

Structural Dynamics
構造動力学
(4)

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Chapter 6 Response to General Dynamic Loading: Superposition Methods一般的な荷重に対する応答 (重ね合わせ法に基づく)

This chapter is described based on “Dynamics of Structures” by Shelton Cherry.

6.1 Impulsive Excitation: The Impulse Function インパルス応答(インパルス関数)

- Consider an idealized SDOF viscous damped system (under critically damped system) at rest at $t=0$ at which time a force F_0 is applied to the mass for a very short interval Δt .
- Therefore at time $t=0$, the system is subjected to a single impulse $I=F_0\Delta t$ as shown in Fig. 6.1

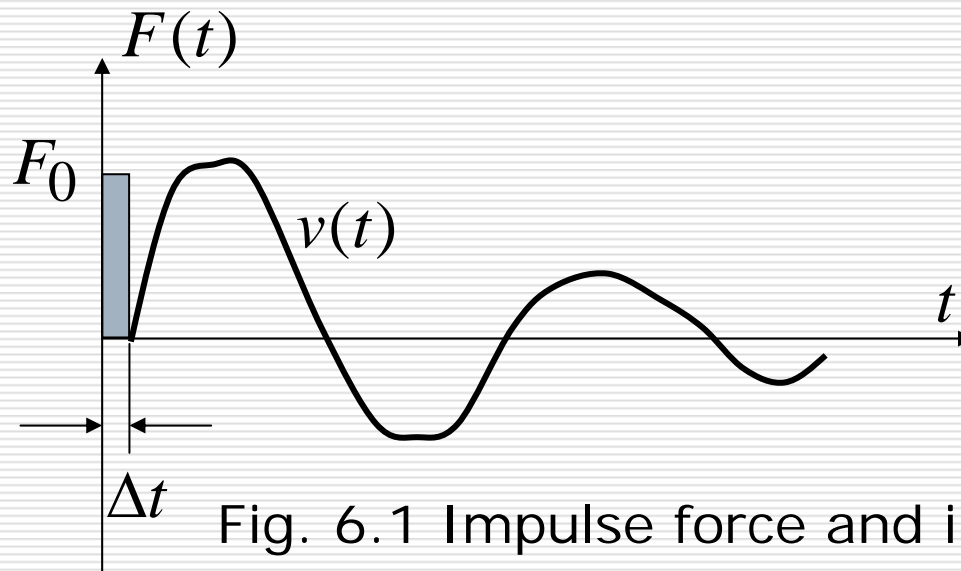


Fig. 6.1 Impulse force and impulse response

- From impulse-momentum principle, we have

$$F_0 \Delta t = m \dot{v}_0 \quad (6.1)$$

- Eq. (6.1) implies that as a result of the impulse I , the mass of the system acquires an instantaneous velocity

$$\dot{v}_0 = \frac{F_0 \Delta t}{m} = \frac{I}{m} \quad (6.2)$$

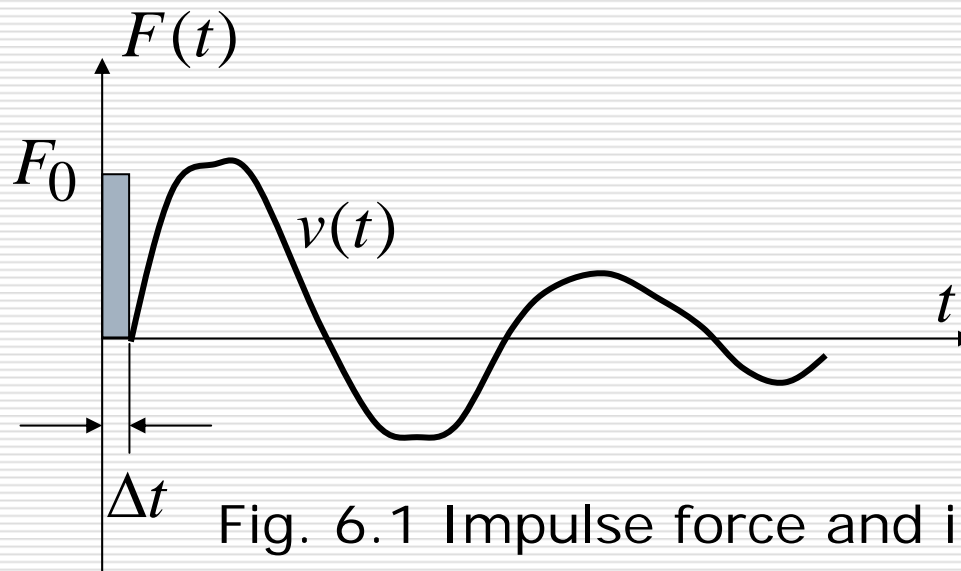


Fig. 6.1 Impulse force and impulse response

- Since $F(t)=0$ for $t>\tau$, there is no further excitation and hence no particular integral in the solution of the basic equation of motion.
- The system there executes free vibration
- If we measure time from the end of the impulse and assume no appreciable change in displacement during the short time Δt , the initial conditions are $v(0)=v_0=0$ and $\dot{v}(0)=\dot{v}_0$.

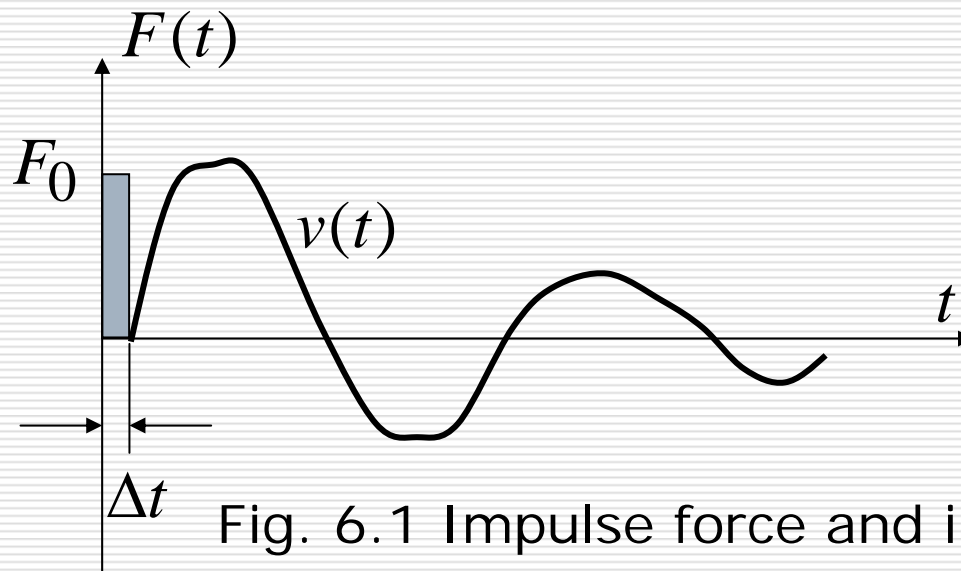


Fig. 6.1 Impulse force and impulse response

- Eq. (2.50) represents the damped free vibration of the system with $V_0=0$ and $v_0 = l/m$.

$$v(t) = \rho \cos(\omega_D t + \theta) e^{-\xi \omega t} \quad (2.50)$$

$$\rho = \sqrt{v(0)^2 + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D} \right)^2} \quad (2.51)$$

$$\theta = -\tan^{-1} \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D v(0)} \right) \quad (2.52)$$

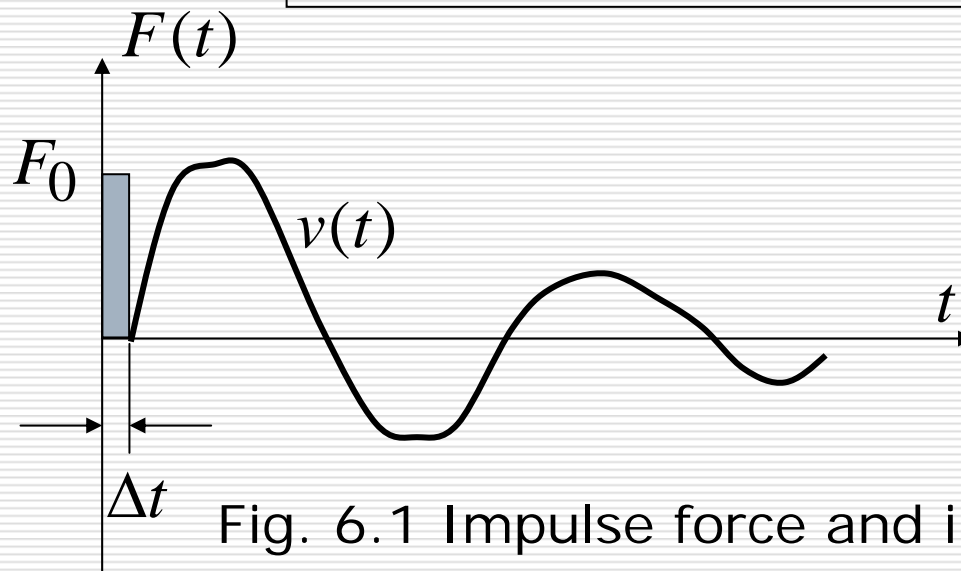


Fig. 6.1 Impulse force and impulse response

- Denoting from Eq.(2.52), $\theta = -\pi/2$, Eq. (2.50) becomes

$$v(t) = \frac{\dot{v}_0}{\omega_D} \cos(\omega_D t - \frac{\pi}{2}) e^{-\omega \xi t}$$

or

$$v(t) = \frac{I}{m\omega_D} e^{-\omega \xi t} \sin \omega_D t \quad (6.3)$$

$$v(t) = \rho \cos(\omega_D t + \theta) e^{-\xi \omega t} \quad (2.50)$$

$$\rho = \sqrt{v(0)^2 + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D} \right)^2} \quad (2.51)$$

$$\theta = -\tan^{-1} \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D v(0)} \right) \quad (2.52)$$

- Eq.(6.3) shows a damped sine-wave of circular frequency ω_D and decay of amplitude by $\exp(-\xi\omega t)$

$$v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin \omega_D t \quad (6.3)$$

- If we define the notation $h(t)$ to represent the response to a unit impulse, then Eq. (6.3) becomes

$$v(t) = I \cdot h(t) \quad (6.4)$$

where, $h(t)$ is referred to as **the impulse-response function (インパルス応答関数)**.

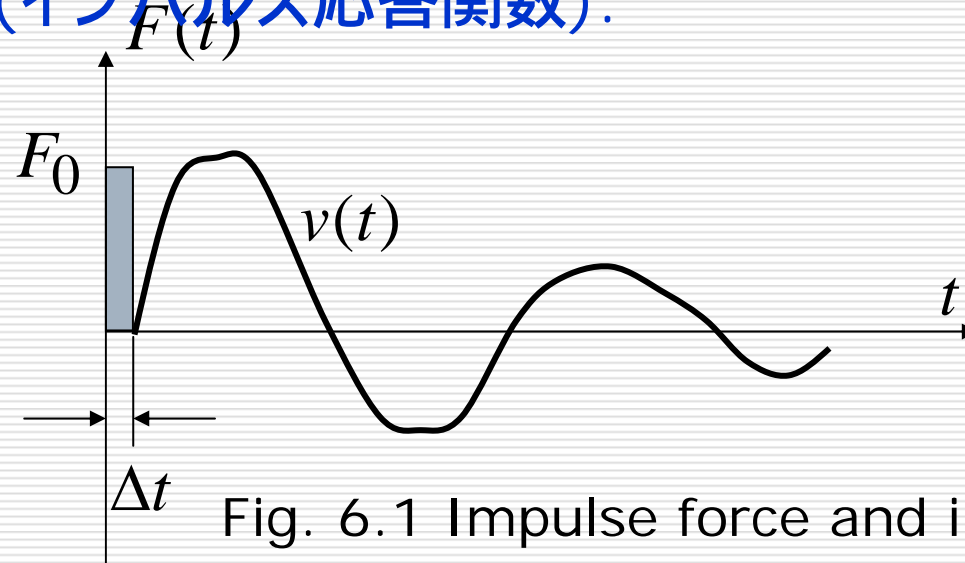


Fig. 6.1 Impulse force and impulse response

6.2 Response to the General Transient (過渡応答)

- Transient response of a SDOF system subjected to an arbitrary external force $F(t)$ may be considered that it is developed as a sequence of infinitesimal impulses each of which contributes to the total motion according to Eq. (6.3).

$$v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin \omega_D t \quad (6.3)$$

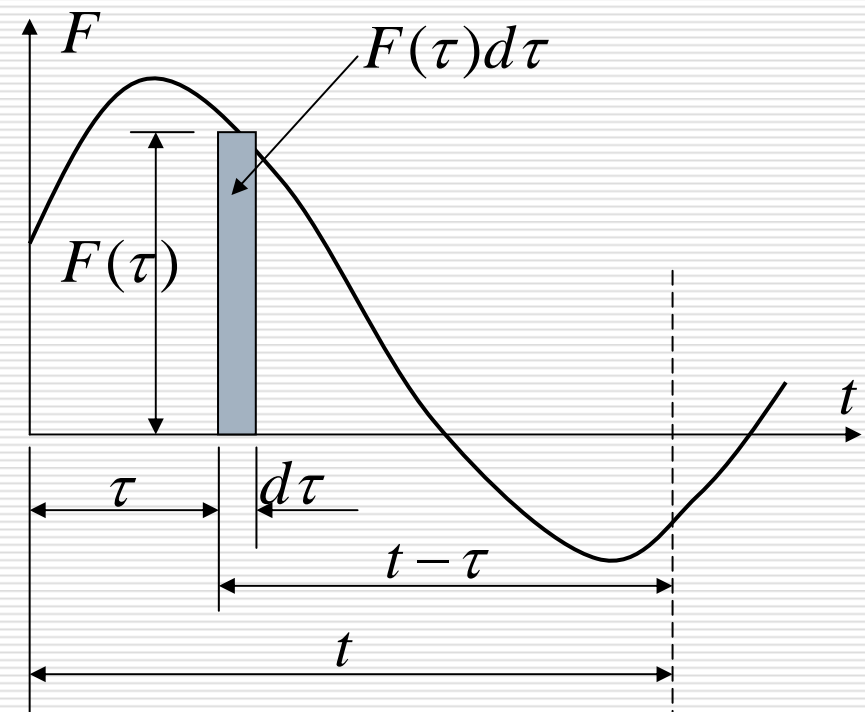


Fig. 6.2 Transient response ⁹
as a sequence of impulses

- Assuming $v(0) = \dot{v}(0) = 0$ at $t=0$ (this restriction shall later be removed), the response due to any one impulse in the absence of all others may be written from Eq. (6.3) as

$$dv(t) = \frac{F(\tau)d\tau}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin \omega_D(t-\tau) \quad (6.5)$$

- In Eq. (6.5) we have to carefully distinguish between t (the time at which the response is desired) and τ (the time at which the impulse applies)

$$v(t) = \frac{I}{m\omega_D} e^{-\omega\xi t} \sin \omega_D t \quad (6.3)$$

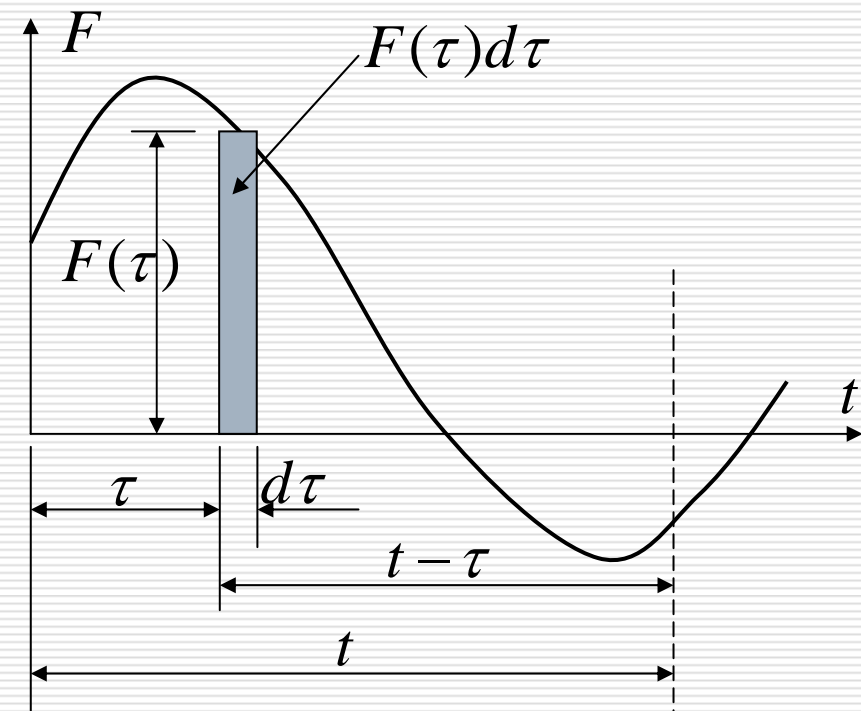


Fig. 6.2 Transient response as a sequence of impulses 10

- The effect of all impulses in the interval $t=0$ and $t=t$ contribute to the total response at time t .
- Because we now consider the linear system, the total response due to all impulses may be evaluated by superposition of Eq. (6.5) as

$$v(t) = \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin \omega_D(t-\tau) d\tau \quad (6.6)$$

where τ is a dummy variable which disappears on integration.

- Eq. (6.6) is known as **Duhamel integral (デュアメル積分)**, superposition integral or **convolution integral (畳み込み積分)**.

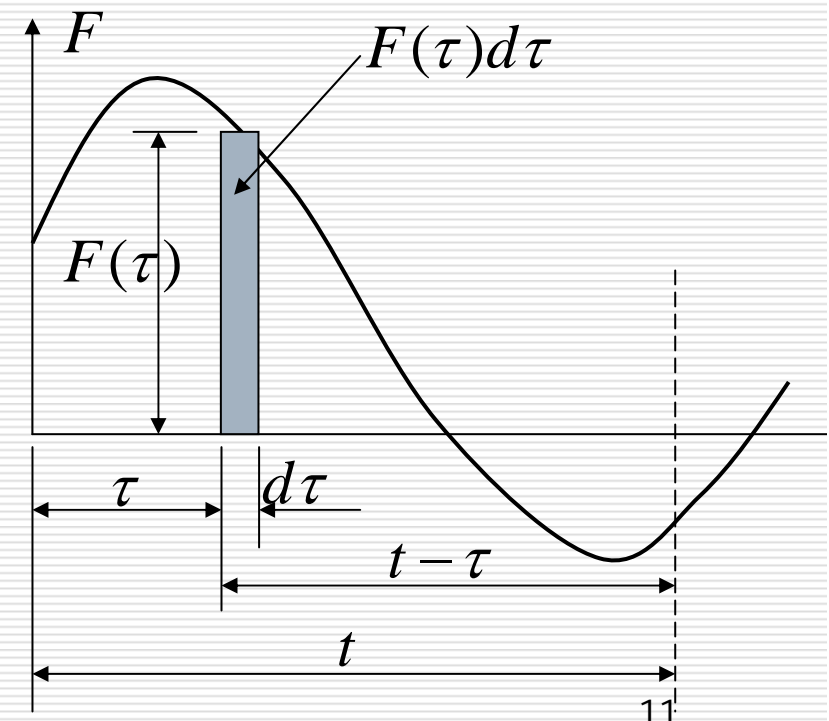


Fig. 6.2

- If the system has initial condition $v(0)=v_0$, and $\dot{v}(0) = \dot{v}_0$, the complete solution of the response becomes from Eq. (2.49) and Eq. (6.6) as

$$v(t) = \left\{ v(0) \cos \omega_D t + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D} \right) \sin \omega_D t \right\} e^{-\xi\omega t} + \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin \omega_D(t-\tau) d\tau \quad (6.7)$$

in which, the first term represents the free vibration of the system.

$$v(t) = \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin \omega_D(t-\tau) d\tau \quad (6.6)$$

$$v(t) = \left\{ v(0) \cos \omega_D t + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D} \right) \sin \omega_D t \right\} e^{-\xi\omega t} \quad (2.49)$$

- Eq. (6.6) can be rewritten using the impulse response function given by Eq. (6.4) as

$$v(t) = \int_0^t F(\tau) h(t - \tau) d\tau \quad (6.8)$$

where

$$h(t - \tau) = \frac{1}{m\omega_D} \left\{ e^{-\xi\omega(t-\tau)} \sin \omega_D(t - \tau) \right\} \quad (6.9)$$

- Depending on the complexity of the system, it may or may not be easy to find $h(t-\tau)$ analytically. It is sometimes possible to obtain good approximations to $h(t-\tau)$ experimentally.

$$v(t) = I \cdot h(t) \quad (6.4)$$

$$v(t) = \int_0^t \frac{F(\tau)}{m\omega_D} e^{-\xi\omega(t-\tau)} \sin \omega_D(t - \tau) d\tau \quad (6.6)$$