Structural Dynamics
構造動力学
(2)

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2.6 DAMPED FREE VIBRATIONS

- If damping is present in the system, the solution of Eq. (2.25) is

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2} \qquad (2.39)$$

$$\boxed{s^2 + \frac{c}{m}s + \omega^2 = 0 \qquad (2.25)}$$

- Three types of motion are represented by this expression, according to whether the quantity under the square-root sign is positive, negative or zero.

Critically-Damped Systems （臨界減衰システム）

- If the radical term in Eq. (2.39) is set equal to zero, it is evident that $c/2m = \omega$; thus, the critical value of the damping coefficient (**臨界減衰係数**), c_c, is

$$c_c = 2m\omega = 2\sqrt{mk} \qquad (2.40)$$

$$\boxed{s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2} \qquad (2.39)}$$

- Then,

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega \qquad (2.41)$$

●The solution of Eq. (2.20) in this special case must now be of the form

$$v(t) = (G_1 + G_2 t)e^{-\omega t} \qquad (2.42)$$

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = 0 \qquad (2.20)$$

Because $\exp(-\omega t)$ is a real function, the constants G_1 and G_2 must also be real.

●Using the initial conditions $v(0)$ and $\dot{v}(0)$, these constants can be evaluated leading to

$$v(t) = \{v(0)(1-\omega t) + \dot{v}(0)t\}e^{-\omega t} \qquad (2.43)$$

●Fig. 2.9 shows the response for positive values of $v(0)$ and $\dot{v}(0)$.

$$v(t) = \{v(0)(1-\omega t) + \dot{v}(0)t\}e^{-\omega t} \qquad (2.43)$$

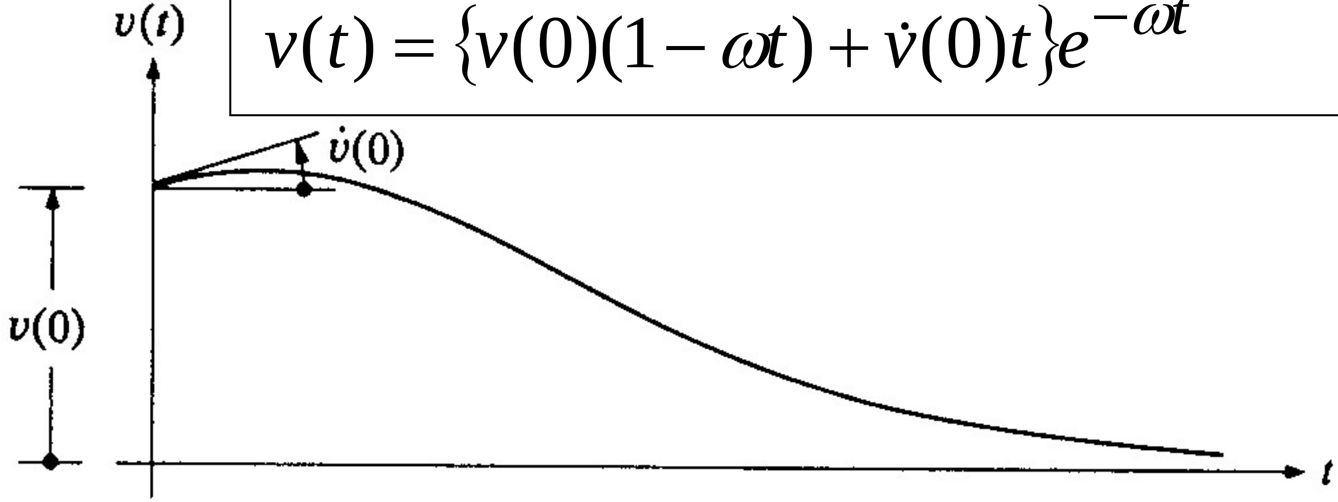

FIGURE 2-9
Free-vibration response with critical damping.

- Note that this free response of a critically-damped system（**臨界減衰システム**）does not include oscillation about the zero-deflection position; instead it simply returns to zero asymptotically in accordance with the exponential term of Eq. (2.43).
- However a single zero-displacement crossing would occur if the signs of the initial velocity and displacement were different from each other.

$$v(t) = \{v(0)(1-\omega t) + \dot{v}(0)t\}e^{-\omega t} \qquad (2.43)$$

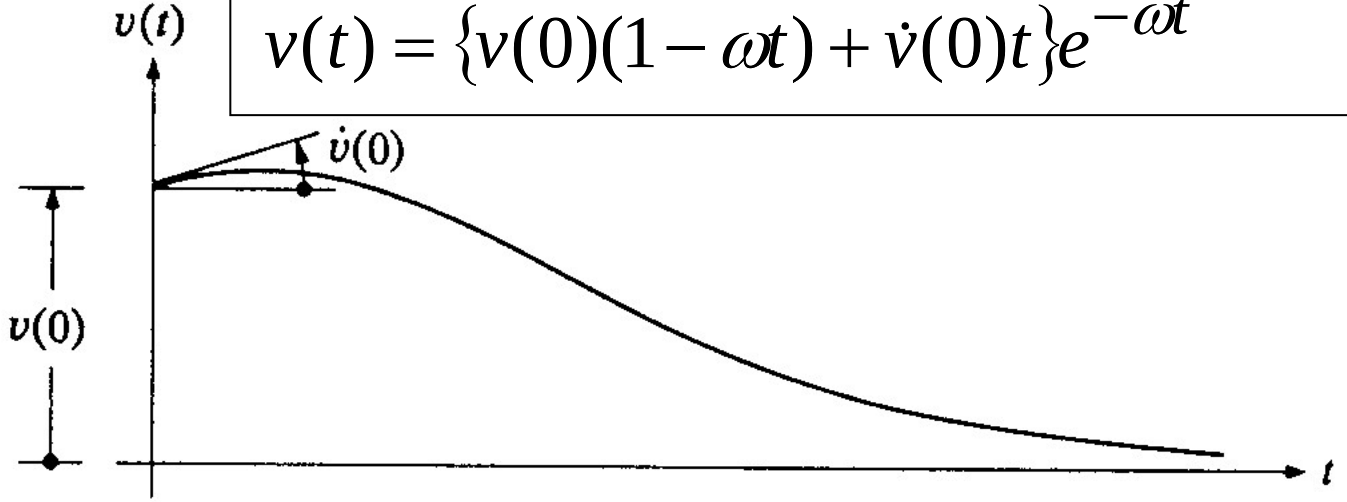

FIGURE 2-9
Free-vibration response with critical damping.

- A very useful definition of the critically-damped condition is that it represents the smallest amount of damping for which no oscillation occurs in the free-vibration response.

Undercritically-Damped Systems

- If damping is less than critical, that is, if $c<c_c$ (i.e., $c < 2m\omega$, it is apparent that the quantity under the radical sign in Eq. (2.39) is negative.

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2} \qquad (2.39)$$

- To evaluate the free-vibration response in this case, it is convenient to express damping in terms of a damping ratio ξ (**減衰定数**) which is the ratio of the given damping to the critical value;

$$\xi \equiv \frac{c}{c_c} = \frac{c}{2m\omega} \qquad (2.44)$$

● Introducing Eq. (2.44) into Eq. (2.39) leads to

$$s_{1,2} = -\xi\omega \pm i\omega_D \tag{2.45}$$

where

$$\omega_D \equiv \omega\sqrt{1-\xi^2} \tag{2.46}$$

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2} \tag{2.39}$$

$$\xi \equiv \frac{c}{c_c} = \frac{c}{2m\omega} \tag{2.44}$$

is the free-vibration frequency of the damped system (damped angular natural frequency, **減衰角固有振動数**).

Damping ratio of structures

Standard bridges $\xi = 0.05 - 0.07$

Long span bridges $\xi = 0.01 - 0.05$

Suspension bridges and cable stayed bridges

$$\xi = 0.005 - 0.02$$

Sloshing of liquid $\xi = 0.001 - 0.01$

● Note that for low damping values which are typical of most practical structures, $\xi < 20\%$, the frequency ratio ω_D / ω as given by Eq. (2.46) is nearly unity. The relation between damping ratio and frequency is represented in Fig. 2.10.

$$\omega_D \equiv \omega\sqrt{1-\xi^2} \qquad (2.46)$$

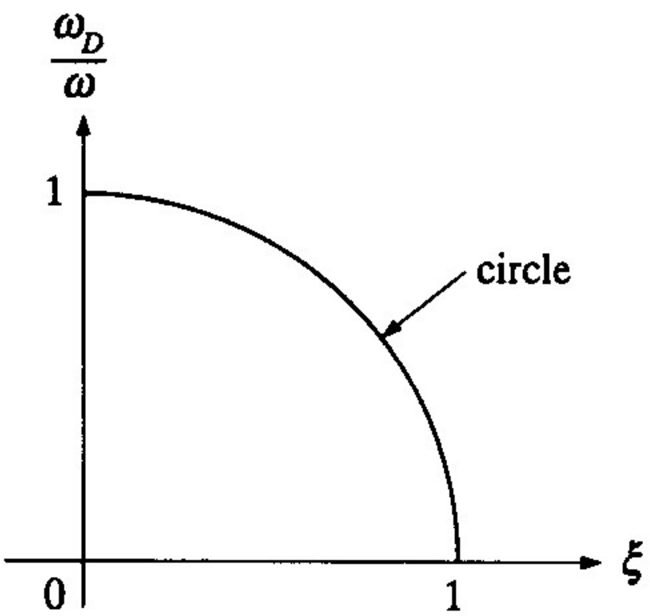

FIGURE 2-10
Relationship between frequency ratio and damping ratio.

- Using Eq. (2.21) and two values of s given by Eq. (2.45), the free-vibration response becomes

$$v(t) = \left\{ G_1 e^{i\omega_D t} + G_2 e^{-i\omega_D t} \right\} e^{-\xi\omega t} \qquad (2.47)$$

in which the constants G_1 and G_2 must be complex conjugate pairs for the response *v(t)* to be real, i.e.,

$$G_1 = G_R + iG_I$$
$$G_2 = G_R - iG_I$$

$$v(t) = Ge^{st} \qquad (2.21)$$

$$s_{1,2} = -\xi\omega \pm i\omega_D \qquad (2.45)$$

- The response by Eq. (2.47) can be represented by vectors in the complex plane similar to those shown in Fig. 2.6 for undamped case
- Only difference is that the damped circular frequency ω_D must be substituted for the undamped circular frequency ω and the magnitude of the vectors must be forced to decay exponentially with time in accordance with the term outside of the bracket, $e^{-\xi\omega t}$

$$v(t) = \left(G_1 e^{i\omega_D t} + G_2 e^{-i\omega_D t}\right) e^{-\xi\omega t} \qquad (2.47)$$

$$\boxed{v(t) = G_1 e^{i\omega t} + G_2 e^{-i\omega t} \qquad (2.27)}$$

$$v(t) = \left\{ (G_R + iG_I) e^{i\omega_D t} + (G_R - iG_I) e^{-i\omega_D t} \right\} e^{-\xi\omega t}$$

$$\boxed{v(t) = (G_R + iG_I) e^{i\omega t} + (G_R - iG_I) e^{-i\omega t} \qquad (2.29)}$$

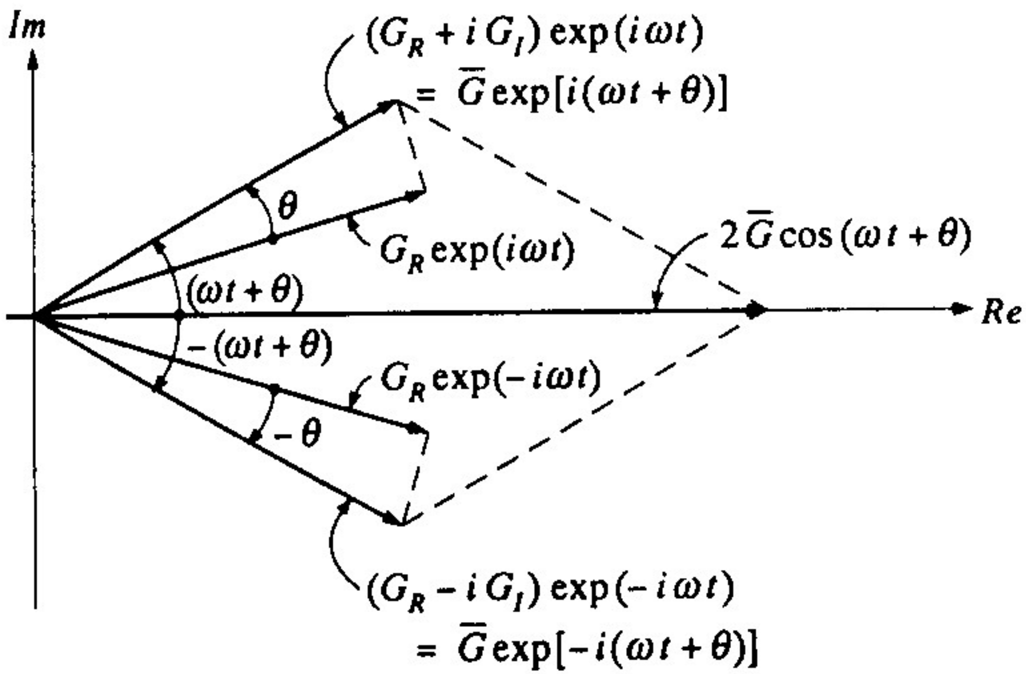

FIGURE 2-6
Total free-vibration response.

●The same procedure used in arriving Eq. (2.31) results in the equivalent trigonometric form

$$v(t) = \{A\cos\omega_D t + B\sin\omega_D t\}e^{-\xi\omega t} \qquad (2.48)$$

where $A=2G_R$ and $B=-2G_I$.

$$\boxed{v(t) = A\cos\omega t + B\sin\omega t \qquad (2.31)}$$

●Using the initial conditions v(0) and ,
constants A and B can be evaluated leading to

$$v(t) = \left\{v(0)\cos\omega_D t + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D}\right)\sin\omega_D\right\}e^{-\xi\omega t}$$

(2.49)

- Eq. (2.49) can be written as

$$v(t) = \rho \cos(\omega_D t + \theta) e^{-\xi \omega t} \qquad (2.50)$$

where,

$$\rho = \sqrt{v(0)^2 + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D} \right)^2} \qquad (2.51)$$

$$\theta = -\tan^{-1}\left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D v(0)} \right) \qquad (2.52)$$

$$\boxed{v(t) = \left\{ v(0)\cos\omega_D t + \left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D} \right) \sin\omega_D \right\} e^{-\xi\omega t} \qquad (2.49)}$$

- A plot of the response of an undercritically-damped system subjected to an initial displacement v(0) but starting with zero velocity is shown in Fig. 2.11.
- The undedamped system oscillates about the neutral position, with a constant circular frequency ω_D .

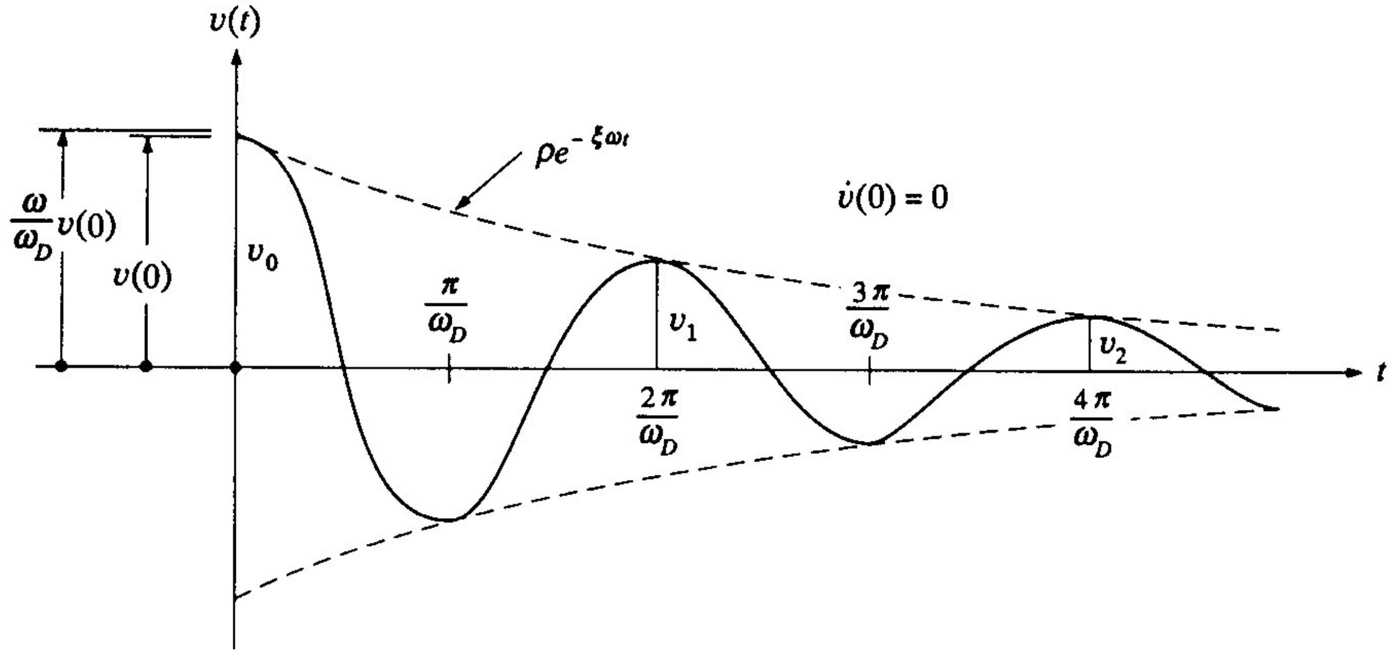

FIGURE 2-11
Free-vibration response of undercritically-damped system.

Evaluation of damping ratio of structures based on free-oscillation

- Consider any two successive positive peaks such as v_n and v_{n+1} which occur at time $n(2\pi/\omega_D)$ and $(n+1)(2\pi/\omega_D)$, respectively.

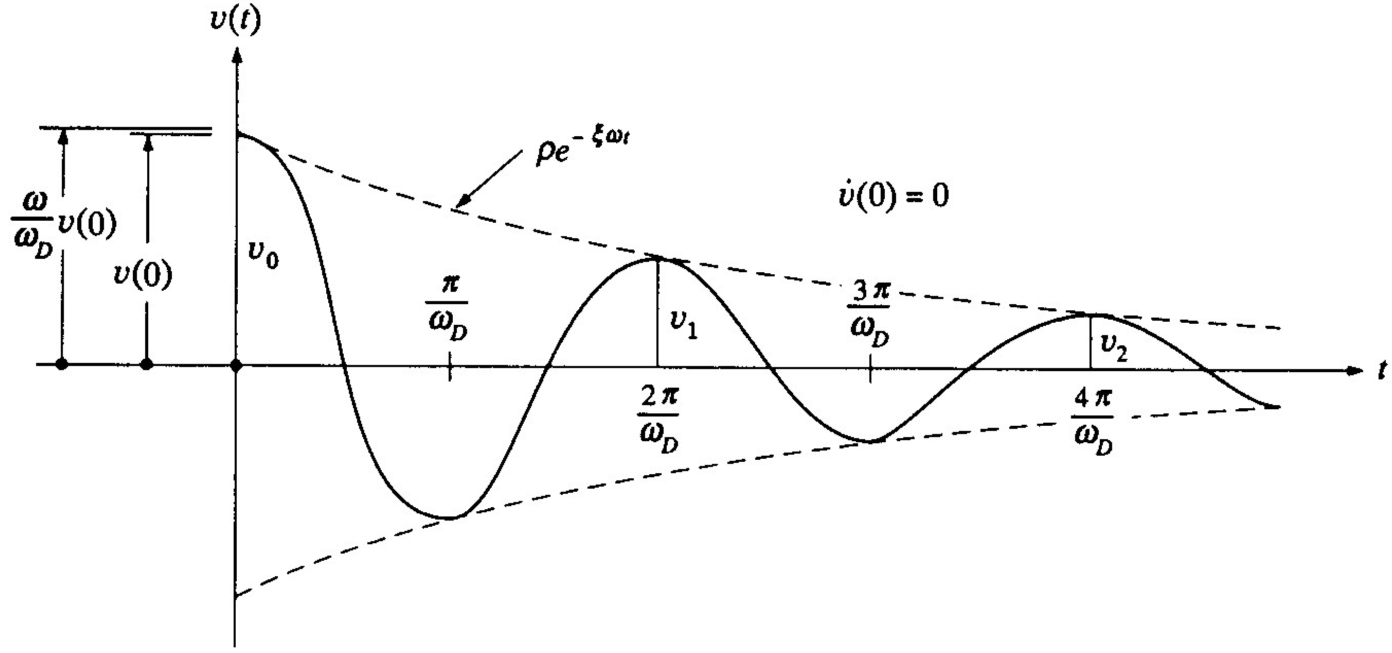

FIGURE 2-11
Free-vibration response of undercritically-damped system.

Using Eq. (2.50), the ratio of these two successive values is given by

$$\frac{v_n}{v_{n+1}} = e^{2\pi\xi\omega/\omega_D} \qquad (2.53)$$

$$\boxed{v(t) = \rho\cos(\omega_D t + \theta)e^{-\xi\omega t} \qquad (2.50)}$$

- Defining the natural logarithm of Eq. (2.53), one obtains

$$\delta \equiv \ln\frac{v_n}{v_{n+1}} = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \qquad (2.54)$$

δ is generally called logarithmic decrement of damping (**対数減衰率**)

- For low values of damping ratio, Eq. (2.54) can be approximated by

$$\delta \approx 2\pi\xi \qquad (2.55)$$

$$\boxed{\delta \equiv \ln\frac{v_n}{v_{n+1}} = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \qquad (2.54)}$$

- In Eq. (2.54),

$$\frac{v_n}{v_{n+1}} = e^{\delta} \approx 1 + 2\pi\xi + \frac{(2\pi\xi)^2}{2!} + \cdots \qquad (2.56)$$

- Sufficient accuracy is obtained by retaining only the first two terms in the Taylor's series expression, in which case

$$\xi \approx \frac{v_n - v_{n+1}}{2\pi v_{n+1}} \qquad (2.57)$$

- For lightly damped systems, greater accuracy in evaluating the damping ratio can be obtained by considering response peaks which are several cycles apart, say m cycles; then

$$\ln \frac{v_n}{v_{n+m}} = \frac{2m\pi\xi}{\sqrt{1-\xi^2}} \qquad (2.58)$$

$$\boxed{\delta \equiv \ln \frac{v_n}{v_{n+1}} = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \qquad (2.54)}$$

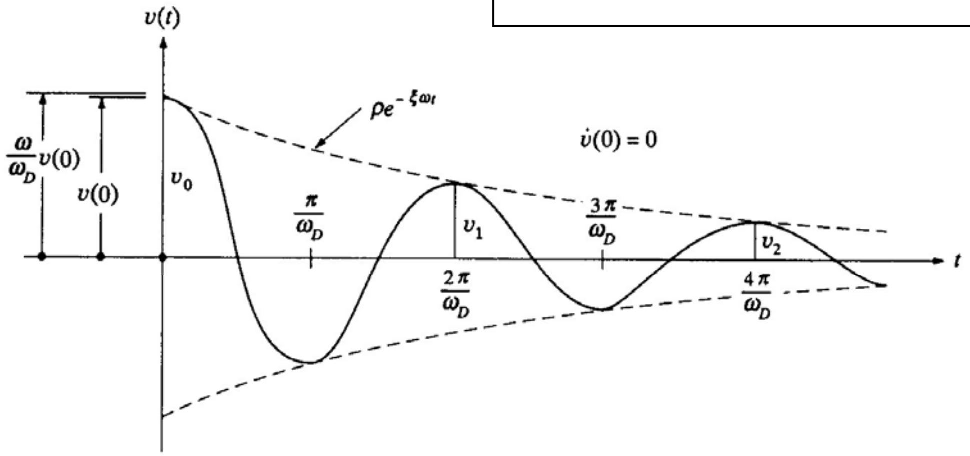

FIGURE 2-11
Free-vibration response of undercritically-damped system.

●Eq. (2.58) can be simplified for low damping to an approximation relation equivalent to Eq. (2.57):

$$\xi \approx \frac{v_n - v_{n+m}}{2m\pi v_{n+m}} \qquad (2.59)$$

$$\ln \frac{v_n}{v_{n+m}} = \frac{2m\pi\xi}{\sqrt{1-\xi^2}} \qquad (2.58)$$

$$\xi \approx \frac{v_n - v_{n+1}}{2\pi v_{n+1}} \qquad (2.57)$$

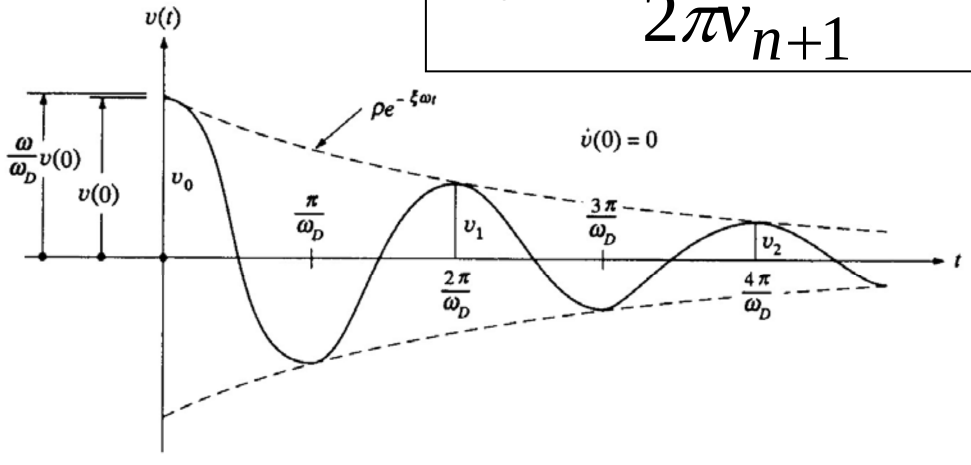

FIGURE 2-11
Free-vibration response of undercritically-damped system.

Why Does Eq. (2.59) provide more accurate estimation than Eq. (2.57)?

Measured free oscillation decay (**自由減衰振動**) is not necessarily smooth, but includes noise as shown in the following figure.

- When damped free vibrations are observed experimentally, a convenient method for estimating the damping ratio is to count the number of cycles required to give a 50% reduction in amplitude.
- The relation to be used in this case is presented in Fig. 2.13.

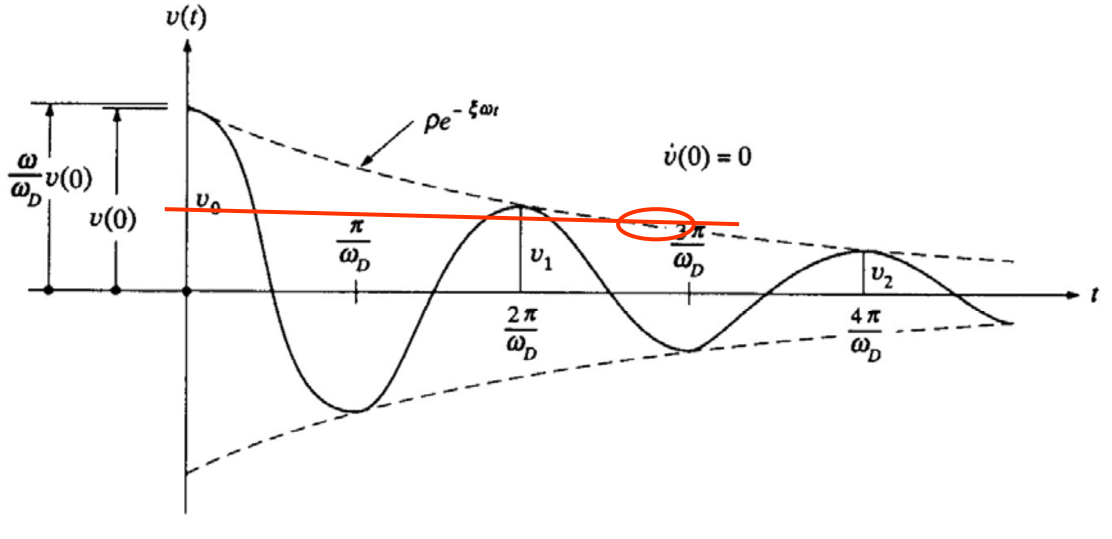

FIGURE 2-11
Free-vibration response of undercritically-damped system.

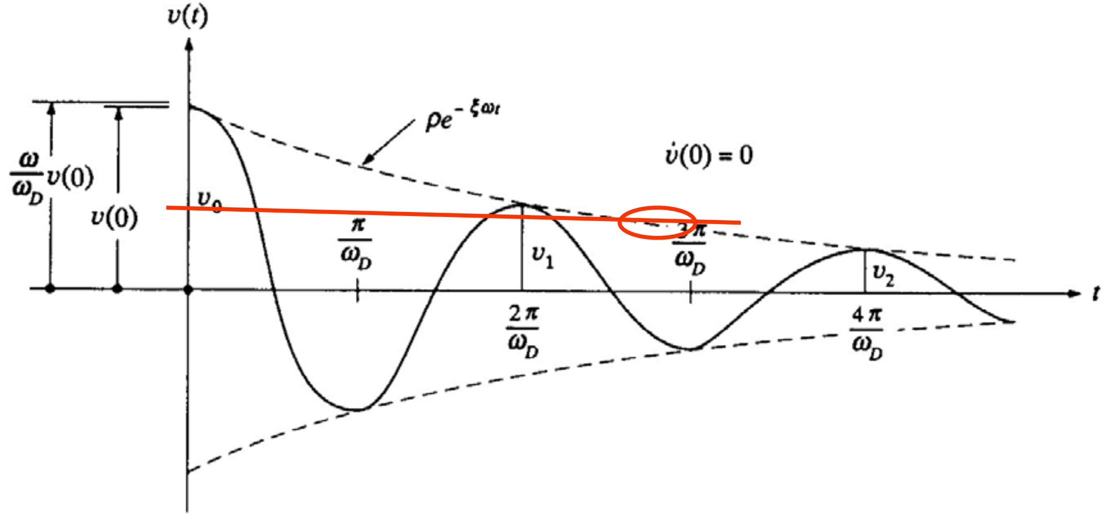

FIGURE 2-11
Free-vibration response of undercritically-damped system.

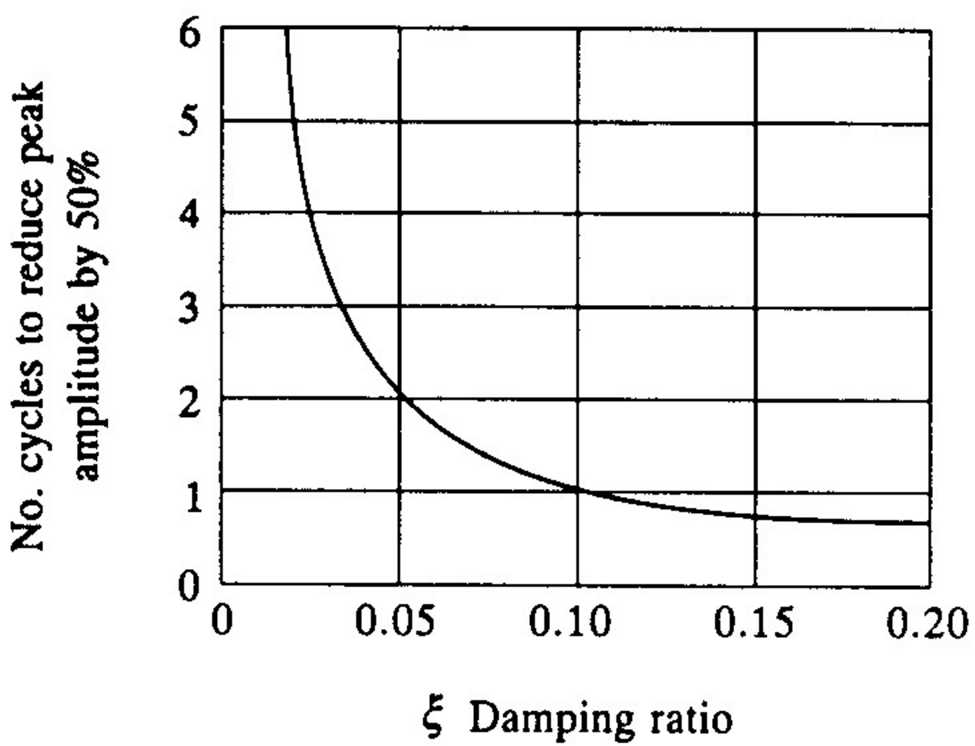

FIGURE 2-13
Damping ratio vs. number of cycles required to reduce peak amplitude by 50 percent.

Example 2.1

- A bridge is idealized as a rigid girder supported by weightless columns as shown in Fig. E2.1.
- In order to evaluate the dynamic properties of this structure, a free-vibration test is made, in which the girder is displaced laterally by a hydraulic jack and then suddenly released.

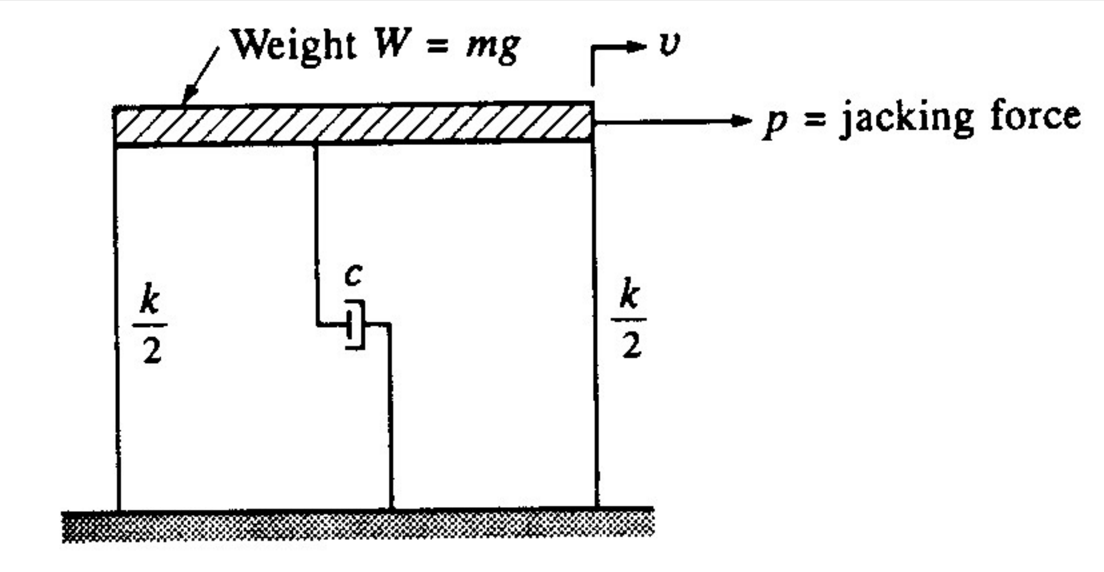

Fig. E2.1 Vibration test of a simple bridge

- During the jacking operation, it is observed that a force of 20 kips (9072kg) is required to displace the girder 0.2 in (0.508cm). After the instantaneous release of this initial displacement, the maximum displacement on the first return swing is only 0.16 in (0.406cm) and the period of this displacement cycle is T=1.40 sec.

- From these data, the following dynamic properties are determined:

- Effective weight of the girder:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{W}{gk}} = 1.40s$$

Hence $870.9 \times 10^3 kg$

$$W = \left(\frac{1.40}{2\pi}\right)^2 gk = 0.0496 \times 386 \times \frac{20}{0.2} = 1920kips$$

- Undamped frequency of vibration:

$$f = \frac{1}{T} = \frac{1}{1.40} = 0.714Hz$$

$$\omega = 2\pi f = 4.48 rad/s$$

- Damping properties:

Logarithmic decrement $\delta = \ln \frac{0.20}{0.16} = 0.223$

Damping ratio $\xi \approx \frac{\delta}{2\pi} = 3.55\%$

Damping coefficient

$$c = \xi c_c = \xi 2m\omega = 0.0355 \times \frac{2 \times 1{,}920}{386} \times 4.48$$

Damped frequency

$$\omega_D = \omega\sqrt{1-\xi^2} = \omega\sqrt{1-0.00355^2} \approx \omega$$

- Amplitude after six cycles:

$$v_6 = \left(\frac{v_1}{v_0}\right)^6 v_0 = \left(\frac{0.16}{0.20}\right)^6 \times 0.20 = 0.0524 in$$

Overcritically damped systems

- Because $\xi \equiv \dfrac{c}{c_c} > 1$, solutions of Eq. (2.39) are

$$s_{1,2} = -\xi\omega \pm \omega\sqrt{\xi^2 - 1} = -\xi\omega \pm \tilde{\omega} \qquad (2.60)$$

in which

$$\tilde{\omega} = \omega\sqrt{\xi^2 - 1} \qquad (2.61)$$

$$\boxed{s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2} \qquad (2.39)}$$

- Substituting Eq. (2.60) into Eq. (2.21) and simplifying leads to

$$v(t) = \{A \sinh \tilde{\omega} t + B \cosh \tilde{\omega} t\} e^{-\xi \omega t} \qquad (2.62)$$

$$\boxed{v(t) = G e^{st} \qquad (2.21)}$$

- The real constants A and B can be evaluated using the initial conditions $v(0)$ and $\dot{v}(0)$ as

$$A = \frac{\dot{v}_0(0) + v_o(0)\xi\omega}{\tilde{\omega}}$$

$$B = v_0(0) \qquad (2.63)$$

● Substituting Eq. (2.63) into Eq.(2.62) leads to

$$v(t) = \left\{ \frac{\dot{v}_0(0) + v_0(0)\xi\omega}{\tilde{\omega}} \sinh \tilde{\omega}t + v_0(0) \cosh \tilde{\omega}t \right\} e^{-\xi\omega t} \tag{2.64}$$

$$v(t) = \{A \sinh \tilde{\omega}t + B \cosh \tilde{\omega}t\} e^{-\xi\omega t} \tag{2.62}$$

$$A = \frac{\dot{v}_0(0) + v_o(0)\xi\omega}{\tilde{\omega}}$$

$$B = v_0(0) \tag{2.63}$$

● From the form of Eq. (2.64), the response of an over-damped system is similar to the motion of a critically-damped system as shown in Fig. 2.9.

$$v(t) = \left\{ \frac{\dot{v}_0(0) + v_0(0)\xi\omega}{\tilde{\omega}} \sinh \tilde{\omega} t + v_0(0) \cosh \tilde{\omega} t \right\} e^{-\xi\omega t} \tag{2.64}$$

$$v(t) = \{v(0)(1-\omega t) + \dot{v}(0)t\} e^{-\omega t} \tag{2.43}$$

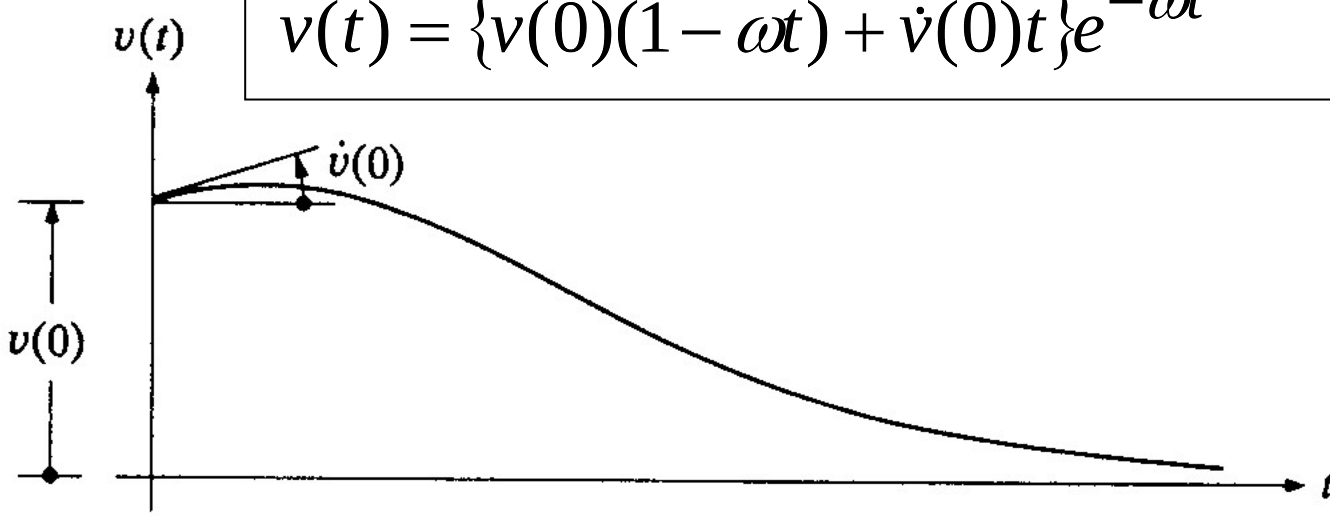

FIGURE 2-9
Free-vibration response with critical damping.

Numerical example

● Free-vibration decay using Eq. (2.64) as well as Eq. (2.43) was computed for a simple bridge shown in Fig. E2.1. The following parameters were assumed here based on Example E2.1,

$$T = 1.4s$$

$$v(0) = 0.2inch = 0.508cm$$

$$\dot{v}(0) = 0$$

$$\xi = 1.0, 1.5, 2.0, 10.0$$

● Fig. E2.2 shows free decays of critically damped system (Eq. (2.43)) and over-damped system (Eq. (2.64)).

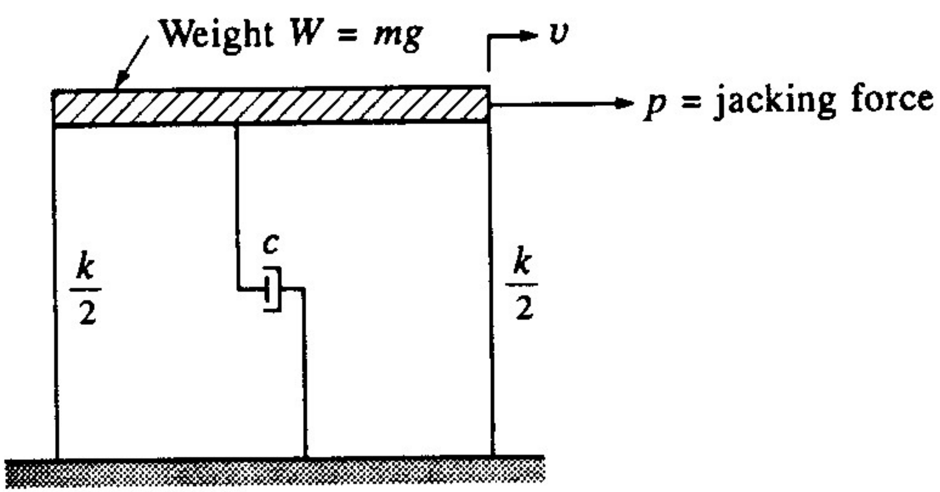

Fig. E2.1 Vibration test of a simple bridge

$$v(t) = \{v(0)(1-\omega t) + \dot{v}(0)t\}e^{-\omega t} \qquad (2.43)$$

$$v(t) = \left\{ \frac{\dot{v}_0(0) + v_0(0)\xi\omega}{\tilde{\omega}} \sinh \tilde{\omega}t + v_0(0)\cosh \tilde{\omega}t \right\} e^{-\xi\omega t} \qquad (2.64)$$

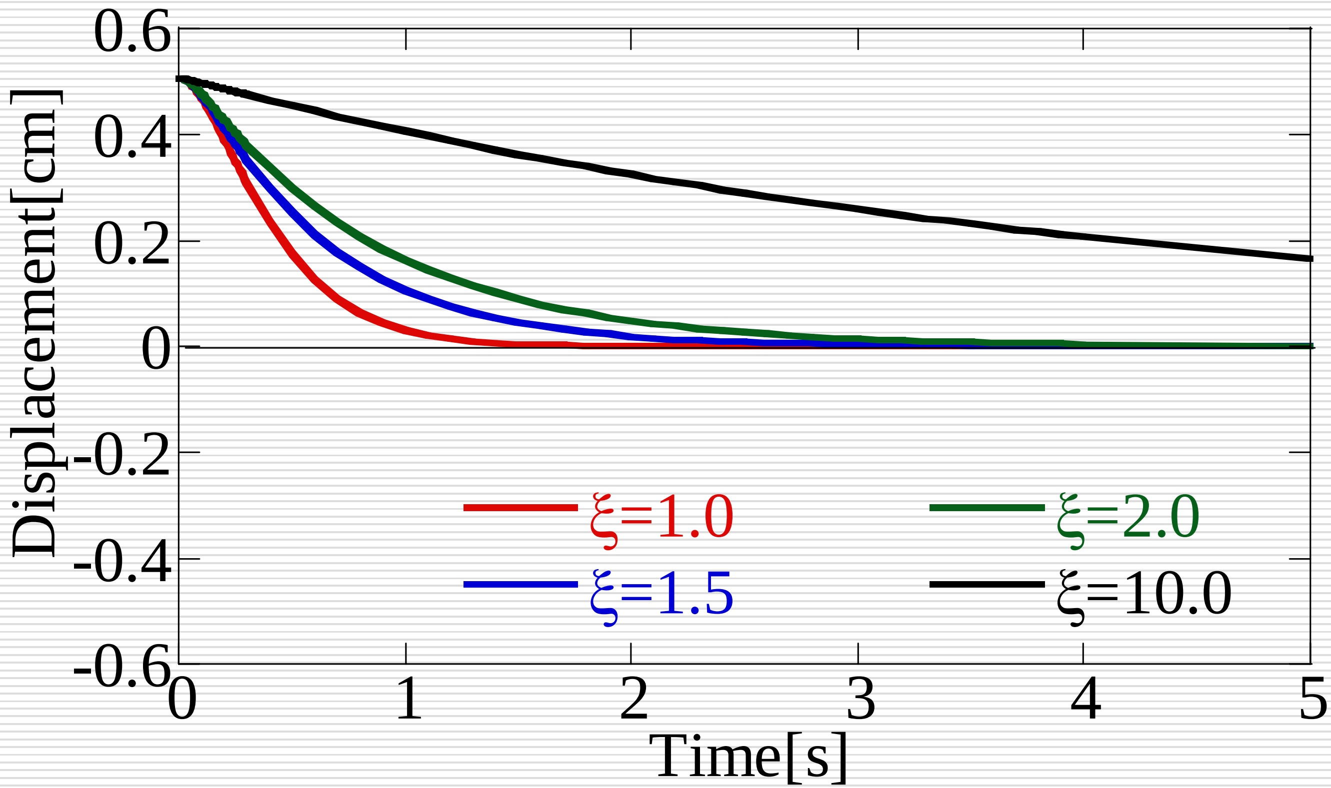

Fig. E2.2 Comparison of critically damped and over-damped system

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- However the asymptotic return to the zero-displacement position is slower depending on the amount of damping. This will be shown in the next numerical example.