EXTENSIVE GAMES WITH PERFECT INFORMATION AND BACKWARD INDUCTION

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For the mathematical discussion regarding trees, refer to the mathematical note on Decision Analysis.

1. Extensive Games with Perfect Information

The notations in this note are largely based on Osborne and Rubinstein[1].

Definition 1 (Extensive Game with Perfect Information)**.** An extensive game with perfect information is a structure $\langle \mathcal{K}, P, u \rangle$ where:

- $\mathcal{K} = \langle H, A \rangle$ is a tree with:
  - H a set of histories, and
  - A a set of actions
- $P : H \setminus Z \to N$ is a **player function** assigning to each non-terminal history $h \in H \setminus Z$ a player $P(h)$ who takes an action after the history h.
- $u_i : Z \to \Re$ is a utility function for player $i \in N$.

Since an extensive game with perfect information is a natural generalization of a decision tree, it is often called a game tree.

The set of decision nodes for player $i \in N$ is defined by $H_i = P^{-1}(\{i\})$.

A strategy of a player in an extensive game is a plan that specifies the action chosen by the player at *every* decision node of hers.

Definition 2 (Strategy)**.** Player i's strategy is a function $s_i : H_i \to A$ that satisfies $(\forall h \in H_i))s_i(h) \in A(h)$.

Denote S_i the set of all strategies of player i.

A strategy profile specifies the action chosen by the players *even for histories that are never reached* by the actual play.

When $s \in S$ is given, a unique outcome is well-defined as the result of each player i following the precepts of s_i.

Definition 3 (Outcome)**.** Outcome $O(s) \in Z$ of a strategy profile $s \in S$ is the sequence $(a^1, ..., a^K)$ defined as follows:

- $a^1 = s_{P(h^t)}(\varnothing)$
- For $1 \leq t < K$, $a^{t+1} = s_{P(a^1,...,a^t)}(a^1, ..., a^t)$

Definition 4 (Normal-Form Representation)**.** The normal form of the extensive game with perfect information $\Gamma = \langle \mathcal{K}, P, u \rangle$ is the normal form game $\langle N, S, u' \rangle$ in which for $\forall i \in N$:

- S_i is the set of strategies of player i in Γ, and
- $(\forall s \in S) u'_i(s) = u_i(O(s))$

Definition 5 (Nash Equilibrium)**.** $s^* \in S$ of an extensive game with perfect information $\Gamma = \langle \mathcal{K}, P, u \rangle$ is a Nash equilibrium iff it is a Nash equilibrium of the normal form game derived from Γ.

Date: May 16, 2011.

Definition 6 (Backwards Induction)**.** A backward induction solution $s^* \in S$ of a game tree $\Gamma = \langle \mathcal{K}, P, u \rangle$ is obtained by the following algorithm:

- Start from the terminal node. For $\forall z \in Z$, let $U(z) := u(z)$.
- For each non-terminal history $h \in H \setminus Z$, when U of all the direct successors are determined, $U(h) := U(h, s^*(h))$ such that $s^*(h) \in \arg\max_{a \in A(h)} U_{P(h)}(h, a)$.

Definition 7 (Extensive Game with Perfect Information and Chance Moves)**.** An extensive game with perfect information and chance moves is a structure $\Gamma = \langle \mathcal{K}, P, f_c, u \rangle$ where:

- $\mathcal{K} = \langle H, A \rangle$ is a tree,
- $P : H \setminus Z \to N \oplus \{c\}$ is a player function (player c is called *chance* or *nature*),
- $f_c(\cdot|h) \in \Delta(A(h))$ is a probability measure on $A(h)$ for each chance node $h \in P^{-1}(\{c\})$,
- $u_i : Z \to \Re$ is a utility function for player $i \in N$.

Definition 8 (Backwards Induction with Chance Moves))**.** A backward induction solution $s^* \in S$ of a game tree $\Gamma = \langle \mathcal{K}, P, f_c, u \rangle$ is obtained by the following algorithm:

- Start from the terminal node. For $\forall h \in Z$, let $U(h) := u(h)$.
- For each non-terminal node $h \in H \setminus A$, when U of all the direct successors are determined:
  - For $h \in P^{-1}(\{c\})$, $U(h) := \sum_{\omega \in A(h)} f_c(\omega|h) U(h, \omega)$
  - For $h \in P^{-1}(N)$, $U(h) := U(h, s^*(h))$ such that $s^*(h) \in \arg\max_{a \in A(h)} U_{P(h)}(h, a)$

2. The Extension with Simultaneous Moves and Subgame-Perfect Nash Equilibria (SPNE)

Definition 9 (Extensive Game with Perfect Information and Simultaneous Moves)**.** An extensive game with perfect information is a structure $\Gamma = \langle \mathcal{K}, P, u \rangle$ where:

- $\mathcal{K} = \langle H, A \rangle$ is a tree,
- $P : H \setminus Z \to \mathcal{P}(N)$ is a function assigning *a subset* of players who take an action after the history h,
- A collection of action spaces $\{A_i(h)\}_{i \in P(h)}$ for which $A(h) = \times_{i \in P(h)} A_i(h)$,
- $u_i : Z \to \Re$ is a utility function for player i.

Players in $P(h)$ play simultaneously at h. $a \in A(h)$ now becomes an action profile. $h \in H$ can be represented likewise by a sequence of action profiles.

The decision nodes of player $i \in N$ is now defined as $H_i = \{h \in H | i \in P(h)\}$. Notice that H_i and H_j for two players i and j are no longer disjoint.

Definition 10 (Strategy)**.** Player i's strategy is a function that assigns an action in $A_i(h)$ to every $h \in H_i$.

Backwards induction solution is named *subgame-perfect Nash equilibrium (SPNE)* and is obtained by the following algorithm, replacing utility maximization at each node by Nash equilibrium at each node.

Definition 11 (Subgame-Perfect Nash Equilibrium (SPNE))**.** A subgame-perfect Nash equilibrium (SPNE) $s^* \in S$ of a game tree $\Gamma = \langle \mathcal{K}, P, u \rangle$ is obtained by the following algorithm:

- Start from the terminal node. For $\forall z \in Z$, let $U(z) := u(z)$.
- For each non-terminal history $h \in H \setminus Z$, when U of all the direct successors are determined, define a normal-form game $\langle N, A(h), \tilde{U} \rangle$ by $\tilde{U}(a) = U(h, a)$ for all $a \in A(h)$.
- Let $s^*(h) \in A(h)$ be a Nash equilibrium of $\langle N, A(h), \tilde{U} \rangle$ and $U(h) := \tilde{U}(s^*(h))$.

References

[1] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory.* MIT Press, Cambridge, 1994.

VALDES, Tokyo Institute of Technology