Structural Dynamics
構造動力学
(3)

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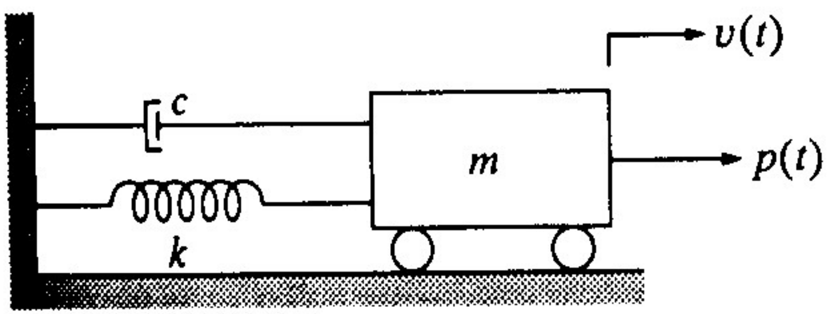
CHAPTER 3 RESPONSE TO HARMONIC LOADING (調和振動外力を受け場合の振動)

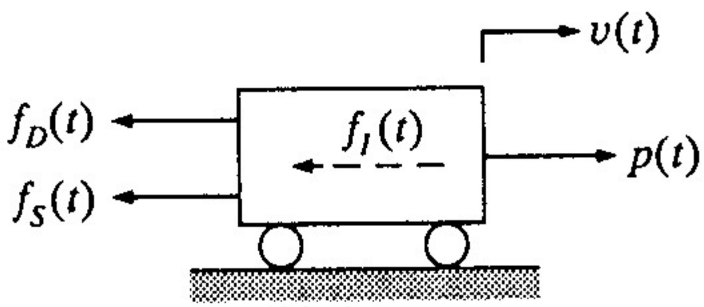
3.1 Undamped System

1) Complementary Solution (Free Vibration Solution; 自由振動解)

● Assume that the system of Fig. 2.1 is subjected to a harmonic varying load p(t) of sine-wave form having an amplitude p_0 and circular frequency ω (角振動数) .



(a) Basic components



(b) Forces in equilibrium

Fig. 2.1 Idealized SDOF system

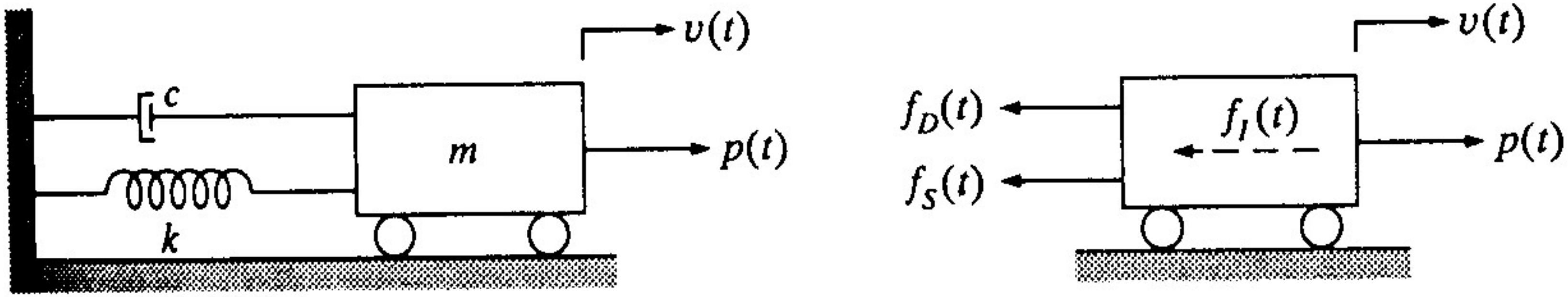
● Substituting $p(t) = p_0 \sin \omega t$ into Eq. (2.3), one obtains

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = p_0 \sin \overline{\omega} t \qquad (3.1)$$

$$\boxed{m\ddot{v}(t) + c\dot{v}(t) + kv(t) = p(t) \qquad (2.3)}$$

● Before considering damped vibration, it is instructive to examine the behavior of an undamped system

$$m\ddot{v}(t) + kv(t) = p_0 \sin \overline{\omega} t \qquad (3.2)$$



(a) Basic components (b) Forces in equilibrium

Fig. 2.1 Idealized SDOF system

● Eq. (3.2) has a complementary solution (Free Vibration solution;**自由振動解**) of the free-vibration from of Eq. (2.31).

$$v_c(t) = A\cos\omega t + B\sin\omega t \qquad (3.3)$$

$$m\ddot{v}(t) + kv(t) = p_0 \sin\overline{\omega}t \qquad (3.2)$$

$$v(t) = A\cos\omega t + B\sin\omega t \qquad (2.31)$$

2) Particular Solution (特解、Forced vibration、強制振動応答)

- Under a harmonic excitation, it is reasonable to assume that the corresponding motion is harmonic and in phase (位相も荷重と同じ) with the loading; thus the particular solution is

$$v_p(t) = C\sin\overline{\omega}t \qquad (3.4)$$

in which the amplitude C is to be evaluated.

- Substituting Eq. (3.4) into Eq. (3.2) gives

$$-m\overline{\omega}^2 C\sin\overline{\omega}t + kC\sin\overline{\omega}t = p_0\sin\overline{\omega}t \qquad (3.5)$$

$$\boxed{m\ddot{v}(t) + kv(t) = p_0\sin\overline{\omega}t \qquad (3.2)}$$

- Dividing through by $\sin\bar{\omega}t$ and by k and noting that $k/m = \omega^2$, one obtains after some rearrangement

$$C = \frac{p_0}{k}\frac{1}{1-\beta^2} \qquad (3.6)$$

in which β is defined as the ratio of the applied loading frequency $\bar{\omega}$ to the natural free-vibration frequency ω , i.e.,

$$\beta \equiv \frac{\bar{\omega}}{\omega} \qquad (3.7)$$

3) General Solution (一般解)

- The general solution of Eq. (3.2) is now obtained by combining the complementary solution (free vibration, **自由振動解**) and particular solution (**特解、** forced excitation vibration solution; **強制振動解**) as

$$v(t) = v_c(t) + v_p(t)$$

$$= A\cos\omega t + B\sin\omega t + \frac{p_0}{k}\frac{1}{1-\phi^2}\sin\overline{\omega}t \qquad (3.8)$$

$$\boxed{m\ddot{v}(t) + kv(t) = p_0\sin\overline{\omega}t \qquad (3.2)}$$

- For the system starting from rest, i.e., $v(0) = \dot{v}(0) = 0$, it is easy to show that

$$A = 0 \qquad B = -\frac{p_0\beta}{k}\frac{1}{1-\beta^2} \qquad (3.9)$$

- In this case, Eq. (3.8) becomes

$$v(t) = \frac{p_0}{k} \frac{1}{1-\beta^2} \left(\sin \overline{\omega} t - \beta \sin \omega t \right) \tag{3.10}$$

Displacement which would be developed by the load p_0 applied statically （**静的変位**）

Magnification factor (MF) (**増幅率**)
representing the amplification effect of the harmonically applied loading

$$v(t) = \frac{p_0}{k} \frac{1}{1-\beta^2} \left(\sin \overline{\omega} t - \beta \sin \omega t \right) \qquad (3.10)$$

$\sin \overline{\omega} t$ represents the response component at the frequency of the applied loading, and it is called the steady-state response (**定常応答**)

$\beta \sin \omega t$ is the response component at the natural period of the structure (**構造物の固有振動数**) and is the free-vibration (**自由振動**) effect controlled by the initial condition. Since in a practical case, damping will cause the term to vanish eventually, it is termed the transient response (**過渡応答**).

4) Response Ratio (応答比)

- A convenient measure of the influence of dynamic loading is provided by the ratio of the dynamic displacement response (動的応答変位) to the displacement produced by static load p_0 (静的荷重によって生じる静的変位), i.e.,

$$R(t) \equiv \frac{v(t)}{v_{st}} = \frac{v(t)}{p_0/k} \qquad (3.11)$$

$$= \frac{1}{1-\beta^2}\left(\sin\overline{\omega}t - \beta\sin\omega t\right) \qquad (3.12)$$

$$= MF\left(\sin\overline{\omega}t - \beta\sin\omega t\right)$$

$$R(t) = R_p(t) + R_s(t)$$

Transient response (過渡応答)

Steady-state response (定常応答)

● Fig. 3.1(a) shows the steady-state response, and Fig. 3.1(b) represents transient response while Fig. 3.1(c) represents the total response. It is assumed here $\beta \equiv \overline{\omega}/\omega = 2/3$

●The two components get in phase and then out of phase again, causing a "beating" effect (ビート現象) in the total response.

● At t=0, $v(0)=0$ and $\dot{v}(t)=0$

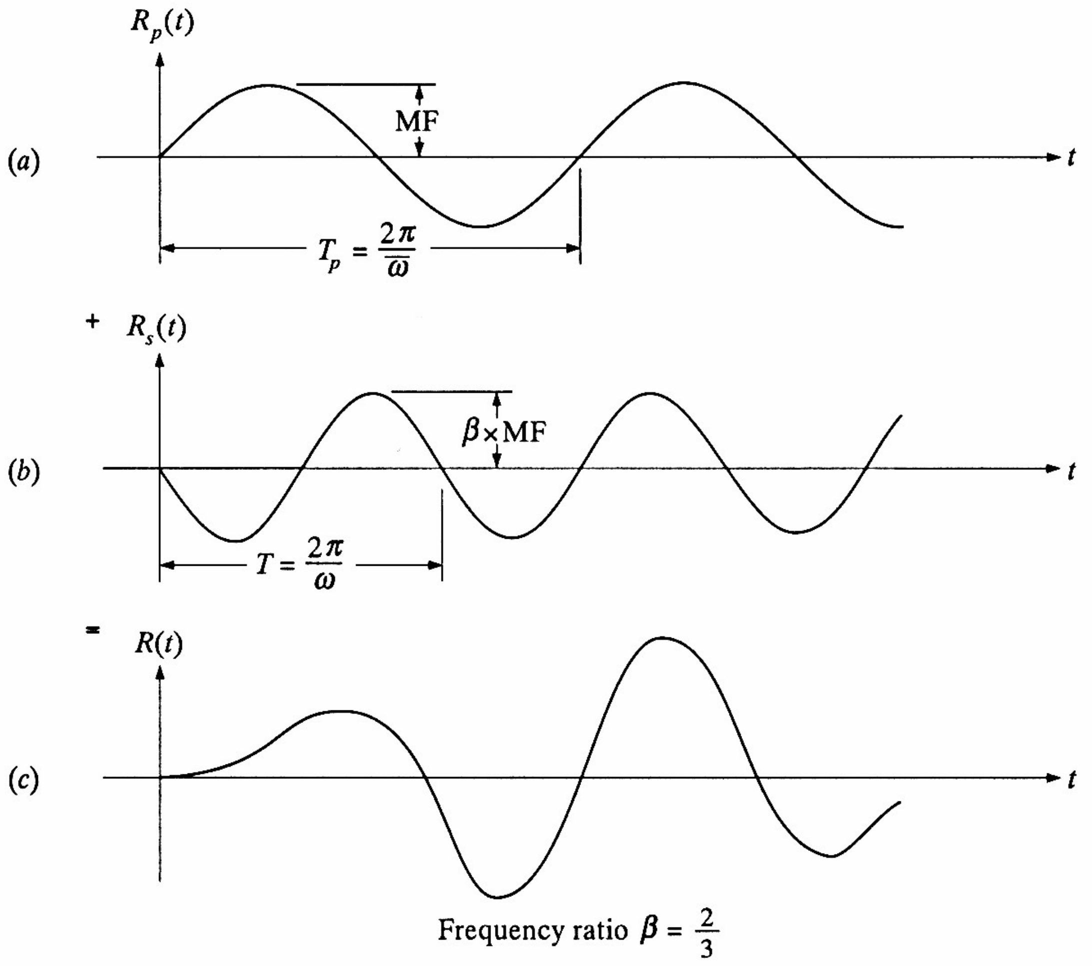


FIGURE 3-1
Response ratio produced by sine wave excitation starting from at-rest initial conditions: (*a*) steady state; (*b*) transient; (*c*) total $R(t)$.

3.2 System with viscous damping (粘性減衰のある系)

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = p_0 \sin \overline{\omega} t \qquad (3.1)$$

- Dividing Eq. (3.1) by m and noting that $c/m = 2\xi\omega$ leads to

$$\ddot{v}(t) + 2\xi\omega\dot{v}(t) + \omega^2 v(t) = \frac{p_0}{m} \sin \overline{\omega} t \qquad (3.13)$$

- The complementary solution (transient solution; 過渡応答解) is the damped free-vibration response (減衰自由振動) by

$$v_c(t) = \left(A \cos \omega_D t + B \sin \omega_D t\right) e^{-\xi\omega t} \qquad (3.14)$$

- The particular solution (特解、steady-state solution; 定常解) is given as

$$v_p(t) = G_1 \cos \overline{\omega} t + G_2 \sin \overline{\omega} t \qquad (3.15)$$

● It should be noted in Eq. (3.15) that the cosine term is required as well as the sine term because , in general, the response of a damped system is not in phase with the loading (refer to Eq. (3.4) for the response of an undamped system)

$$v_p(t) = G_1 \cos\overline{\omega}t + G_2 \sin\overline{\omega}t \qquad (3.15)$$

$$v_p(t) = C \sin\overline{\omega}t \qquad (3.4)$$

● Substituting Eq. (3.15) into Eq. (3.13) leads to

$$\left\{-G_1\overline{\omega}^2 + G_2\overline{\omega}(2\xi\omega) + G_1\omega^2\right\}\cos\overline{\omega}t$$

$$+\left\{-G_2\overline{\omega}^2 - G_1\overline{\omega}(2\xi\omega) + G_2\omega^2 - \frac{p_0}{m}\right\}\sin\overline{\omega}t = 0$$

$$(3.16)$$

$$\ddot{v}(t) + 2\xi\omega\dot{v}(t) + \omega^2 v(t) = \frac{p_0}{m}\sin\overline{\omega}t \qquad (3.13)$$

- In order to satisfy this equation for all values of t, it is necessary that each of the two bracket quantities equal to zero;

$$G_1(1-\beta^2)+G_2(2\xi\beta)=0$$
$$G_2(1-\beta^2)-G_1(2\xi\beta)=\frac{p_0}{k} \qquad (3.17)$$

- Solving these two equations simultaneously yields

$$G_1=\frac{p_0}{k}\frac{-2\xi\beta}{(1-\beta^2)^2+(2\xi\beta)^2}$$
$$G_2=\frac{p_0}{k}\frac{1-\beta^2}{(1-\beta^2)^2+(2\xi\beta)^2} \qquad (3.18)$$

● Based on those results, the total displacement is obtained in the form

$$v(t) = \left(A\cos\omega_D t + B\sin\omega_D t\right)e^{-\xi\omega t} + \frac{p_0}{k}\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\left\{(1-\beta^2)\sin\overline{\omega}t - 2\xi\omega\cos\overline{\omega}t\right\} \tag{3.19}$$

The transient response (**過渡応答**), which damps out (**減衰していく**) in accordance with $\exp(-\xi\omega t)$

The steady-state harmonic response which will continue indefinitely (**無限に続く定常応答**)

● The constants A and B can be evaluated for any given initial conditions, $v(0)$ and $\dot{v}(0)$

Steady-state response (定常応答)

- Of great interest is the steady-state harmonic response.

$$v_p(t) = \frac{p_0}{k}\frac{1}{(1-\beta^2)^2+(2\xi\beta)^2}\left\{(1-\beta^2)\sin\overline{\omega}t - 2\xi\omega\cos\overline{\omega}t\right\} \tag{3.20}$$

$$v_p(t) = G_1\cos\overline{\omega}t + G_2\sin\overline{\omega}t \tag{3.15}$$

$$G_1 = \frac{p_0}{k}\frac{-2\xi\beta}{(1-\beta^2)^2+(2\xi\beta)^2}$$

$$G_2 = \frac{p_0}{k}\frac{1-\beta^2}{(1-\beta^2)^2+(2\xi\beta)^2} \tag{3.18}$$

● The steady-state displacement can be interpreted easily by plotting two rotating vectors in the complex plane as shown in Fig. 3.2.

$$v_p(t) = G_1 \cos \overline{\omega} t + G_2 \sin \overline{\omega} t \qquad (3.15)$$

$$G_1 = \frac{p_0}{k} \frac{-2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

$$G_2 = \frac{p_0}{k} \frac{1-\beta^2}{(1-\beta^2)^2 + (2\xi\beta)^2} \qquad (3.18)$$

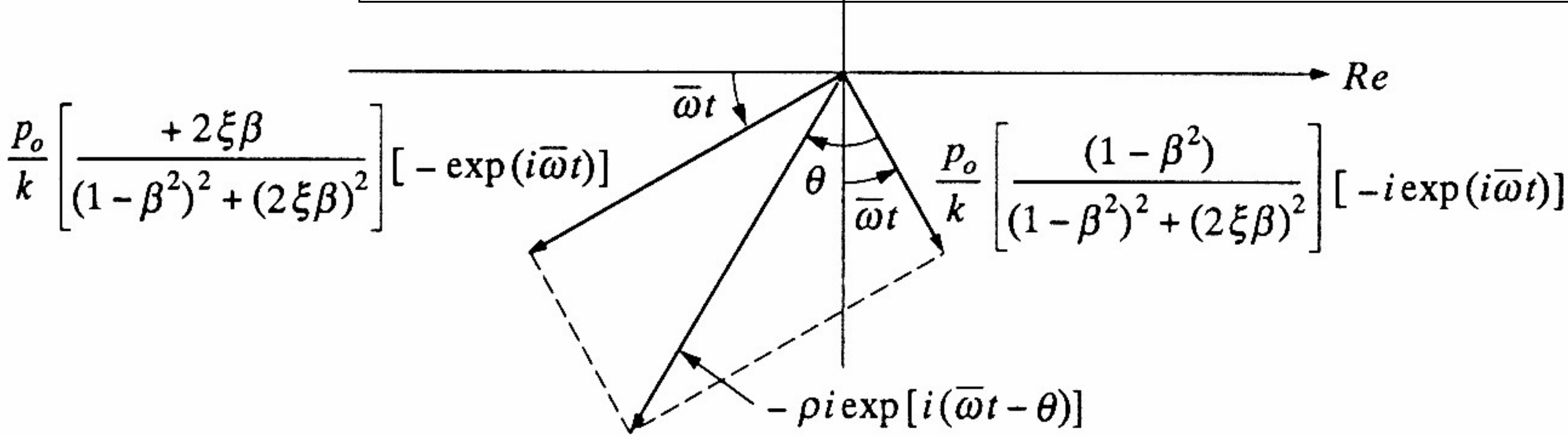


Fig. 3.2 Steady-state displacement response

● The real component of the resultant vector $-\rho i \exp\{i(\overline{\omega}t-\theta)\}$ gives the steady-state response in the form

$$v_p(t) = \rho \sin(\overline{\omega}t-\theta) \qquad (3.21)$$

$$\rho \equiv \frac{p_0}{k}\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}} \qquad (3.22)$$

$$\theta \equiv \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right) \qquad (3.23)$$

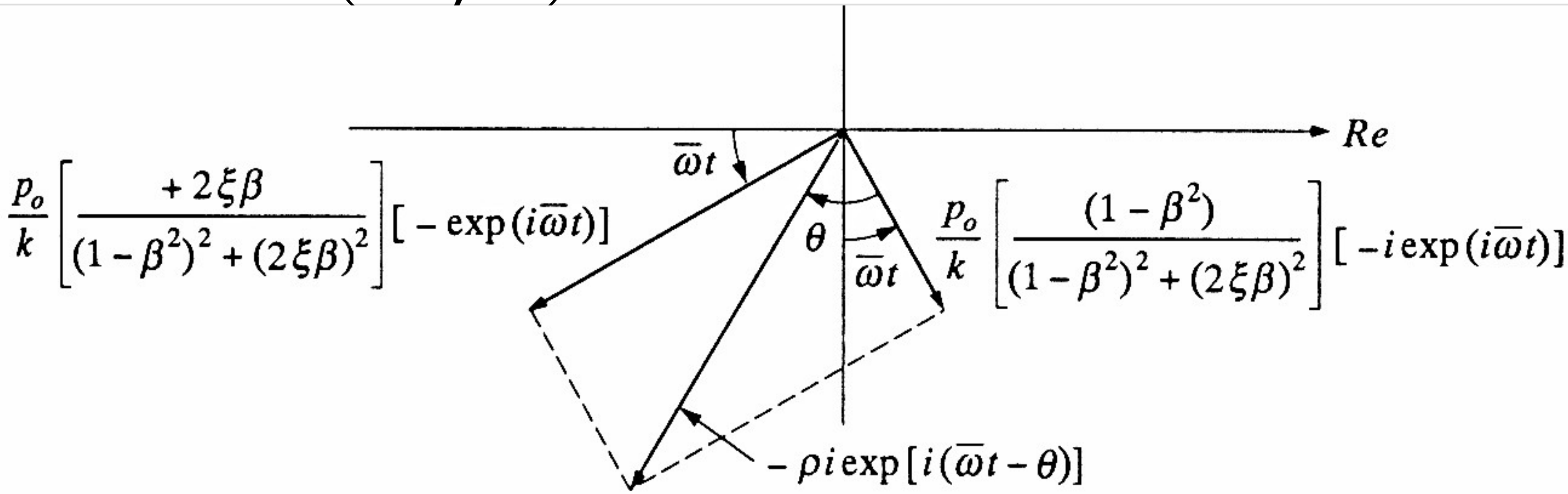


Fig. 3.2 Steady-state displacement response

● The ratio of the resultant harmonic amplitude to the static displacement which would be produced by the force p_0 will be called the dynamic magnification factor D (**動的增幅率**), thus

$$D \equiv \frac{\rho}{p_0/k} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \qquad (3.24)$$

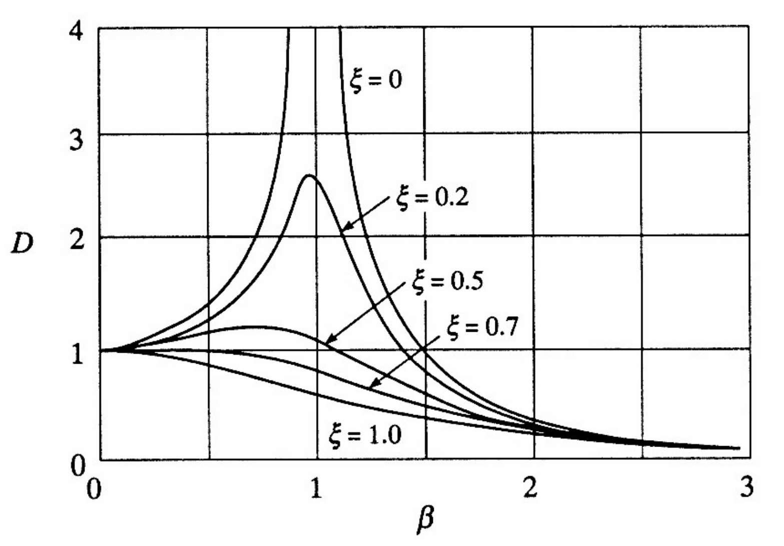


Fig. 3.3 Dynamic magnification factor vs. damping and frequency

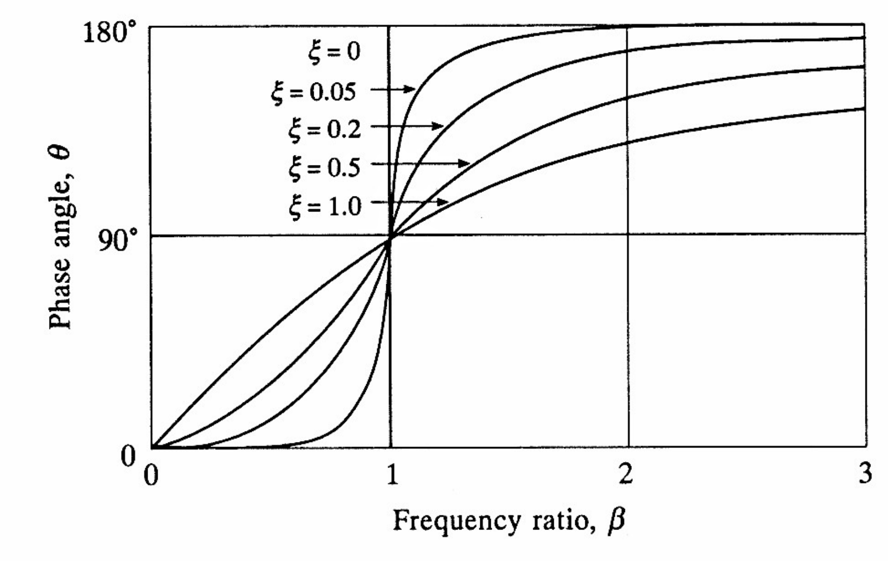
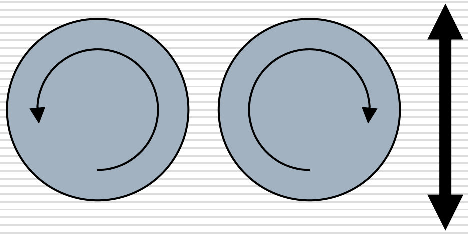


Fig. 3.4 Phase angle vs. damping and frequency

Example E3-1



Harmonic-loading machine

- A portable harmonic-loading machine （起振機） provides an effective means for evaluating the dynamic properties of structures in the field.
- By operating the machine at two different frequencies and measuring the resulting structural response amplitude and phase relationship in each case, it is possible to determine the mass, damping and stiffness of a SDOF structure.

- In a test of this type on a simply supported bridge, the shaking machine was operated at a frequency of $\overline{\omega}_1 = 16 rad/s$ and $\overline{\omega}_2 = 25 rad/s$, with a forced amplitude $500 lb (226.8 kgf)$ in each case.

- The response amplitudes and phase relationship measured in the two cases were

$$\rho_1 = 7.2 \times 10^{-3} in (18.3 \times 10^{-3} cm)$$

$$\theta_1 = 15 \deg ree \qquad \cos\theta_1 = 0.966 \qquad \sin\theta_1 = 0.259$$

$$\rho_2 = 14.5 \times 10^{-3} in (36.8 \times 10^{-3} cm)$$

$$\theta_2 = 55 \deg ree \qquad \cos\theta_2 = 0.574 \qquad \sin\theta_2 = 0.819$$

$$\rho_1 = 7.2\times10^{-3} in (18.3\times10^{-3} cm)$$
$$\theta_1 = 15 \deg ree \qquad \cos\theta_1 = 0.966 \qquad \sin\theta_1 = 0.259$$

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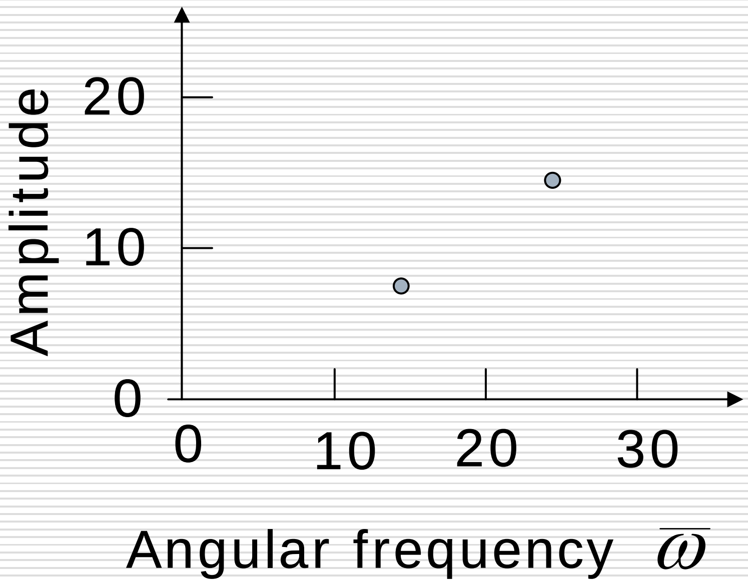
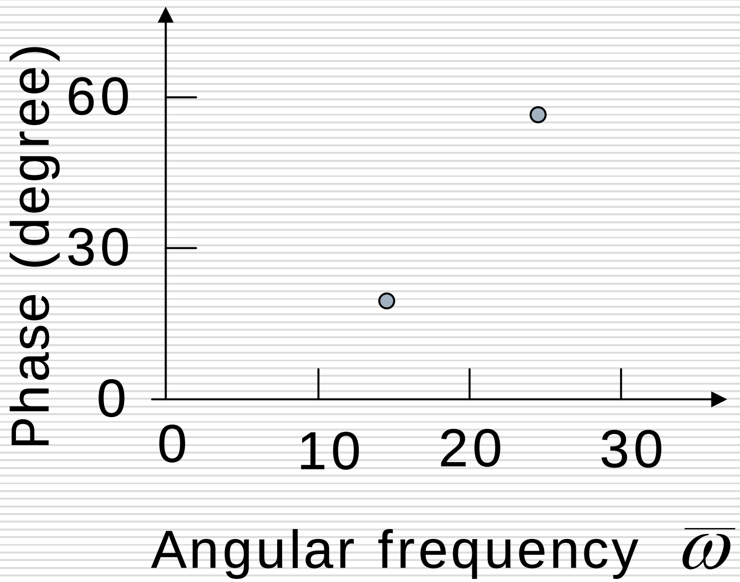




- It is convenient to rewrite Eq. (3.22) as

$$\rho = \frac{p_0}{k}\frac{1}{1-\beta^2}\frac{1}{\sqrt{1+\left\{2\xi\beta/(1-\beta^2)\right\}^2}}$$

$$\boxed{\rho \equiv \frac{p_0}{k}\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}} \qquad (3.22)}$$

- Because $\tan\theta$ is provided as based on Eq. (3.23)

$$\tan\theta = \frac{2\xi\beta}{1-\beta^2}$$

$$\boxed{\theta \equiv \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right) \qquad (3.23)}$$

- ρ is written as

$$\rho = \frac{p_0}{k}\frac{\cos\theta}{1-\beta^2} \qquad \text{(a)}$$

- With further algebraic simplification,

$$k(1-\beta^2) = k\left\{1-\left(\frac{\overline{\omega}}{\omega}\right)^2\right\} = k - \overline{\omega}^2 m$$

$$\boxed{\rho = \frac{p_0}{k}\frac{\cos\theta}{1-\beta^2}}$$

$$= \frac{p_0 \cos\theta}{\rho}$$

$$\begin{bmatrix} 1 & -16^2 \\ 1 & -25^2 \end{bmatrix}\begin{Bmatrix} k \\ m \end{Bmatrix} = 500lb\begin{Bmatrix} \dfrac{0.966}{7.2\times10^{-3}} \\ \dfrac{0.574}{14.5\times10^{-3}} \end{Bmatrix}$$

$$\rho_1 = 7.2\times10^{-3} in (18.3\times10^{-3} cm)$$

$$\theta_1 = 15 \deg ree \qquad \cos\theta_1 = 0.966$$

$$\rho_2 = 14.5\times10^{-3} in (36.8\times10^{-3} cm)$$

$$\theta_2 = 55 \deg ree \qquad \cos\theta_2 = 0.574$$

Evaluation of stiffness and mass

- This equation can be solved to give

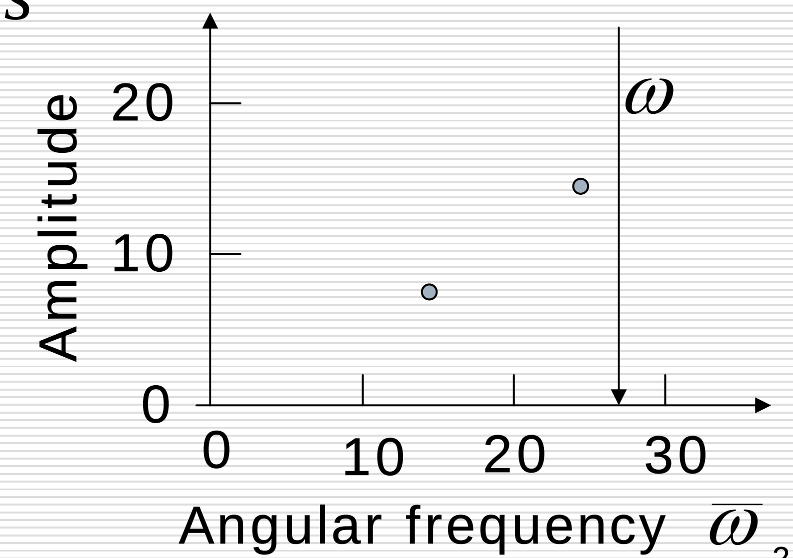
$$k = 100 \times 10^3 lb/in(17.8 \times 10^3 kgf\ /cm)$$

$$m = 128.5 lb \cdot s^2 /in^2 (22.95 kgf\ \cdot s^2 /cm)$$

Evaluation of natural frequency

- The natural frequency is given as

$$\omega = \sqrt{\frac{k}{m}} = 27.9 rad/s$$



Evaluation of damping ratio

- From $\rho = \frac{p_0}{k}\frac{\cos\theta}{1-\beta^2}$ (a)

$$1-\beta^2 = \frac{p_0 \cos\theta}{\rho k}$$

- Substituting this into $\tan\theta = \frac{2\xi\beta}{1-\beta^2}$

$$\xi = \frac{\tan\theta}{2\beta}\frac{p_0\cos\theta}{\rho k} = \frac{p_0 \sin\theta}{2\beta\rho k} = \frac{p_0\sin\theta}{c_c\overline{\omega}\rho}$$

- Thus with the data of the first test

$$c = \xi c_c = \frac{p_0\sin\theta}{\overline{\omega}\rho} = \frac{500\times 0.259}{16\times 7.2\times 10^{-3}} = 1{,}125 lb\cdot s/in$$

- From the second test

$$c = \frac{500\times 0.819}{25\times 15.5\times 10^{-3}} = 1{,}056 lb\cdot s/inch$$

- It should be noted that the damping coefficient c obtained from the second test is close with engineering accuracy with c obtained from the first test.

- The damping ratio therefore is

$$\xi = \frac{c}{2k/\omega} = \frac{1{,}125 \times 27.9}{2 \times 100 \times 10^3} = 15.7\%$$

3.3 Resonance Response

- From Eq. (3.12), it is apparent that the steady-state response amplitude of an undamped system tends to toward infinitively as the frequency ratio approached unity.

$$R(t)=\frac{1}{1-\beta^2}\left(\sin\overline{\omega}t-\beta\sin\overline{\omega}t\right) \qquad (3.12)$$

$$\beta\equiv\frac{\overline{\omega}}{\omega} \qquad (3.7)$$

- This tendency can be seen in Fig. 3.3 for the case of $\xi=0$

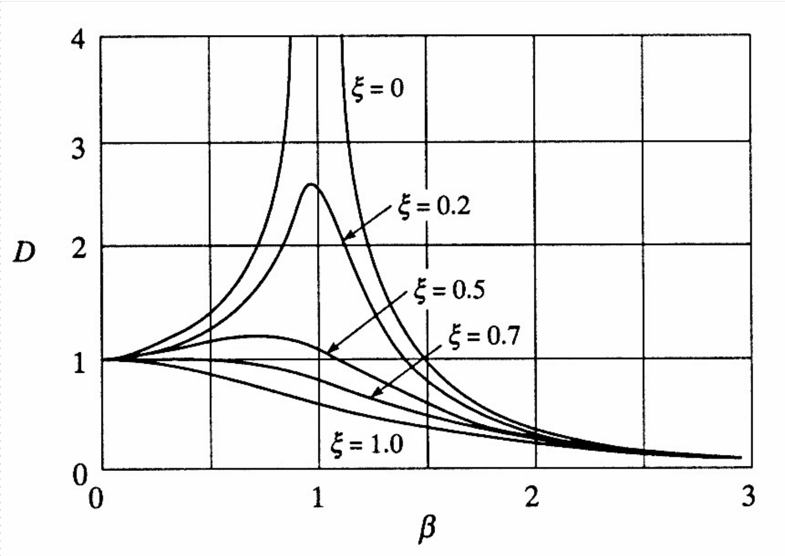


Fig. 3.3 Dynamic magnification factor vs. damping and frequency

● From Eq. (3.24), it is seen that the dynamic magnification factor (動的增幅率) D under this condition ($\beta = 1$) is

$$D_{\beta=1} = \frac{1}{2\xi} \qquad (3.31)$$

$$\boxed{D \equiv \frac{\rho}{p_0/k} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \qquad (3.24)}$$

● To find the maximum or peak value of dynamic magnification factor, one must differentiate Eq. (3.24) with respect to β and solve the resulting expression for β obtaining

$$\beta_{peak} = \sqrt{1 - 2\xi^2} \qquad (3.32)$$

● Eq. (3.32) yields positive real value for damping ratio $\xi < 1/\sqrt{2}$

- Substituting Eq. (3.32) into Eq. (3.24),

$$D_{\max} = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\xi}\frac{\omega}{\omega_D} \qquad (3.33)$$

$$D \equiv \frac{\rho}{p_0/k} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \qquad (3.24)$$

$$\beta_{peak} = \sqrt{1-2\xi^2} \qquad (3.32)$$

- For typical values of structural damping, say $\xi < 0.1$, the difference between Eq. (3.33) and the simpler Eq. (3.31) is very small.

$$D_{\beta=1} = \frac{1}{2\xi} \qquad (3.31)$$

● For more complete understanding of the nature of the resonant response (共振応答) of a structure subjected to harmonic loading, consider the general response Eq. (3.19), which includes the transient response (過渡応答) as well as the steady response (定常応答).

$$v(t) = (A\cos\omega_D t + B\sin\omega_D t)e^{-\xi\omega t} + \frac{p_0}{k}\frac{\cos\omega t}{2\xi} \qquad (3.34)$$

$$\boxed{\begin{aligned} v(t) &= (A\cos\omega_D t + B\sin\omega_D t)e^{-\xi\omega t} \\ &+ \frac{p_0}{k}\frac{1}{(1-\beta^2)^2+(2\xi\beta)^2}\left\{(1-\beta^2)\sin\overline{\omega}t - 2\xi\omega\cos\overline{\omega}t\right\} \\ &\qquad (3.19) \end{aligned}}$$

- Assuming that the system starts from rest, i.e., $v(0)=0$ and $\dot{v}(0)=0$ the constants are

$$A=\frac{p_0}{k}\frac{1}{2\xi} \qquad B=\frac{p_0}{k}\frac{\omega}{2\omega_D}=\frac{p_0}{k}\frac{1}{2\sqrt{1-\xi^2}} \qquad (3.35)$$

- Thus, Eq. (3.34) becomes

$$v(t)=\frac{1}{2\xi}\frac{p_0}{k}\left\{\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t+\cos\omega_D t\right)e^{-\xi\omega t}-\cos\omega t\right\}$$

(3.36)

- For the amounts of damping to be expected in structural system, the term $\sqrt{1-\xi^2}$ is nearly unity . In this case, Eq. (3.36) can be approximated

$$R(t)\approx\frac{v(t)}{p_0/k}=\frac{1}{2\xi}\left\{\left(e^{-\xi\omega t}-1\right)\cos\omega t+\xi e^{-\xi\omega t}\sin\omega t\right\} \quad (3.37)$$

- For zero damping, the approximation by Eq. (3.37) is indeterminate; but when L'Hospital's rule is applied, the response ratio for the undamped system is found to be

$$R(t) \approx \frac{1}{2}(\sin \omega t - \omega t \cos \omega t) \qquad (3.38)$$

- Plot of Eq. (3.37) are shown in Fig. 3.7.

L'Hopital's rule (or, Bernoulli's rule)

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

- Note that because the terms containing $\sin\omega t$ contribute little to the response, the peak values build up linearly for the undamped system, changing by an amount of π in each cycle; however they build up in accordance with $(1/2\xi)\{\exp(-\xi\omega t)-1\}$ for the damped system.

- This latter envelope function is plotted against frequency in Fig. 3.8.

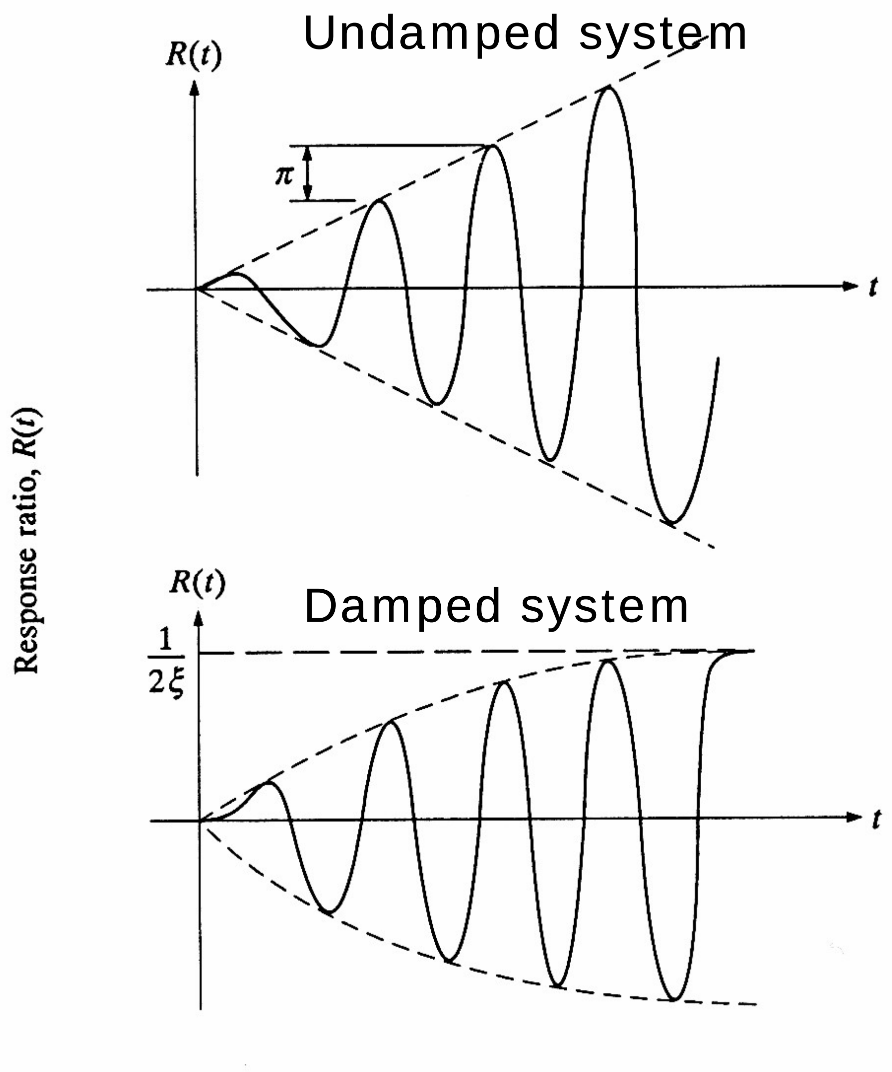


FIGURE 3-7
Response to resonant loading $\beta = 1$ for at-rest initial conditions.

● This latter envelope function is plotted against frequency in Fig. 3.8.

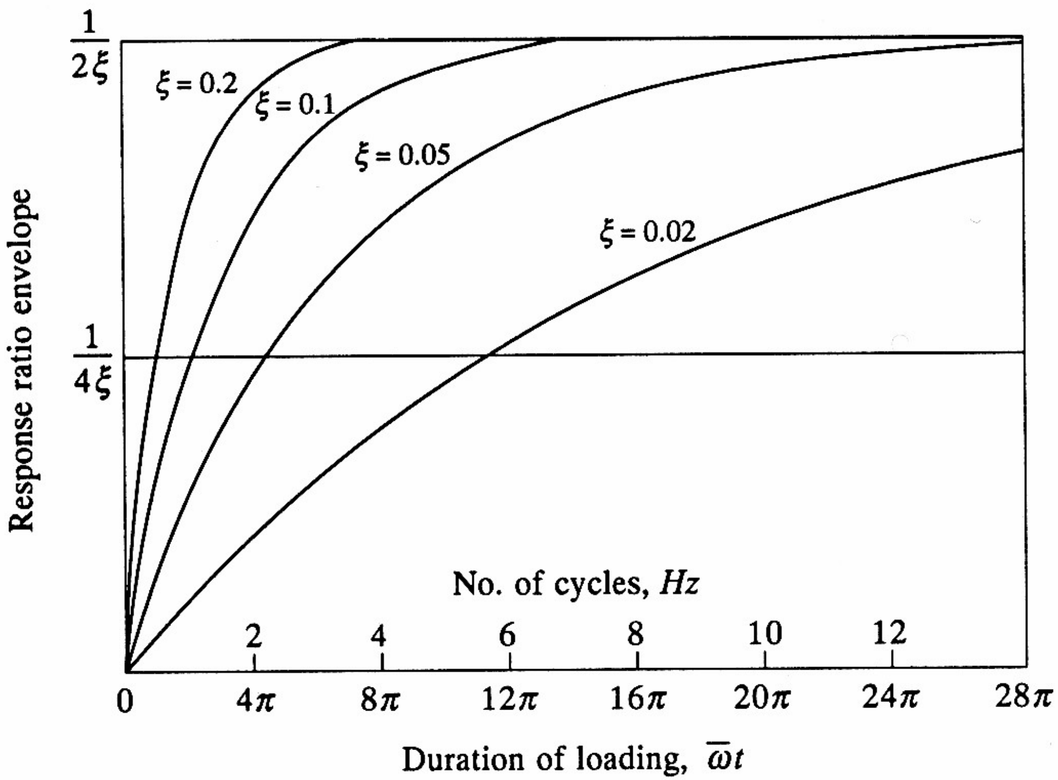


FIGURE 3-8
Rate of buildup of resonant response from rest.