Bayesian Nash Equilibrium

Incomplete information game

→ structure of the game is not completely known

Harsanyi's approach :

players' <u>preferences</u> are determined

by realization of a random variable.

ex ante distribution of the r.v. → common knowledge

incomplete information game → complete information game

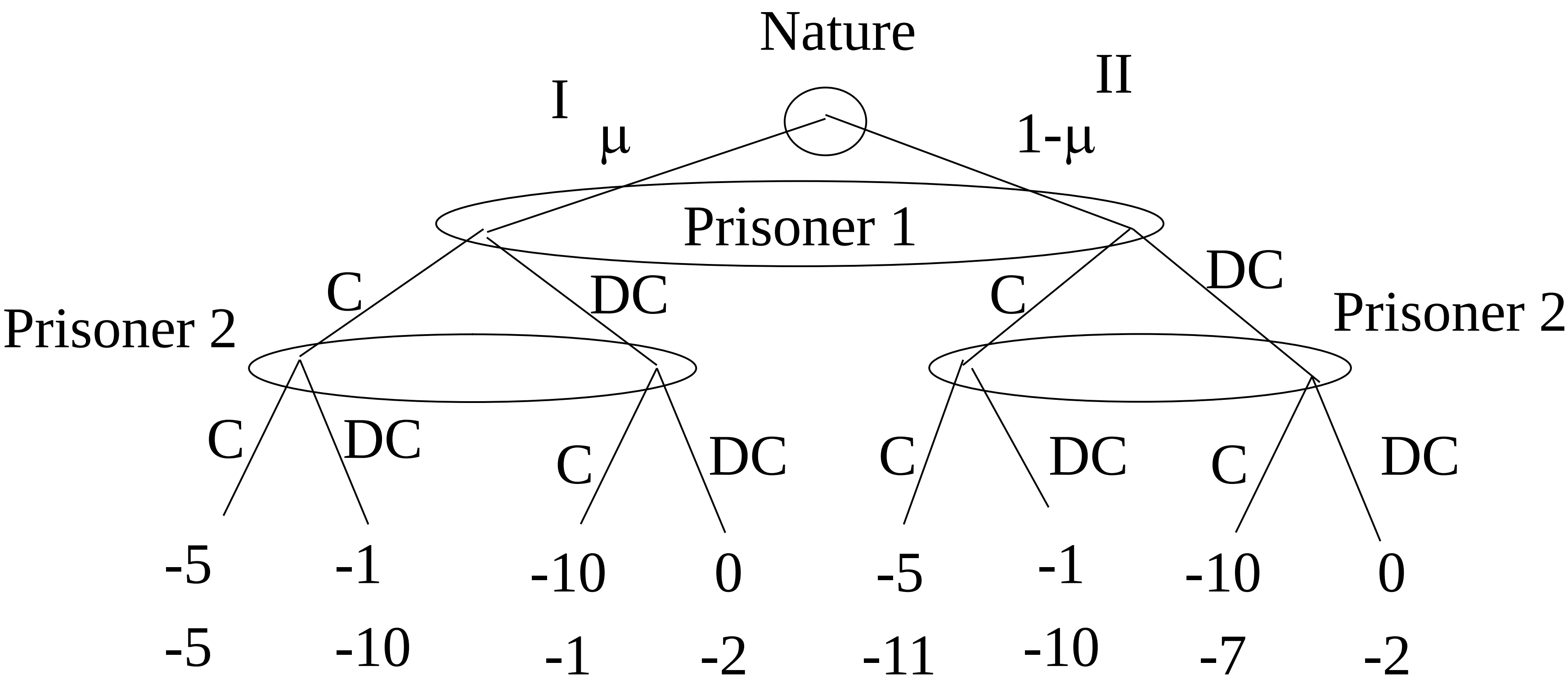
w/ imperfect information

Nature chooses realization of r.v.

that determines player's preference type

<u>Bayesian game</u>

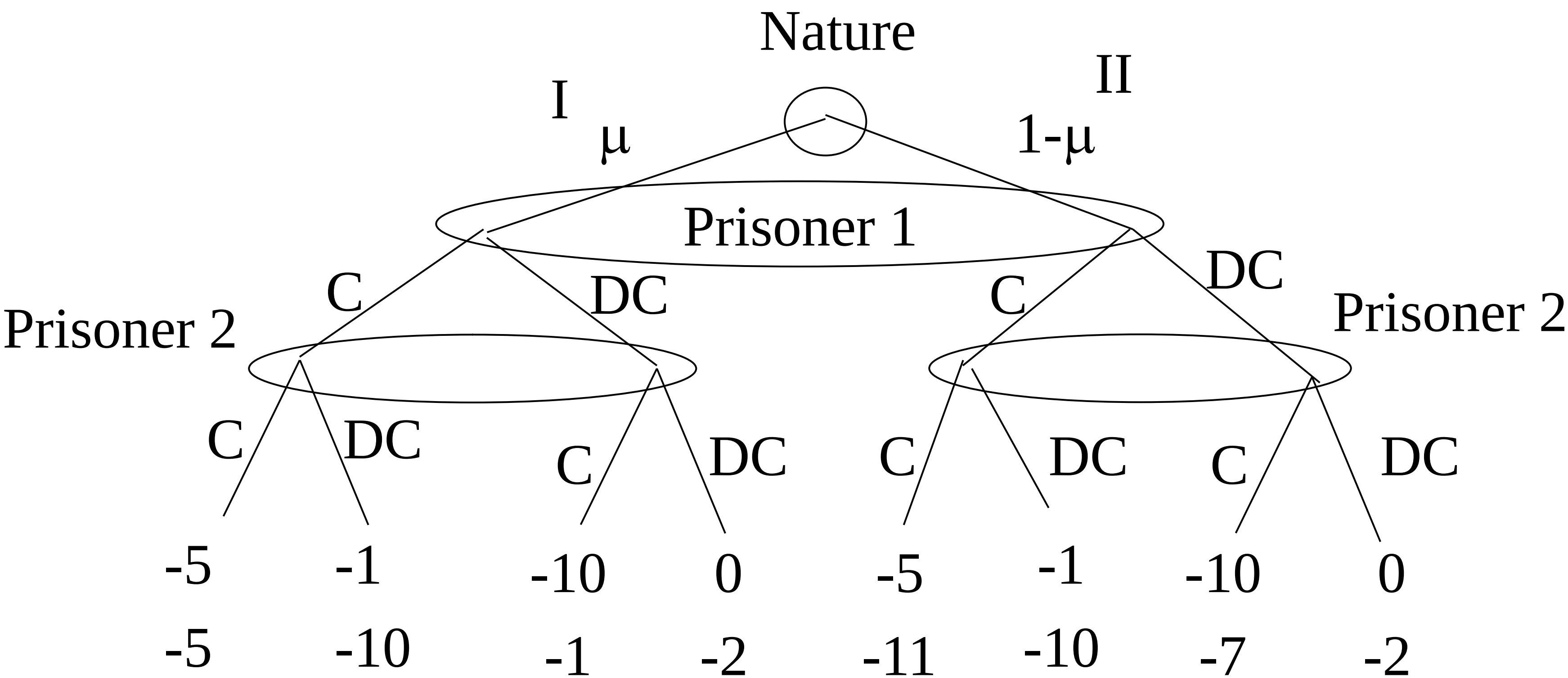
Example 8.E.1 (Modification of DA's Brother)



	DC	C
DC	0, -2	-10, -1
C	-1, -10	-5, -5

	DC	C
DC	0, -2	-10, -7
C	-1, -10	-5, -11

Example 8.E.1 (Modification of DA's Brother)



Prisoner 2 has four strategies: (C,C), (C, DC), (DC, C), (DC, DC)

(type I, type II)

Prisoner 1 has two strategies: C, DC

Bayesian Game

$[N=\{0,1,\ldots,I\}, \{S_i\}, \{u_i\}, \Theta, F]$

$u_i : S(=S_1\times\ldots\times S_I)\times\Theta_i \rightarrow \Re$ utility function

$\Theta = \Theta_1\times\ldots\times\Theta_I$ set of profiles of types

$F : \Theta_1\times\ldots\times\Theta_I \rightarrow \Re$ joint distribution function

S_i : set of actions (so far called strategies)

i's pure strategy $\boldsymbol{s}_i : \Theta_i \rightarrow S_i$

set of i's pure strategies $\boldsymbol{S}_i$

i's expected payoff under $(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_I)$

$\tilde{u}_i(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_I) = E_\theta[u_i(\boldsymbol{s}_1(\theta_1), \ldots, \boldsymbol{s}_I(\theta_I), \theta_i)]$

Bayesian Nash Equilibrium

Definition 8.E.1: $(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_I)$ is a Bayesian Nash equilibrium

for $[N=\{0,1,\ldots,I\}, \{S_i\}, \{u_i\}, \Theta, F]$

if $u^{\sim}_i(\boldsymbol{s}_i, \boldsymbol{s}_{-i}) \geq u^{\sim}_i(\boldsymbol{s}'_i, \boldsymbol{s}_{-i}) \ \forall \boldsymbol{s}'_i \in \boldsymbol{S}_i$

(i,.e., $(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_I) \in \boldsymbol{S} = \boldsymbol{S}_1 \times \ldots \times \boldsymbol{S}_I$

is a Nash eq. of $[N=\{1,\ldots,I\}, \{\boldsymbol{S}_i\}, \{u^{\sim}_i\}]$)

Proposition 8.E.1:

$(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_I) \in \boldsymbol{S}$ is a Bayesian Nash eq.

for $[N=\{0,1,\ldots,I\}, \{S_i\}, \{u_i\}, \Theta, F]$

$\Leftrightarrow \ \forall i = 1, \ldots, I, \ \forall \theta^{\sim}_i \in \Theta_i$ occurring with positive prob.

$E_{\theta\text{-}i}[u_i(\boldsymbol{s}_i(\theta^{\sim}_i), \boldsymbol{s}_{-i}(\theta_{-i}), \theta^{\sim}_i) \mid \theta^{\sim}_i] \geq E_{\theta\text{-}i}[u_i(s'_i, \boldsymbol{s}_{-i}(\theta_{-i}), \theta^{\sim}_i) \mid \theta^{\sim}_i]$

$\forall s'_i \in S_i$

Bayesian Nash Equilibrium

<u>Proposition 8.E.1</u>:

$(\boldsymbol{s}_1, \dots, \boldsymbol{s}_I) \in \boldsymbol{S}$ is a Bayesian Nash eq.

for $[N=\{0,1,\dots,I\}, \{S_i\}, \{u_i\}, \Theta, F]$

$\Leftrightarrow \; \forall\, I = 1, \dots, I, \; \forall \theta^{\sim}_i \in \Theta_i$ occurring with positive prob.

$$E_{\theta\text{-}i}[u_i(\boldsymbol{s}_i(\theta^{\sim}_i), \boldsymbol{s}_{-i}(\theta_{-i}), \theta^{\sim}_i) \mid \theta^{\sim}_i] \geq E_{\theta\text{-}i}[u_i(s'_i, \boldsymbol{s}_{-i}(\theta_{-i}), \theta^{\sim}_i) \mid \theta^{\sim}_i]$$

$$\forall\, s'_i \in S_i$$

Pf.: $\rightarrow$) Suppose $\exists\, i \in N, \; \exists\, \theta^{\sim}_i \in \Theta_i$ occurring with positive prob. and $\exists\, s'_i \in S_i$ such that

$$E_{\theta\text{-}i}[u_i(\boldsymbol{s}_i(\theta^{\sim}_i), \boldsymbol{s}_{-i}(\theta_{-i}), \theta^{\sim}_i) \mid \theta^{\sim}_i] < E_{\theta\text{-}i}[u_i(s'_i, \boldsymbol{s}_{-i}(\theta_{-i}), \theta^{\sim}_i) \mid \theta^{\sim}_i].$$

Let $\boldsymbol{s}''_i \in \boldsymbol{S}_i$ be such that $\boldsymbol{s}''_i(\theta_i) = \boldsymbol{s}_i(\theta_i) \; \forall \theta_i \neq \theta^{\sim}_i$ and $\boldsymbol{s}''_i(\theta^{\sim}_i) = s'_i$.

Then $u^{\sim}_i(\boldsymbol{s}_i, \boldsymbol{s}_{-i}) < u^{\sim}_i(\boldsymbol{s}''_i, \boldsymbol{s}_{-i})$.

Bayesian Nash Equilibrium

Proposition 8.E.1:

$(\boldsymbol{s}_1, \dots, \boldsymbol{s}_I) \in \boldsymbol{S}$ is a Bayesian Nash eq.

for $[N=\{1,\dots,I\}, \{S_i\}, \{u_i\}, \Theta, F]$

$\Leftrightarrow \ \forall\, i = 1, \dots, I, \ \forall \theta^{\sim}_i \in \Theta_i$ occurring with positive prob.

$$E_{\theta\text{-}i}[u_i(\boldsymbol{s}_i(\theta^{\sim}_i), \boldsymbol{s}_{-i}(\theta_{-i}), \theta^{\sim}_i) \mid \theta^{\sim}_i] \geq E_{\theta\text{-}i}[u_i(s'_i, \boldsymbol{s}_{-i}(\theta_{-i}), \theta^{\sim}_i) \mid \theta^{\sim}_i] \quad \forall s'_i \in S_i$$

Pf.: $\leftarrow$) Take any strategy $\boldsymbol{s}'_i \in \boldsymbol{S}_i$.

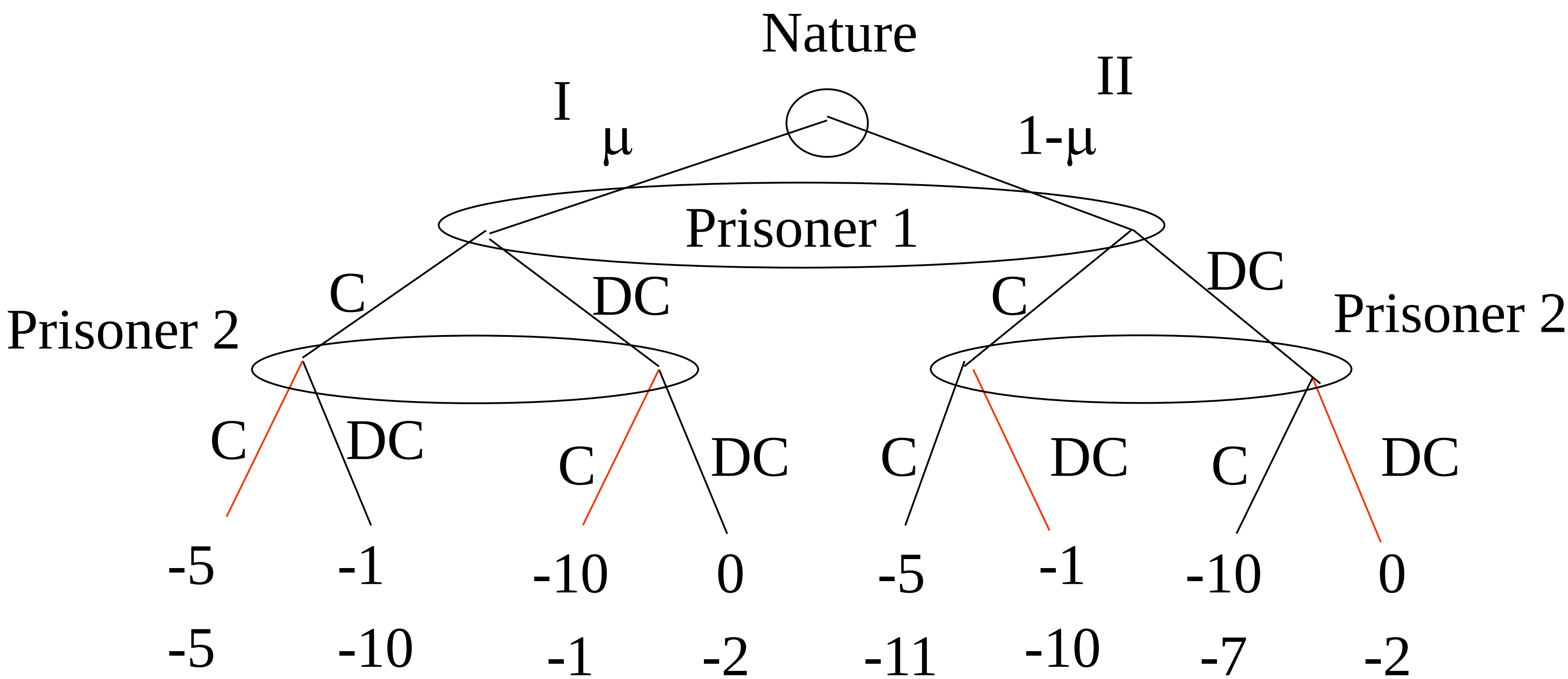
$$u^{\sim}_i(\boldsymbol{s}_i, \boldsymbol{s}_{-i}) = E_\theta[u_i(\boldsymbol{s}_i(\theta_i), \boldsymbol{s}_{-i}(\theta_{-i}), \theta_i)]$$

$$= \Sigma_{\theta i} \text{Prob}(\theta_i)\, E_{\theta\text{-}i}[u_i(\boldsymbol{s}_i(\theta_i), \boldsymbol{s}_{-i}(\theta_{-i}), \theta_i) \mid \theta_i]$$

$$\geq \Sigma_{\theta i} \text{Prob}(\theta_i)\, E_{\theta\text{-}i}[u_i(\boldsymbol{s}'_i(\theta_i), \boldsymbol{s}_{-i}(\theta_{-i}), \theta_i) \mid \theta_i]$$

$$= E_\theta[u_i(\boldsymbol{s}'_i(\theta_i), \boldsymbol{s}_{-i}(\theta_{-i}), \theta_i)] = u^{\sim}_i(\boldsymbol{s}'_i, \boldsymbol{s}_{-i})$$

Example 8.E.1 (Modification of DA's Brother)



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For 2, C dom DC

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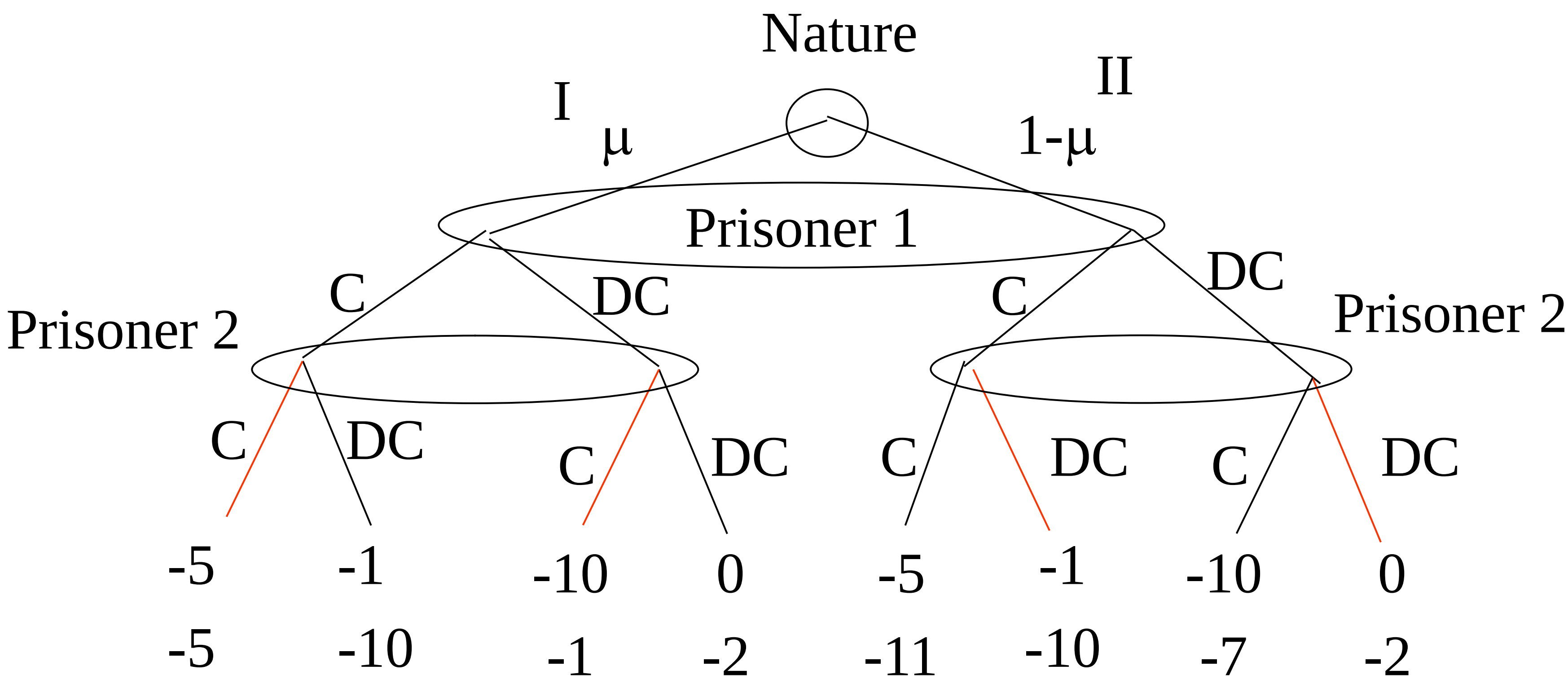
For 2, DC dom C

Example 8.E.1 (Modification of DA's Brother)



For 1:

$C \rightarrow -5\mu + (-1)(1-\mu) = -1-4\mu$, $DC \rightarrow -10\mu + 0(1-\mu) = -10\mu$

$-1-4\mu > -10\mu$ $(\mu > 1/6) \rightarrow$ C

$-1-4\mu < -10\mu$ $(\mu < 1/6) \rightarrow$ DC

$-1-4\mu = -10\mu$ $(\mu = 1/6) \rightarrow$ C, DC indifferent

Example 8.E.2

AB R&D consortium : firm 1 and firm 2

new tech. Z   <u>cost</u> c $(0 < c < 1)$

<u>benefit</u> depends on the type θ_i (uniformly distributed on [0, 1])

and given by θ_i^2

<u>Bayesian NE</u>:

strategy: $s_i(\theta_i) = 1$ if i develops Z; $s_i(\theta_i) = 0$ if not

develop $\rightarrow$ $\theta_i^2 - c$; not develop $\rightarrow$ $\theta_i^2 \, \text{Prob}(s_j(\theta_j) = 1)$

i develops, i.e. $s_i(\theta_i) = 1$

iff $\theta_i \geq (c / (1 - \text{Prob}(s_j(\theta_j) = 1))^{1/2}$ (<u>cutoff rule</u>)

Only one firm: develop iff $\theta_i^2 \geq c$, i.e., $\theta_i \geq c^{1/2}$

In the consortium: the cutoff value is larger

possibility to <u>free-ride</u>

Example 8.E.2

Bayesian NE: $s_i(\theta_i) = 1$ if i develops Z; $s_i(\theta_i) = 0$ if not

develop $\rightarrow$ θ_i^2 - c; not develop $\rightarrow$ θ_i^2 Prob $(s_j(\theta_j) = 1)$

i develops iff $\theta_i \geq (c / (1 - \text{Prob}(s_j(\theta_j) = 1))^{1/2}$ (cutoff rule)

$\theta_1^\wedge, \theta_2^\wedge$ cutoff values in Bayesian NE

$\text{Prob}(s_j(\theta_j) =1) = 1 - \theta_j^\wedge$

$\theta_i^\wedge = (c/\theta_j^\wedge)^{1/2}$, i.e., $\theta_i^{\wedge 2}\theta_j^\wedge = c$, similarly $\theta_i^\wedge\theta_j^{\wedge 2} = c$

Thus $\theta_1^\wedge = \theta_2^\wedge = c^{1/3}$

In the Bayesian NE,

Prob (both develop) $= (1 - c^{1/3})^2$

Prob (one develops) $= 2c^{1/3}(1-c^{1/3})$

Prob (neither develops) $= c^{2/3}$

Mixed Strategies and Bayesian NE

Complete information game with a unique NE

Introduce many different types of each player.

Realization of various players' types are statistically independent.

all types of a player have identical preferences.

A (pure strategy) Bayesian NE

= a mixed strategy NE of the original game

R. Gibbons, Game Theory for Applied Economists

(Japanese translation), pp.150-153

Types are correlated. → correlated equilibrium

Assignments

Problem Set 5 (due May 19)

Exercises (pp.262-266):

8.E.1 (Assume $s < M < w$.)

8.E.3

Reading Assignment:

Text, Chapter 8, pp.258-261