rightarrow 2$ $Fi \qquad Ac$ $-3 \qquad 2$ $-1 \qquad 1$

		L			
		Fi		Ac	
	Ou	<u>0</u> ,	<u>2</u>	0,	<u>2</u>
	In	-3,	-1	<u>2</u> ,	<u>1</u>

T

 $\frac{\text{Nash eq}}{\rightarrow} \text{ (in pure str.)}$ $\rightarrow \text{ (Ou, Fi), (In, Ac)}$

(Ou, Fi) \rightarrow rational ??? Fi : I's incredible threat If E really plays "In", I will play "Ac". (1 > -1)

Backward Induction



Backward induction

- (1) $1 > -1 \rightarrow I$ plays Ac
- (2) $2 > 0 \rightarrow E$ plays In

(In, Ac)

Games with perfect information

 \rightarrow every information set has <u>one</u> decision point.

Backward Induction (Example 9.B.2)



Backward induction \rightarrow (R, a, (r, r, ℓ)) \rightarrow Nash eq. Other Nash eq. \rightarrow (L, b, (r, ℓ , r))

Other Nash Equilibria (Example 9.B.2)



Nash Equilibria in Games with Perfect Information

<u>Prop. 9.B.1</u> (Zermelo's Theorem) : Every <u>finite</u> game w/ <u>perfect</u> <u>information</u> has a pure strategy Nash equilibrium produced by backward induction. If no player has the same payoffs, then \exists unique Nash eq. derived in this manner.

- <u>Pf</u>: a finite game w/ perfect information
 - \rightarrow backward induction is well-defined
 - no player has the same payoffs
 - \rightarrow a unique strategy combination

Let $(\sigma_1, \dots, \sigma_I)$ be the strategy combination derived thru backward induction

Show $(\sigma_1, \ldots, \sigma_I)$ is a Nash eq.

Proof

Show
$$\forall i \ \forall \sigma_i^{} \quad u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i^{}, \sigma_{-i})$$

Take any σ_i^{n} and define i's strategy $\sigma_i^{n}(n)$ as follows. For each node x,

let $d(x) = \max \#$ of nodes between x and terminal nodes



Proof

Show $u_i(\sigma_i^{(N)} = \sigma_i, \sigma_{-i}) \ge u_i(\sigma_i^{(N)}, \sigma_{-i})$: induction on n

(1)
$$n = 0$$
: $\sigma_{i}^{(0)}(x) = \sigma_{i}(x)$ if $d(x) = 0$
 $\sigma_{i}^{(x)}$ if $d(x) > 0$

 $\sigma_i(x)$ chooses an alternative at x that max i's payoff

$$\rightarrow u_i \left(\sigma_i^{(0)}, \sigma_{-i} \right) \geq u_i \left(\sigma_i^{(0)}, \sigma_{-i} \right)$$

(2) Suppose for n = k-1 $u_i (\sigma_i^{(k-1)}, \sigma_{-i}) \ge u_i (\sigma_i^{(k-1)}, \sigma_{-i})$ holds.

(3) For n = k, <u>show</u> $u_i(\sigma_i^{(k)}, \sigma_{-i}) \ge u_i(\sigma_i^{(k)}, \sigma_{-i})$

Proof

(2) Suppose for n = k - 1, $u_i (\sigma^{\wedge}(k-1), \sigma_{-i}) \ge u_i (\sigma^{\wedge}_i, \sigma_{-i})$ (1) (3) For n = k, <u>show</u> $u_i (\sigma^{\wedge}(k), \sigma_{-i}) \ge u_i (\sigma^{\wedge}_i, \sigma_{-i})$ (2) d(x)=k d(x')=k-1 x' $\sigma^{\wedge}(k)(x) = \sigma_i(x')$ $\sigma_i(x'') \cdots$

 $\sigma_{i}^{(k-1)(x)} = \sigma_{i}^{(x)} \qquad \sigma_{i}^{(k-1)(x')} = \sigma_{i}^{(x')} \qquad \sigma_{i}^{(x'')}$

By the definition of σ_i , $u_i (\sigma_i^{(k)}, \sigma_{-i}) \ge u_i (\sigma_i^{(k-1)}, \sigma_{-i})$ 3 (1) and (3) \rightarrow (2) holds.

 $\label{eq:eventually} \begin{array}{lll} u_i(\sigma_i, \ \sigma_{\text{-}i}) \ = \ u_i \ (\sigma^{\wedge}_i(N), \ \sigma_{\text{-}i} \) \ \geq \ u_i \ (\sigma^{\wedge}_i, \ \sigma_{\text{-}i} \) \quad Q.E.D. \end{array}$

A Game with Imperfect Information (Example 9.B.3)



		Ι				
		Ac	Fi			
	Ou Ac	0, <u>2</u>	<u>0</u> , <u>2</u>			
E	Ou Fi	0, <u>2</u>	<u>0</u> , <u>2</u>			
	In Ac	<u>3, 1</u>	-2, -1			
	In Fi	1, -2	-3, <u>-1</u>			
	Nash eq. ((Ou Ac), Fi),					
	((Ou, Fi), Fi),					
		<u>((In, Ac), Ac)</u>				

Nash eq. (Ac, Ac)

<u>Defn. 9.B.1</u>: A subgame of an extensive form game is a subset of the game having the following properties:

- (1) It begins with an information set containing only one node.
- (2) It contains all successors of the node and no other node.
- (3) For each successor, any node, in the information set that contains the successor, is in the subset.

<u>Note</u>: (1) whole game \rightarrow a subgame

- (2) Fig.9.B.1 \rightarrow two subgames
- (3) Fig.9.B.3 \rightarrow five subgames

(games with perfect information

 \rightarrow each node initiates a subgame)

- (4) Fig.9.B.4 \rightarrow two subgames
- (5) Fig.9.B.5 \rightarrow parts of the game that are not subgames

<u>Defn. 9.B.2</u>: A strategy profile $\sigma = (\sigma_1, \dots, \sigma_I)$ of an extensive form game is <u>SPNE</u> if it induces a Nash equilibrium in every subgame of the game.

<u>Note</u>: (1) SPNE \rightarrow Nash equilibrium (whole game is a subgame.)

- (2) SPNE \rightarrow SPNE of each subgame
- (3) Fig.9.B.1 \rightarrow (In, Ac)
- (4) Fig.9.B.2 \rightarrow (R, a, (r, r, ℓ))
- (5) Fig.9.B.3 \rightarrow ((In, Ac), Ac)

<u>Prop. 9.B.2</u> : Every <u>finite</u> game w/ <u>perfect information</u> has a pure strategy SPNE. If no player has the same payoffs, then \exists unique SPNE

<u>Pf</u>: clear from Prop. 9.B.1 and the definition of SPNE

Assignments

Problem Set 7 (due June 2) Exercises (pp.301-305) 9.B.3, 9.B.5

Reading Assignment:

Text, Chapter 9, pp.277-282