DECISION ANALYSIS

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1. Decision Problem and Value of Information

Definition 1 ((Prototype) Decision Problem)**.** A decision problem is a structure $\langle A, \Omega, u \rangle$, where:

- A is a set of alternatives
- Ω is a state space of uncertainty
- $p \in \Delta(\Omega)$ is a probability distribution on Ω
- $u : A \times \Omega \to \Re$ is a utility function

Definition 2 (Expected Utility)**.** Expected utility of alternative $a \in A$ is

$$\sum_{\omega \in \Omega} p(\omega) u(a, \omega)$$

Definition 3 (Expected Value of Perfect Information (EVPI))**.** The expected value of obtaining perfect information regarding ω is

$$EVPI = \sum_{\omega \in \Omega} p(\omega) \max_{a \in A} u(a, \omega) - \max_{a \in A} \sum_{\omega \in \Omega} p(\omega) u(a, \omega)$$

Theorem 4 (Knowledge is Power)**.** *$EVPI \geq 0$*

After obtaining information, you may change your choice flexibly according to your knowledge regarding ω, which leads to $EVPI \geq 0$.

There are cases in which $EVPI = 0$.

Definition 5 (Irrelevance)**.** Uncertainty variable Ω is irrelevant to the decision iff $\forall a \in A, \forall \omega, \omega' \in \Omega$

$$u(a, \omega) = u(a, \omega')$$

Definition 6 (Dominance)**.** $a \in A$ dominates $a' \in A$ iff $\forall \omega \in \Omega$

$$u(a, \omega) \geq u(a', \omega)$$

$a \in A$ is dominant iff $(\forall a' \in A)$ a dominates a'.

2. Tree

Definition 7 (Directed Graph)**.** A directed graph is a structure $\mathcal{K} = \langle V, A \rangle$ with :

- a set of *nodes* (or *vertices*) V and
- a set of *arcs* (or *directed edges*, *arrows*) $A \subset V^2$

Definition 8 (Direct Predecessor, Direct Successor)**.** For an arc $(x, y) \in A$ of a directed graph $\mathcal{K} = \langle V, A \rangle$:

- $x \in V$ is a direct predecessor of $y \in V$ and
- y is a direct successor of x.

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Definition 9 (Path)**.** A (finite) path (of a directed graph) from $x \in V$ to $y \in V$ is a sequence of nodes such that from each of its nodes there is an arc to the next node in the sequence.

Definition 10 (Predecessor, Successor)**.** When there exists a path from node $x \in V$ to node $y \in V$, then x is said to be a predecessor of y and y is said to be a successor of x.

Definition 11 ((Rooted) Tree)**.** A rooted tree is a directed graph $\mathcal{K} = \langle V, A \rangle$ where:

- there exists a unique *root* $v^0 \in V$ for which $\neg(\exists v \in V)(v, v^0) \in A$,
  (= there exists no direct predecessor for v^0)
- $(\forall y \in V \setminus \{v^0\} \exists! x \in V)(v, w) \in A$ and
  (= there exists a *direct predecessor function* $p : V \setminus \{v^0\} \to V$)
- every node in $V \setminus \{v^0\}$ is a successor of v^0.

Using notation p, if x is a predecessor of y, then $(\exists k \in \mathbb{N}) p^k(y) = x$.

Proposition 12. Let $\mathcal{K} = \langle V, A \rangle$ be a rooted tree. Denote $x < y$ when x is a predecessor of y. Then:

K1: $<$ is a strict partial order, that is:
irreflexivity: $(\forall x \in V) x \not< x$
transitivity: $(\forall x, y, z \in V) x < y \wedge y < z \Rightarrow x < z$
K2: $<$ is a strict linear order on the set $\{v' \in V | v' < v\}$ for $\forall v \in V \setminus \{v^0\}$, that is:
completeness: $(\forall x, y \in \{v' \in V | v' < v\}) x \neq y \Rightarrow x < y \vee y < x$
K3: v^0 is the smallest with respect to $<$.

Proof. First, since v^0 is the root, it does not have a predecessor.
(K1) Suppose $(\exists v \in V) v < v$. First, $v \neq v^0$ since v^0 does not have a predecessor. Next, assume $v \in V \setminus \{v^0\}$. By definition, $(\exists l \in \mathbb{N}) v = p^l(v)$. Since v is a successor of v^0,there exists $(\exists k \in \mathbb{N}) p^k(v) = v^0$. If $k = l$, then $v = v^o$, which is contradiction. If $k \leq l$, then $v = p^{l-k}(v_0)$, which is contradictory to $\forall v \in V, v \not< v^0$. If $k > l$, then $p^k(v) = p^{k-l}(p^l(v)) = p^{k-l}(v)$. Repeat the procedure recursively until the case falls in $k \leq l$.
(K2) Assume $x < y$ and $y < z$. By definition, $\exists k, l \in \mathbb{N}$ such that $x = p^k(y)$ and $y = p^l(z)$. For this k and l, $x = p^{k+l}(z)$, which implies $x < z$.
(K3) Take any $v, v' \in \{x \in V | x < y\}$. Then, there exist k, l such that $v = p^k(y)$ and $v' = p^l(y)$. Thus, if $k = l$,then $v = v'$. If $k > l (l < k)$, then $v = p^{k-l}(v')(v' =^{k-l} (v))$ and thus $v < v' (v > v')$. □

Definition 13 (Terminal Node)**.** A node that has no predecessor is called a terminal node. The set of terminal nodes Z of a tree $\mathcal{K} = \langle V, A \rangle$ is given by

$$Z = \{v \in V | \neg(\exists v' \in V)(v, v') \in A\}$$

3. Decision Tree

Henceforth, the terminology particularly useful for decision analysis and game theory is defined on a tree $\mathcal{K} = \langle V, A \rangle$.

Definition 14 (Action)**.** An arc in called an action in game theory. For non-terminal nodes $v \in V \setminus Z$, denote $A(v) = \{(v, v') \in A | v' \in V\}$ the set of actions available at v.

Corresponding to (K2) in Proposition 12, a unique path from the root to each node is well-defined. It is possible to represent a history by the sequence of actions from the root.

Definition 15 (History)**.** For each non-root node $v \in V \setminus \{v^0\}$, let $k \in \mathbb{N}$ be the number satisfying $v^0 = p^k(v)$. The history h is defined by:

- For $1 \leq l \leq k$, $a^l = (p^{k-l+1}(v), p^{k-l}(v)) \in A$.
- $h = h(v) = (a^l)_{l=1}^k$

For ease of notation, an empty sequence (called *initial history* or *null history*) $\varnothing$ that represents v_0 is defined, and together with $V \setminus \{v^0\}$, the set of all histories H corresponds to V. Henceforth, a tree is represented by $\langle H, A \rangle$.

Remark 16 (Redundancy). Due to the unique path condition, configurations that look identical with each other have to be represented separately if the paths reaching those configurations differ.

For example, it is possible that players follow different paths to reach a single configuration in chess.

A decision tree is a generalization of a decision problem.

Definition 17 (Decision Tree)**.** A decision tree is a structure $\Gamma = \langle \mathcal{K}, D, C, f, u \rangle$ where:

- $\mathcal{K} = \langle V, H \rangle$ is a tree,
- $D \oplus C = H \setminus Z$
  - D is a set of decision nodes
  - C is a set of chance nodes
- $f(\cdot|h) \in \Delta(A(h))$ is a probability distribution on the uncertainty space on the chance node $h \in C$ and
- $u : Z \to \Re$ is a utility function

Definition 18 (Backwards Induction)**.** Fold back a decision tree recursively by the following algorithm.

- Start from the terminal node. For $\forall h \in Z$, let $U(h) := u(h)$.
- For each non-terminal node $h \in H \setminus A$, when U of all the direct successors are determined:
  - For $h \in D$, $U(h) := \max_{a \in A(h)} U((h, a))$
  - For $h \in C$, $U(h) := \sum_{\omega \in A(h)} f(\omega|h) U((h, \omega))$