Structural Dynamics 構造動力学 (9)

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9.6 Equations of Motion of MDOF System Subjected to Earthquake Ground Motion (地震 動を受ける多自由度系の運動方程式)

• Based on Eq. (9.16), the equations of motion of a MDOF system subjected to a ground motion \ddot{u}_g are

$$[M]{\ddot{u}} + [K]{u} = -\ddot{u}_g[M]{I}$$
(9.16)

• Based on Eq. (9.42c), the generalized force for the r-th mode P_r^* is

$$P_r^* = \{\phi_r\}^T \{P\} = -\ddot{u}_g \{\phi_r\}^T [M] \{I\}$$
(9.64)

$$\begin{bmatrix} M \\ \{\ddot{u}\} + [K] \{u\} = \{P\} - \ddot{u}_g [M] \{I\}$$
(9.16)
$$P_r^* = \{\phi_r\}^T \{P\}$$
(9.42c)

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 Based on Eq. (9.60), the equation of motion for the r-th mode is written in the form of Eq. (9.62), and its solution is given by Eq. (9.63).

• Because it is likely for a MDOF system under a seismic ground motion that the initial velocity and displacements are zero, Eq. (9.63) becomes

$$q_{r} = \frac{1}{M_{r}^{*}\omega_{dr}} \int_{0}^{t} P_{r}^{*}(\tau) e^{-h\omega_{nr}(t-\tau)} \sin \omega_{dr}(t-\tau) d\tau \quad (9.65)$$

$$\left| q_{r}(t) = e^{-h_{r}\omega_{nr}t} \left\{ q_{r}(0)\cos\omega_{d}t + (\frac{\dot{q}_{r}(0)}{\omega_{dr}} + \frac{h_{r}\omega_{nr}q_{r}(0)}{\omega_{dr}})\sin\omega_{dr}t \right\} + \frac{1}{M_{r}^{*}\omega_{dr}} \int_{0}^{t} P_{r}^{*}(\tau)e^{-h\omega_{nr}(t-\tau)}\sin\omega_{dr}(t-\tau)d\tau \quad (9.63)$$

• Substitution of Eq. (9.64) into Eq. (9.65) leads to

$$q_r = -\frac{\beta_r}{\omega_{dr}} \int_0^t \ddot{u}_g(\tau) e^{-\xi_r \omega_{nr}(t-\tau)} \sin \omega_{dr}(t-\tau) d\tau$$
(9.66a)

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where

$$\beta_{r} = \frac{\{\phi_{r}\}^{T} [M]\{I\}}{M_{r}^{*}} = \frac{\{\phi_{r}\}^{T} [M]\{I\}}{\{\phi_{r}\}^{T} [M]\{\phi_{r}\}} = \frac{\sum m_{i}\phi_{ir}}{\sum i=1}$$
(9.66b)

in which β_r is called mode participation factor of the r-th mode (r次の刺激係数)

$$P_{r}^{*} = \{\phi_{r}\}^{T}\{P\} = -\ddot{u}_{g}\{\phi_{r}\}^{T}[M]\{I\}$$
(9.64)
$$q_{r} = \frac{1}{M_{r}^{*}\omega_{dr}} \int_{0}^{t} P_{r}^{*}(\tau) e^{-\xi_{r}\omega_{nr}(t-\tau)} \sin \omega_{dr}(t-\tau) d\tau (9.65)$$

• Denoting Eq. (9.66a) below

$$\widetilde{q}_r = -\frac{1}{\omega_{dr}} \int_0^t \ddot{u}_g(\tau) e^{-\xi_r \omega_{nr}(t-\tau)} \sin \omega_{dr}(t-\tau) d\tau$$
(9.66a)

represents solution of a SDOF system subjected to a ground motion $\ddot{u}_g(t)$, Eq. (9.66a) can be rewritten

$$q_r(t) = \beta_r \tilde{q}_r(t) \tag{9.67}$$

• It should be noted that $\tilde{q}_r(t)$ can be computed by the direct integration method (Newmark's β method)

• Eq. (9.67) shows that the mode participation factor of the r-th mode represents degree of contribution of the r-th mode to the total response. •When $|\beta_r|$ is large, the contribution of the r-th mode is predominant

• The generalize mass M_r^* (一般化された質量) given by Eq. (9.42a) is also important to show how the rth mode is predominant to the total response.

$$q_{r}(t) = \beta_{r} \tilde{q}_{r}(t)$$
(9.67)
$$M_{r}^{*} = \{\phi_{r}\}^{T} [M] \{\phi_{r}\}$$
(9.42a)

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• Substitution of Eq. (9.67) into Eqs. (9.38),
(9.39) and (9.48) leads to

$$\{u\} = [\Phi]\{q\} = \{\phi_1\}q_1 + \{\phi_2\}q_2 + \dots + \{\phi_r\}q_r + \dots + \{\phi_n\}q_n \\
\{\ddot{u}\} = [\Phi]\{\ddot{q}\} = \{\phi_1\}\ddot{q}_1 + \{\phi_2\}\ddot{q}_2 + \dots + \{\phi_r\}\ddot{q}_r + \dots + \{\phi_n\}\ddot{q}_n \\
(9.39) \\
\{\dot{u}\} = [\Phi]\{\dot{q}\} = \{\phi_1\}\dot{q}_1 + \{\phi_2\}\dot{q}_2 + \dots + \{\phi_r\}\dot{q}_r + \dots + \{\phi_n\}\dot{q}_n \\
(9.48) \\
q_r(t) = \beta_r \tilde{q}_r(t) \\
(9.67) \\$$

$\{u\} = [\Phi] \{q\} = \{\phi_1\} \beta_1 \tilde{q}_1 + \{\phi_2\} \beta_2 \tilde{q}_2 + \dots \{\phi_r\} \beta_r \tilde{q}_r + \dots \{\phi_n\} \beta_n \tilde{q}_n$ $\{\ddot{u}\} = [\Phi] \{\ddot{q}\} = \{\phi_1\} \beta_1 \tilde{\ddot{q}}_1 + \{\phi_2\} \beta_2 \tilde{\ddot{q}}_2 + \dots \{\phi_r\} \beta_r \tilde{\ddot{q}}_r + \dots \{\phi_n\} \beta_n \tilde{\ddot{q}}_n$ (9.68) (9.69) $\{\dot{u}\} = [\Phi] \{\dot{q}\} = \{\phi_1\} \beta_1 \tilde{\dot{q}}_1 + \{\phi_2\} \beta_2 \tilde{\dot{q}}_2 + \dots \{\phi_r\} \beta_r \tilde{\dot{q}}_r + \dots \{\phi_n\} \beta_n \tilde{\dot{q}}_n$ (9.70)

• It is noted that \tilde{q}_r , \tilde{q}_r and $\tilde{\ddot{q}}_r$ are the response displacement, velocity and acceleration of SDOF system with natural frequency ω_r and damping ratio ξ_r . •Hence, \tilde{q}_r , $\tilde{\dot{q}}_r$ and $\tilde{\ddot{q}}_r$ can be easily computed by Excel Sheet (LDRA-2) based on $_9$ Newmark β method • It should be noted that the acceleration response $\{\ddot{u}\}\$ by Eq. (9.69) is the relative acceleration. Because absolute acceleration $\{\ddot{u}_{abs}\}\$ is important for evaluation of the inertia force, $\{\ddot{u}_{abs}\}\$ has to be computed by

$$\{\ddot{u}_{abs}\} = \{\ddot{u}\} + \{I\}\ddot{u}_g \tag{9.71}$$

$$\{\ddot{u}\} = [\Phi]\{\ddot{q}\} = \{\phi_1\}\beta_1\tilde{q}_1 + \{\phi_2\}\beta_2\tilde{q}_2 + \cdots \{\phi_r\}\beta_r\tilde{q}_r + \cdots \{\phi_n\}\beta_n\tilde{q}_n$$
(9.69)



Example 9.5 Evaluate the mode participation factor for a 3 DOFS structure shown





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• The characteristic equation is

$$\begin{vmatrix} k_1 - \omega^2 m & -k_1 & 0 \\ -k_1 & k_1 + k_2 - \omega^2 m & -k_2 \\ 0 & -k_2 & k_2 + k_3 - \omega^2 m \end{vmatrix} = 0$$

• Hence

 $k - \omega^2 150/9.8$ -k \cap $-k \qquad 2k - \omega^2 150/9.8$ -k= 0 $-k \qquad 2k - \omega^2 150/9.8$ ()

Natural periods and natural mode shapes are

 $\omega_1 = 6.28 rad / s$ $T_1 = 1.0s$ $\omega_2 = 17.6 rad / s$ $T_2 = 0.357 s$ $\omega_3 = 25.4 rad / s$ $T_3 = 0.247 s$





• Modal matrix $\begin{bmatrix} \Phi \end{bmatrix} = \begin{bmatrix} 1.00 & -0.802 & -0.445 \\ 0.802 & 0.445 & 1.00 \\ 0.445 & 1.00 & -0.802 \end{bmatrix}$

Check of the orthogonal condition

$$[\Phi]^{T}[M][\Phi] = [\Phi]^{T} m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\Phi]$$

$$= 1.841m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Generalized mass

 $M_1^* = M_2^* = M_3^* = 1.841m = 1.841 \times 150/9.8 = 28.18$

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Check of the orthogonal condition

$$\begin{bmatrix} \Phi \end{bmatrix}^{T} \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} \Phi \end{bmatrix} = \begin{bmatrix} \Phi \end{bmatrix}^{T} k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \Phi \end{bmatrix}$$
$$= k \begin{bmatrix} 0.365 & 0 & 0 \\ 0 & 2.863 & 0 \\ 0 & 0 & 5.978 \end{bmatrix}$$

• Generalized stiffness $K_1^* = 0.365k = 1,113kN / m$ $K_2^* = 2.863k = 8,735kN / m$ $K_3^* = 5.978k = 18,238kN / m$

• Check of Eq. (9.44)

$$\omega_{1} = \sqrt{\frac{K_{1}^{*}}{M_{1}^{*}}} = \sqrt{\frac{1113}{28.79}} = 6.28 rad / s \iff 6.28 rad / s$$

$$OK$$

$$\omega_{2} = \sqrt{\frac{K_{2}^{*}}{M_{2}^{*}}} = \sqrt{\frac{8735}{28.79}} = 17.6 rad / s \iff 17.4 rad / s$$

$$OK$$

$$\omega_3 = \sqrt{\frac{K_3^*}{M_3^*}} = \sqrt{\frac{18238}{28.79}} = 25.4/s \quad \longleftrightarrow \quad 25.4 \text{ rad/s}$$
 OK

• Compute mode participation factor $\beta_{r} = \frac{\{\phi_{r}\}^{T} [M]\{I\}}{M_{r}^{*}} = \frac{\{\phi_{r}\}^{T} [M]\{I\}}{\{\phi_{r}\}^{T} [M]\{\phi_{r}\}} = \frac{\sum_{i=1}^{n} m_{i}\phi_{ir}}{\sum_{i=1}^{n} m_{i}\phi_{ir}^{2}}$ (9.66b) i=1 $\beta_1 = \{\phi_1\}^T [M] \{I\} / M_1^*$ $= \{1.0 \quad 0.802 \quad 0.445\} \times m \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} / M_{1}^{*}$ 2.247*m*

$$\beta_{2} = \{\phi_{2}\}^{T} [M] \{I\} / M_{2}^{*}$$

$$= \{-0.802 \quad 0.445 \quad 1.00\} \times m \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} / M_{2}^{*}$$

$$= \frac{0.643m}{1.841m} = 0.349$$

$$\beta_{3} = \{\phi_{3}\}^{T} [M] \{I\} / M_{3}^{*}$$

$$= \{-0.445 \quad 1.00 \quad -0.802\} \times m \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} / M_{3}^{*}$$

$$= \frac{-0.247m}{1.841m} = -0.134$$

Example 9.6 Compute response displacement, velocity and acceleration (absolute acceleration) of the 3 DOFS structure of Example 9.5 under JR JMA Kobe ground motion. Assume that damping ratio is 0.05 for the three modes

 Compute response displacement, velocity and acceleration (absolute acceleration) of 3 DOF system based on Eqs.(9.68), (9.69), (9.70), and (9.71) using LDRA-2 as follows.

 The mode participation factors are already computed in Example 9.5.

• Compute \tilde{q}_r , \tilde{q}_r , \tilde{q}_r , \tilde{q}_r using LDRA-2. Note that mass M_r^* , natural period $T_r = 2\pi/\omega_r$ and damping ratio ξ_r have to be set in LDRA-2.



$$q_r(t) = \beta_r \tilde{q}(t)$$
$$\dot{q}_r(t) = \beta_r \tilde{\dot{q}}(t)$$
$$\ddot{q}_r(t) = \beta_r \tilde{\dot{q}}(t)$$

$$\{u(t)\} = \{u_1(t)\} + \{u_2(t)\} + \{u_3(t)\}$$

$$= \{\phi_1\}q_1(t) + \{\phi_2\}q_2(t) + \{\phi_3\}q_3(t)$$

$$\{\dot{u}(t)\} = \{\dot{u}_1(t)\} + \{\dot{u}_2(t)\} + \{\dot{u}_2(t)\}$$

$$= \{\phi_1\}\dot{q}_1(t) + \{\phi_2\}\dot{q}_2(t) + \{\phi_3\}\dot{q}_3(t)$$

$$\{\ddot{u}(t)\} = \{\ddot{u}_1(t)\} + \{\ddot{u}_2(t)\} + \{\ddot{u}_2(t)\}$$

$$= \{\phi_1\}\ddot{q}_1(t) + \{\phi_2\}\ddot{q}_2(t) + \{\phi_3\}\ddot{q}_3(t)$$

• Compute absolute acceleration by

$$\{\ddot{u}_{abs}\} = \{\ddot{u}\} + \{I\}\ddot{u}_g \qquad (9.71)$$

• Consider a 3 DOF structure shown below.

m m k_1 m k_2 k_3

m = 150kN / g = 150/9.8 $k_1 = k_2 = k_3 = k = 3050.9kN / m$







Peak Response Displacements and Time when they Occur

Mode	1st Mode	2nd Mode	3rd mode	Total
Mass 1	0.462m	0.0234m	-0.00133m	0.465m
	(5.18s)	(7.35s)	(8.54s)	(5.18s)
Mass 2	0.370m	-0.0130m	0.00298m	0.369m
	(5.18s)	(7.35s)	(8.54s)	(5.18s)
Mass 3	0.205m	-0.0292m	-0.00239m	0.200m
	(5.18s)	(7.35s)	(8.54s)	(5.18s)

Peak Response Velocity and Time when they Occur

Mode	1st Mode	2nd Mode	3rd mode	Total
Mass 1	2.99m/s	-0.388m/s	-0.0270m/s	3.00m/s
	(4.96s)	(7.44s)	(8.47s)	(4.96s)
Mass 2	2.40m/s	0.215m/s	0.0607m/s	2.38m/s
	(4.96s)	(7.44s)	(8.46s)	(4.96s)
Mass 3	1.33m/s	0.484m/s	-0.0487m/s	0.132m/s
	(4.96s)	(7.44s)	(8.47s)	(4.96s)

Peak relative acceleration response and time when they occur

Mode	1st Mode	2nd Mode	3rd mode	Total
Mass 1	-22.3m/s ²	-6.34m/s²	0.628m/s ²	-22.7m/s ²
	(5.18s)	(7.35s)	(8.53s)	(5.18s)
Mass 2	-17.9m/s ²	3.52m/s ²	-1.41m/s ²	-18.1m/s ²
	(5.18s)	(7.35s)	(8.53s)	(5.15s)
Mass 3	-9.94m/s²	7.90m/s ²	1.13m/s ²	-12.5m/s ²
	(5.18s)	(7.35s)	(8.53s)	(7.17s)

Comparison of Peak Relative Acceleration and the Absolute acceleration

Mode	Relative acceleration	Absolute acceleration
Mode 1	-22.7m/s ²	-19.5m/s ²
	(5.18s)	(5.18s)
Mode 2	-18.1m/s ²	-14.8m/s ²
	(5.15s)	(5.16s)
Mode 3	-12.5m/s ²	-10.1m/s ²
	(7.17s)	(7.16s)

Distribution of Peak Displacement

Mode shapes







9.7 Earthquake Response Spectra

 Response displacement, velocity and absolute acceleration of a SDOF system subjected to a ground acceleration are given by Eqs. (6.11), (6.12), and (6.13)

$$u(t) = \frac{1}{\omega_D} \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \sin \omega_D (t-\tau) d\tau \quad (6.11)$$

$$\dot{u}(t) = -\int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \cos \omega_D (t-\tau) d\tau + \frac{\xi}{\sqrt{1-\xi^2}} \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \sin \omega_D (t-\tau) d\tau$$

$$\ddot{u}_{abs}(t) = \ddot{u}(t) + \ddot{u}_g(t) \quad (6.12)$$

$$= \omega_D \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \left\{ \left(1 - \frac{\xi^2}{1-\xi^2} \sin \omega_D (t-\tau) \right) + \frac{2\xi}{\sqrt{1-\xi^2}} \cos \omega_D (t-\tau) d\tau \right\} \quad (6.13)$$

• Response acceleration spectrum $S_A(T,\xi)$ is defined as the peak absolute response acceleration of a SDOF oscillator, i.e., from Eq. (6.13)

$$S_A(T,\xi) \equiv \left| \ddot{u}_{abs}(t) \right|_{\max} \tag{9.72}$$

where

$$\ddot{u}_{abs}(t) = \omega_D \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \left\{ \left\{ 1 - \frac{\xi^2}{1 - \xi^2} \sin \omega_D (t-\tau) \right\} + \frac{2\xi}{\sqrt{1 - \xi^2}} \cos \omega_D (t-\tau) d\tau \right\}$$
(6.13)

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• Similarly, response displacement spectrum $S_D(T,\xi)$ and response velocity spectrum $S_V(T,\xi)$ are defined as the peak response displacement and velocity of a SDOF oscillator

$$S_D(T,\xi) \equiv |u(t)|_{\max} \qquad (9.73)$$

where,

$$u(t) = \frac{1}{\omega_D} \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \sin \omega_D (t-\tau) d\tau \quad (6.11)$$

$$S_V(T,\xi) \equiv |\dot{u}(t)|_{\max} \qquad (6.74)$$
where
$$\dot{u}(t) = -\int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \cos \omega_D (t-\tau) d\tau$$

$$+ \frac{\xi}{\sqrt{1-\xi^2}} \int_0^t \ddot{u}_g(\tau) e^{-\xi \omega_n (t-\tau)} \sin \omega_D (t-\tau) d\tau$$

$$(6.12)$$

$$(5.12)$$

$$(5.12)$$

$$(5.12)$$

$$(5.12)$$

$$(5.12)$$









Ground acceleration recorded at Kobe Observatory of Japan Meteorological Agency One of the most significant ground motions ever recorded



Typical Near-Field ground Accelerations (代 表的な断層近傍地震動加速度)



Response Accelerations of Typical Near-Field Acceleration Records



 Design force is generally represented in terms of response acceleration spectra

Design Specifications of Highway Bridges Japan Road Association

 $\xi = 0.05$

Function Evaluation



