3.Plane Wave in Free Space

Starting from Maxwell's equation, we obtain set of differential equations similar to the transmission line equations.

$$rot\mathbf{E} = -j\omega\mathbf{B}$$
$$rot\mathbf{H} = j\omega\mathbf{D} + \mathbf{J}_c + \mathbf{J}_s$$
$$\mathbf{D} = \varepsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}, \mathbf{J}_c = \sigma\mathbf{E}$$

$$\Rightarrow$$

$$rot\mathbf{E} = -j\omega\mu\mathbf{H}$$
$$rot\mathbf{H} = j\omega\varepsilon\mathbf{E}$$
$$\rho = \mathbf{J}_s = 0$$

Assumption: time dependence exp(jωt)
propagation along z direction
no variation in transverse directions (x, y)
no E_y (E_x only)

Then, Maxwell's equations are reduced to

$$0 = -j\omega\mu H_x \qquad -\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x$$
$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y \qquad \frac{\partial H_x}{\partial z} = j\omega\varepsilon E_y = 0$$
$$0 = -j\omega\mu H_z \qquad 0$$

$$\Rightarrow \quad H_x = H_z = 0$$

Plane Wave in Free Space

plane wave

$$\frac{dE_x}{dz} = -j\omega\mu H_y = -(j\omega\mu' + \omega\mu'')H_y$$

$$\frac{dH_y}{dz} = -j\omega\varepsilon E_x = -(j\omega\varepsilon' + \omega\varepsilon'')E_x$$

transmission line

$$\frac{dV(z)}{dz} = -Z_d I(z) \qquad (Z_d = R + j\omega L)$$

$$\frac{dI(z)}{dz} = -Y_d V(z) \qquad (Y_d = G + j\omega C)$$

from the similarity to the transmission line equation

$$E_x = E_{x1}e^{-\gamma z} + E_{x2}e^{+\gamma z}$$

$$H_y = \frac{1}{Z_c}(E_{x1}e^{-\gamma z} - E_{x2}e^{+\gamma z}) \qquad \text{with} \qquad \gamma = j\omega\sqrt{\varepsilon\mu}, \quad Z_c = \sqrt{\frac{\mu}{\varepsilon}}$$

+z propagating wave:

$$E_x^+ = E_{x1}e^{j(\omega t - \beta z)}$$

$$H_y^+ = \frac{1}{Z_c}E_{x1}e^{j(\omega t - \beta z)}$$

wave impedance: $\frac{E_x^+}{H_y^+} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$

phase velocity: $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\varepsilon\mu}}$

4.Transmission Line Composed of Two Conductors

4.1 Electro-static model

The electromagnetic (EM) wave of TEM mode can be determined by an electro-static model.

$$\mathbf{E} = \mathbf{E}_t(x, y)e^{-\gamma z}$$

$$\mathbf{H} = \mathbf{H}_t(x, y)e^{-\gamma z}$$

Use $\nabla = \nabla_t + \mathbf{k}\dfrac{\partial}{\partial z} = \nabla_t - \gamma\mathbf{k}$ in $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$

then, $(\nabla_t - \gamma\mathbf{k}) \times \mathbf{E}_t e^{-\gamma z} = -j\omega\mu\mathbf{H}_t e^{-\gamma z}$ $\Rightarrow$

$$\nabla_t \times \mathbf{E}_t = 0 \rightarrow \boxed{\mathbf{E}_t = -\nabla_t \phi_e}$$

$$-\gamma\mathbf{k} \times \mathbf{E}_t = -j\omega\mu\mathbf{H}_t$$

$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$ $\Rightarrow$

$$\nabla_t \times \mathbf{H}_t = 0 \rightarrow \boxed{\mathbf{H}_t = -\nabla_t \phi_m}$$

$$-\gamma\mathbf{k} \times \mathbf{H}_t = j\omega\varepsilon\mathbf{E}_t$$

$$\downarrow$$

$$-\gamma\mathbf{k} \times (\gamma\mathbf{k} \times \mathbf{E}_t) = -\omega^2\varepsilon\mu\mathbf{E}_t$$

$$\downarrow$$

$$\boxed{\therefore \gamma^2 = -\omega^2\varepsilon\mu}$$

Transmission line with two conductors

$$\mathbf{E}_t = -\nabla_t \phi_e$$
$$div\mathbf{E} = 0$$
$$\longrightarrow \boxed{\nabla_t^2 \phi_e = 0}$$

$$-\gamma\mathbf{k} \times \mathbf{E}_t = -j\omega\mu\mathbf{H}_t \longrightarrow \mathbf{H}_t = \frac{\gamma\mathbf{k} \times \mathbf{E}_t}{j\omega\mu}$$

$$div\,\mathbf{H} = div\,\mathbf{H}_\mathbf{t} = \nabla \cdot \left(\frac{\gamma}{\mathrm{j}\,\omega\mu} \mathbf{k} \times \mathbf{E}_\mathbf{t} \right)$$
$$= \frac{\gamma}{\mathrm{j}\,\omega\mu} \left(\mathbf{E}_\mathbf{t} \cdot \nabla \times \mathbf{k} - \mathbf{k} \cdot \nabla \times \mathbf{E}_\mathbf{t} \right) = \frac{-\gamma}{\mathrm{j}\,\omega\mu} \mathbf{k} \cdot \nabla \times \mathbf{E}_\mathbf{t} = 0$$

$$\boxed{\begin{aligned} &\mathbf{E} = \mathbf{E}_t(x, y)e^{-\gamma z} \\ &\mathbf{H} = \mathbf{H}_t(x, y)e^{-\gamma z} \\ &\qquad \therefore \gamma^2 = -\omega^2 \varepsilon\mu \end{aligned}}$$

Coaxial line and strip line

4.2 Coaxial line

charge density per unit length=q[Cm^{-1}]

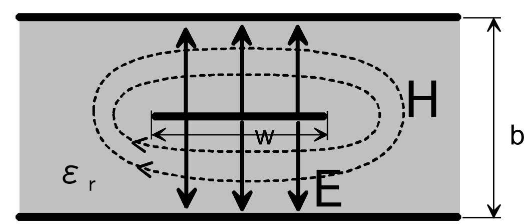
→voltage between inner and outer conductors=V

→capacitance C[Fm^{-1}]

→characteristic impedance $Z_c = \dfrac{V}{I} = \dfrac{\dfrac{q}{C}}{\dfrac{q}{\sqrt{\varepsilon\mu}}} = \dfrac{\sqrt{\varepsilon\mu}}{C}$

4.3 Strip line

Dominant guided mode is a TEM mode.



$$C = 2 \times \varepsilon \frac{w}{\frac{b}{2}} = \frac{4\varepsilon w}{b}$$

$$Z_c = \frac{b}{4w}\sqrt{\frac{\mu}{\varepsilon}}$$

Micro-strip line

4.4 Micro-strip line

a quasi TEM mode

approximate representation of characteristic impedance

0-th order approximation

$$C = \varepsilon \frac{w}{h} \quad \rightarrow \quad \mathrm{Z_c} = \sqrt{\frac{\mu}{\varepsilon}} \frac{\mathrm{h}}{\mathrm{w}}$$

1-st order approximation (Schneider's expression)

$$\varepsilon_{eff} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2} \left(1 + \frac{10h}{w} \right)^{-\frac{1}{2}}$$

$$\beta = \sqrt{\varepsilon_{eff}}\, \beta_0 \quad (\beta_0 = \omega \sqrt{\varepsilon_0 \mu_0}), \quad Z_c = \frac{1}{\sqrt{\varepsilon_{eff}}} Z_{c0}$$

$$Z_{c0} = 60 \ln \left(f \frac{h}{w_0} + \sqrt{1 + \left(\frac{2h}{w_0} \right)^2} \right), \quad f = 6 + 0.283 \exp \left[- \left(30.7 \frac{h}{w_0} \right)^{0.753} \right]$$

$$w_0 = w + \Delta w = w + \frac{t}{\pi} \ln \left(4e / \sqrt{\left(\frac{t}{h} \right)^2 + \frac{1}{\pi^2 (w/t + 1.1)^2}} \right)$$

5.1 Rectangular waveguide

EM field of TE and TM mode

orthogonal relation among eigen modes

$$\gamma E_y = -j\omega\mu H_x \qquad (5.1.a) \qquad\qquad \frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x \qquad (5.1.d)$$

$$-\gamma E_x = -j\omega\mu H_y \qquad (5.1.b) \qquad\qquad -\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \qquad (5.1.e)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad (5.1.c) \qquad\qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \qquad (5.1.f)$$

$$E_x = \frac{-j\omega\mu}{\gamma^2 + \omega^2\varepsilon\mu}\frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial y} \qquad\qquad H_x = \frac{-\gamma}{\beta_c^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x} \qquad\qquad H_y = \frac{-\gamma}{\beta_c^2}\frac{\partial H_z}{\partial y}$$

Rectangular waveguide

$$\frac{j\omega\mu}{\beta_c^2}\frac{\partial^2 H_z}{\partial x^2}+\frac{j\omega\mu}{\beta_c^2}\frac{\partial^2 H_z}{\partial y^2}=-j\omega\mu H_z$$

$$\therefore \frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}=-\beta_c^2 H_z$$

$$\frac{1}{X(x)}\frac{d^2 X}{dx^2}+\frac{1}{Y(y)}\frac{d^2 Y}{dy^2}=-\beta_c^2$$

$$\frac{1}{X(x)}\frac{d^2 X}{dx^2}=-\beta_x^2$$

$$\frac{1}{Y(y)}\frac{d^2 Y}{dy^2}=-\beta_y^2$$

$$\beta_x^2+\beta_y^2=\beta_c^2$$

$$X(x)=A\sin\beta_x x+B\cos\beta_x x$$

$$Y(y)=C\sin\beta_y y+D\cos\beta_y y$$

Rectangular waveguide

$$H_z = X(x)Y(y)$$

$$X(x) = A\sin\beta_x x + B\cos\beta_x x$$

$$Y(y) = C\sin\beta_y y + D\cos\beta_y y$$

boundary condition at x=0, a

$$E_y = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{\beta_c^2}\beta_x(A\cos\beta_x x - B\sin\beta_x x)Y(y)$$

$$A = 0 \qquad (E_y = 0 \quad at \quad x = 0)$$

$$\sin\beta_x a = 0 \qquad (E_y = 0 \quad at \quad x = a)$$

$$\therefore \beta_x = \frac{m\pi}{a}$$

boundary condition for E_x at y=0,b

$$C = 0$$

$$\sin\beta_y b = 0 \qquad \therefore \beta_y = \frac{n\pi}{b}$$

Rectangular waveguide

Field components are determined as

$$H_z = H_0 \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$E_x = \frac{j\omega\mu}{\beta_c^2} H_0 \frac{n\pi}{b} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

$$E_y = \frac{-j\omega\mu}{\beta_c^2} H_0 \frac{m\pi}{a} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_x = \frac{\gamma}{\beta_c^2} H_0 \frac{m\pi}{a} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_y = \frac{\gamma}{\beta_c^2} H_0 \frac{n\pi}{b} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

$$\beta_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 + \omega^2\varepsilon\mu$$

$$\therefore \gamma = \pm\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\varepsilon\mu}$$

Rectangular waveguide

TM mode

$$E_z = E_0 \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \qquad (m, n \geq 1)$$

$$E_x = \frac{-\gamma}{\beta_c^2} E_0 \frac{m\pi}{a}\cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

$$E_y = \frac{-\gamma}{\beta_c^2} E_0 \frac{n\pi}{b}\sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_x = \frac{j\omega\varepsilon}{\beta_c^2} E_0 \frac{n\pi}{b}\sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y$$

$$H_y = \frac{-j\omega\varepsilon}{\beta_c^2} E_0 \frac{m\pi}{a}\cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

$$\gamma = \pm\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\varepsilon\mu}$$

Rectangular waveguide: orthogonality

TE mode (between transverse field component)

$$E_t^e = \frac{j\omega\mu}{\beta_c^2}\left(-\frac{\partial H_z}{\partial y} i + \frac{\partial H_z}{\partial x} j\right)$$

$$H_t^e = \frac{-\gamma}{\beta_c^2}\left(\frac{\partial H_z}{\partial x} i + \frac{\partial H_z}{\partial y} j\right)$$

$$\therefore E_t^e \cdot H_t^e = \frac{-j\omega\mu\gamma}{\beta_c^4}\left(-\frac{\partial H_z}{\partial x}\frac{\partial H_z}{\partial y} + \frac{\partial H_z}{\partial x}\frac{\partial H_z}{\partial y}\right) = 0$$

TM mode (between transverse field component)

$$E_t^m = \frac{-\gamma}{\beta_c^2}\left(\frac{\partial E_z}{\partial x} i + \frac{\partial E_z}{\partial y} j\right)$$

$$H_t^m = \frac{j\omega\varepsilon}{\beta_c^2}\left(\frac{\partial E_z}{\partial y} i - \frac{\partial E_z}{\partial x} j\right)$$

$$\therefore E_t^m \cdot H_t^m = \frac{-j\omega\varepsilon\gamma}{\beta_c^4}\left(\frac{\partial E_z}{\partial x}\frac{\partial E_z}{\partial y} - \frac{\partial E_z}{\partial x}\frac{\partial E_z}{\partial y}\right) = 0$$

5.2 Circular waveguide

$$\nabla \times H = i_r\left(\frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z}\right) + i_\theta\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right) + i_z\frac{1}{r}\left(\frac{\partial (rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)$$

TE mode

$$H_z = H_0 J_n(\beta_c r)\cos n\theta$$

$$E_r = \frac{-j\omega\mu}{\beta_c^2}\frac{1}{r}\frac{\partial H_z}{\partial \theta}$$

$$E_\theta = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial r} \longrightarrow E_\theta = \frac{j\omega\mu}{\beta_c^2}\beta_c H_0 J_n'(\beta_c r)\cos n\theta$$

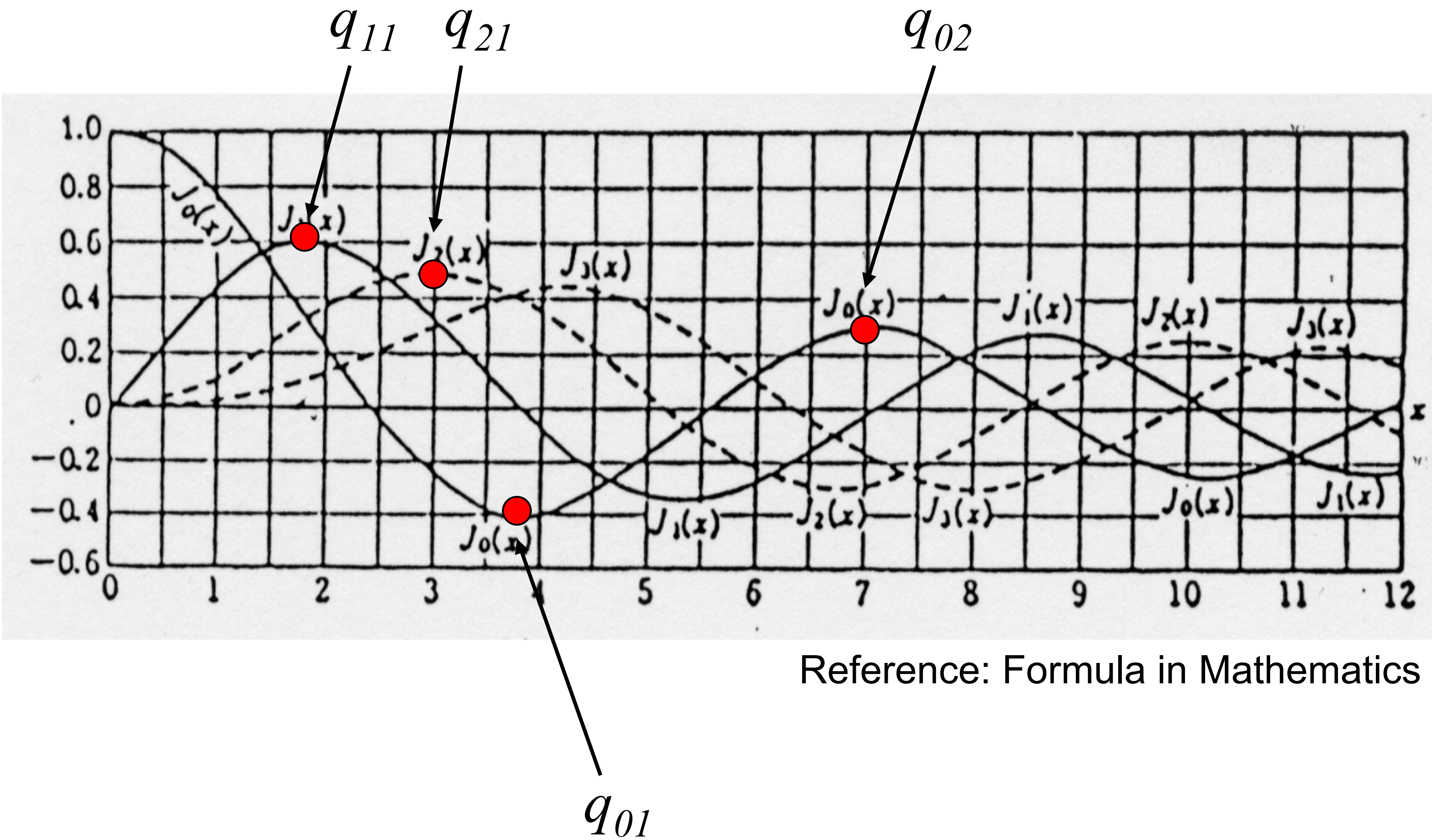
$$H_r = \frac{-\gamma}{\beta_c^2}\frac{\partial H_z}{\partial r}$$

$$H_\theta = \frac{-\gamma}{\beta_c^2}\frac{1}{r}\frac{\partial H_z}{\partial \theta}$$

$$(\beta_c^2 = \gamma^2 + \omega^2\varepsilon\mu)$$

$$\boxed{J_n'(\beta_c a) = 0 \rightarrow q_{nm} \qquad \beta_c = \frac{q_{nm}}{a}, \qquad \gamma = \pm\sqrt{\left(\frac{q_{nm}}{a}\right)^2 - \omega^2\varepsilon\mu}}$$

Bessel function $J_n(x)$



Reference: Formula in Mathematics

Circular waveguide : TM mode

TM mode

$$E_z = E_0 J_n(\beta_c r)\cos n\theta$$

$$E_r = \frac{-\gamma}{\beta_c^2}\frac{\partial E_z}{\partial r}$$

$$E_\theta = \frac{-\gamma}{\beta_c^2}\frac{1}{r}\frac{\partial E_z}{\partial \theta} \longrightarrow E_\theta = \frac{\gamma}{\beta_c^2}\frac{n}{r}E_0 J_n(\beta_c r)\sin n\theta$$

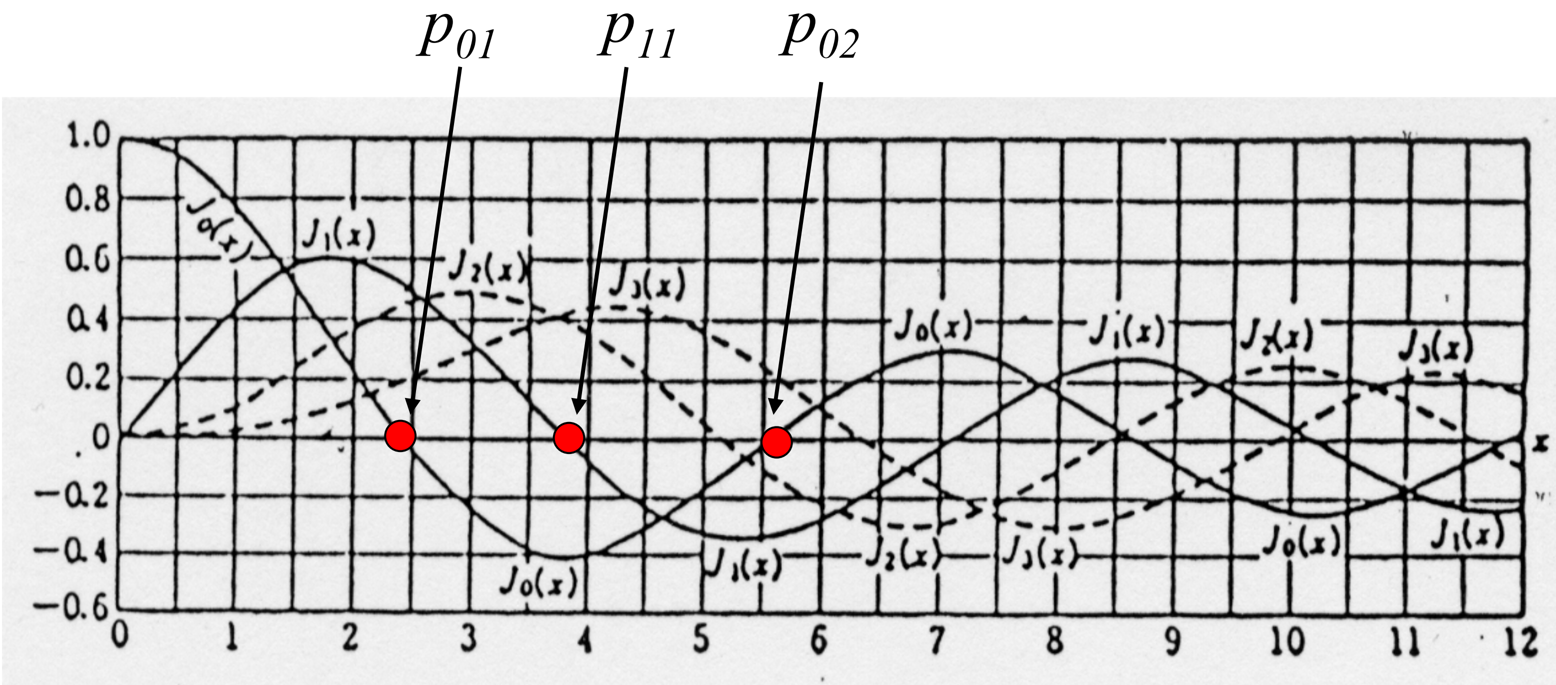
$$H_r = \frac{j\omega\varepsilon}{\beta_c^2}\frac{1}{r}\frac{\partial E_z}{\partial \theta}$$

$$H_\theta = \frac{-j\omega\varepsilon}{\beta_c^2}\frac{\partial E_z}{\partial r}$$

$$(\beta_c^2 = \gamma^2 + \omega^2\varepsilon\mu)$$

$$\boxed{J_n(\beta_c a) = 0 \rightarrow p_{nm} \qquad \beta_c = \frac{p_{nm}}{a}, \quad \gamma = \pm\sqrt{\left(\frac{p_{nm}}{a}\right)^2 - \omega^2\varepsilon\mu}}$$

Bessel function $J_n(x)$



Reference: Formula in Mathematics

5.3 Cut-off

cut-off frequency:

- The frequency at which the propagation constant of the mode concerned γ becomes 0.
- The mode is evanescent below that frequency.

Rectangular (a=2b): listed in the order of lower cut-off frequency

TE_{10}

TE_{20}, TE_{01} (same cut-off frequency)

TE_{11}, TM_{11}

TE_{21}, TM_{21} ......

$$\gamma = \pm\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\varepsilon\mu}$$

Circular : listed in the order of lower cut-off frequency

TE_{11}

TM_{01}

TE_{21}

TE_{01}, TM_{11}, ...

$$\mathrm{TE-mode:}\quad \gamma = \pm\sqrt{\left(\frac{q_{nm}}{a}\right)^2 - \omega^2\varepsilon\mu}$$

$$\mathrm{TM-mode:}\quad \gamma = \pm\sqrt{\left(\frac{p_{nm}}{a}\right)^2 - \omega^2\varepsilon\mu}$$