Structural Dynamics
構造動力学
(1)

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INTRODUCTION

- Structural dynamics is basis for the analysis of structures under non-static loads, that is, dynamic loads.
- The structural dynamics is applied for analysis of structures subjected to earthquake loads, wind loads, vibration control, blasting loads.
- In particular, because earthquake loads control construction of structures in earthquake prone countries including Japan, structural dynamics is essential for mitigating damage of structures and loss of lives.
- In this lecture, basics of structural dynamics is introduced with emphasis on application to seismic design of structures.

SCHEDULE

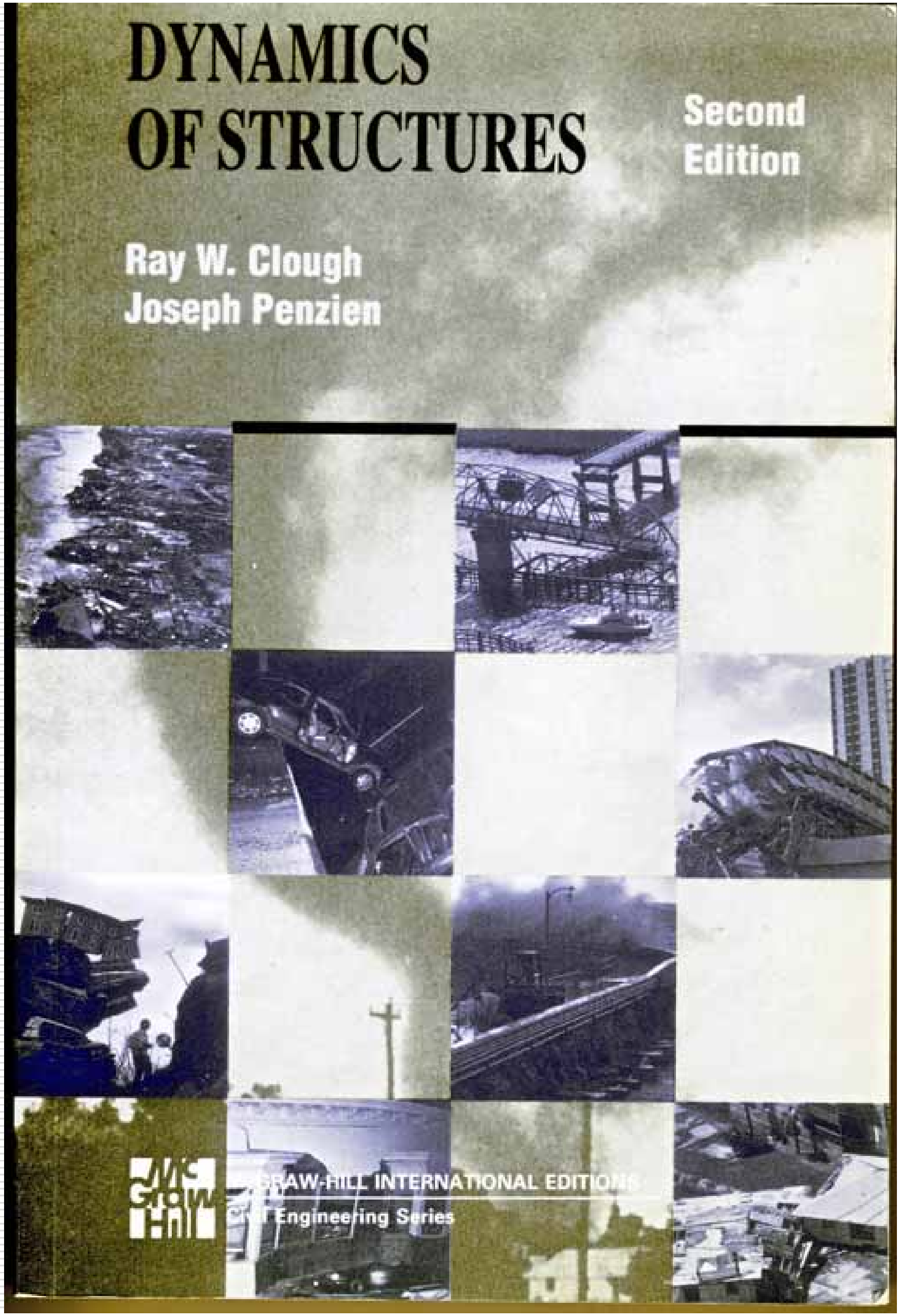
- 1st: April 13 (Tue)
- 2nd: April 20 (Tue)
- 3rd: April 27 (Tue)
- 4th: May 11 (Tue)
- 5th: May 18 (Tue)
- 6th: May 25 (Tue)
- 7th: June 1 (Tue)

- 8th: June 8 (Tue)
- 9th: June 15 (Tue)
- 10th: June 22 (Tue)
- 11th: June 29(Tue)
- 12th: July 6 (Tue)
- 13th: July 13(Tue)
- 14th: July 20(Tue)
- Final Exam: July 27(M)

All classes are provided at 13:20-15:50.

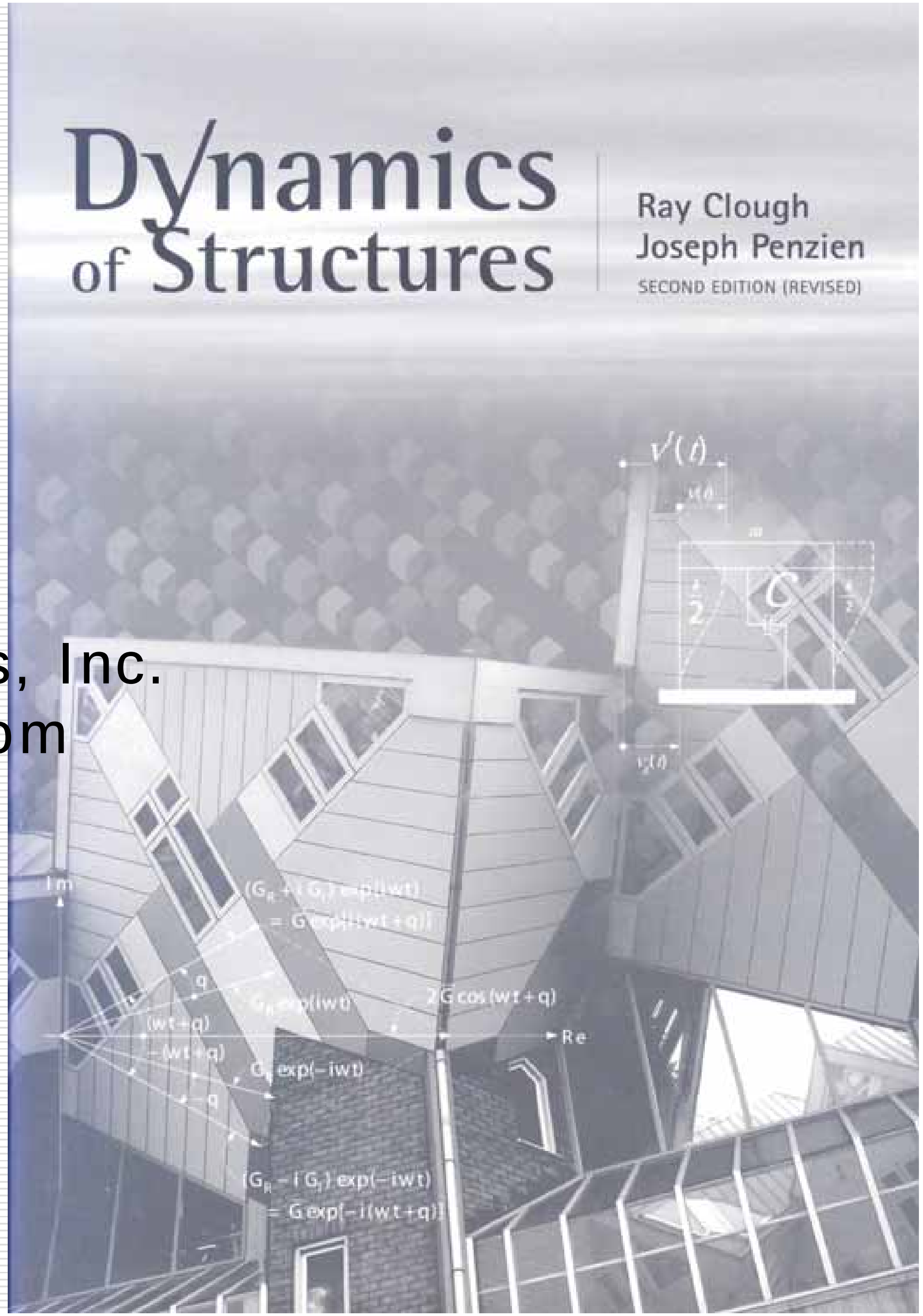
TEXT

Dynamics of Structures
by
Ray W. clough
and Joseph Penzien
University of California,
Berkeley



Revised version

Computers and Structures, Inc.
http://www.csiberkeley.com



CHAPTER 1 OVERVIEW OF STRUCTURAL DYNAMICS

1.1 FUNDAMENTAL OBJECTIVE OF STRUCTURAL DYNAMICS ANALYSIS

- Earthquake loading
- Wind loading
- Bombing
- Vibration and noise pollution
- ….

1.5 DIRECT EQUILIBRIUM USING d'ALEMBERT's PRINCIPLE (ダランベールの法則)

●The equations of motion of any dynamic system can be represented by Newton's second law of motion, which states that the rate of change of momentum of any mass particle m is equal to the force acting on it.

●The Newton's second law of motion is expressed mathematically by the differential equation as

$$\mathbf{p}(t) = \frac{d}{dt}\left(m\frac{d\mathbf{v}}{dt}\right) \qquad (1\text{-}3)$$

where, **p**(t) is the applied force vector and **v**(t) is the position vector of particular mass m.

● For most problems in structural dynamics, it may be assumed that mass does not vary with time, in which case Eq. (1.3) may be written

$$\mathbf{p}(t) = m\frac{d^2\mathbf{v}}{dt^2} = m\ddot{\mathbf{v}}(t) \qquad (1.3a)$$

where the dots represent differentiation with respect to time.

●Eq. (1.3a), indicating that force is equal to the product of mass and acceleration, may be written in the form

$$\mathbf{p}(t) - m\ddot{\mathbf{v}}(t) = 0 \qquad (1.3b)$$

in which, the second term $m\ddot{\mathbf{v}}(t)$ is called the inertial force (**慣性力**) resisting the acceleration of the mass.

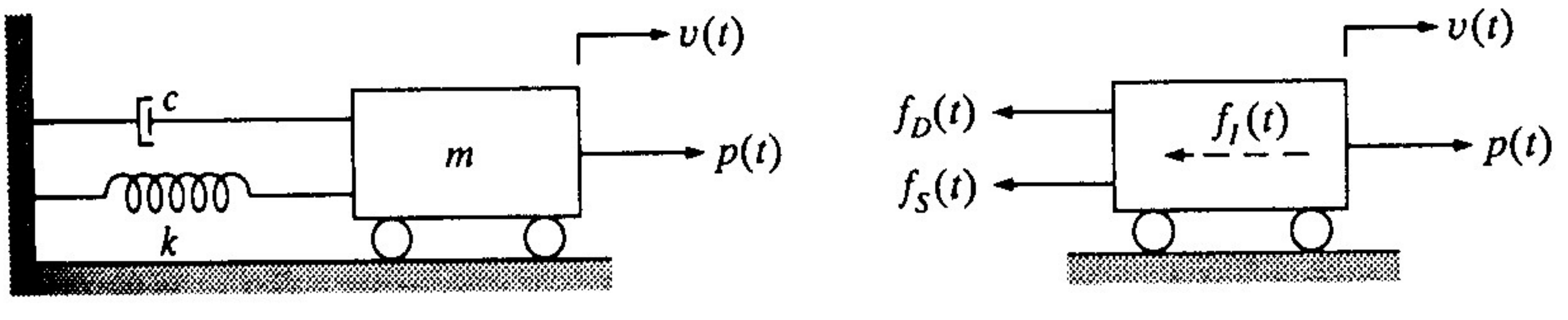
- The concept that a mass develops an inertia force proportional to its acceleration and opposing is known as d'Alembert's principle (**ダランベールの法則**).
- The d'Alembert's principle is a very convenient device in problems of structural dynamics because it permits the equations of motion to be expressed as equations of dynamic equilibrium.
- The force **p**(t) may be considered to include many types of force acting on the mass such as
  - ✓Elastic constraints which oppose displacements
  - ✓Viscous forces which resist velocities
  - ✓Independently defined external loads

- Thus if an inertia force which resists acceleration is introduced, the equation of motion is merely an expression of equilibration of all forces acting on the mass.
- In many simple problems, the most direct and convenient way of formulating the equations of motion is by means of such direct equilibrium.

CHAPTER 2 ANALYSIS OF FREE VIBRATIONS

2.1 COMPONENTS OF THE BASIC DYNAMIC SYSTEM

●The essential physical properties of any linearly elastic structural system subjected to an external source of excitation or dynamic loading are its mass, elastic properties, and energy-loss mechanism (**エネルギー吸収**) or damping (**減衰**) as shown in Fig. 2.1(a).



(a) Basic components (b) Forces in equilibrium

Fig. 2.1 Idealized SDOF system

2.2 EQUATION OF MOTION OF THE BASIC DYNAMIC SYSTEM

- The equation of motion for the simple system of Fig. 2.1 (a) is most easily formulated by directly expressing the equilibrium of all forces acting on the mass using d'Alembert's principle.

- The forces acting in the direction of the displacement degree of freedom are applied load p(t) and the three resisting forces resulted from the motion, i.e., the inertia force $f_I(t)$ (**慣性力**), the damping force $f_D(t)$ (**減衰力**), and the spring force $f_S(t)$.

- The equation of motion is merely an expression of the equilibrium of these forces as given by

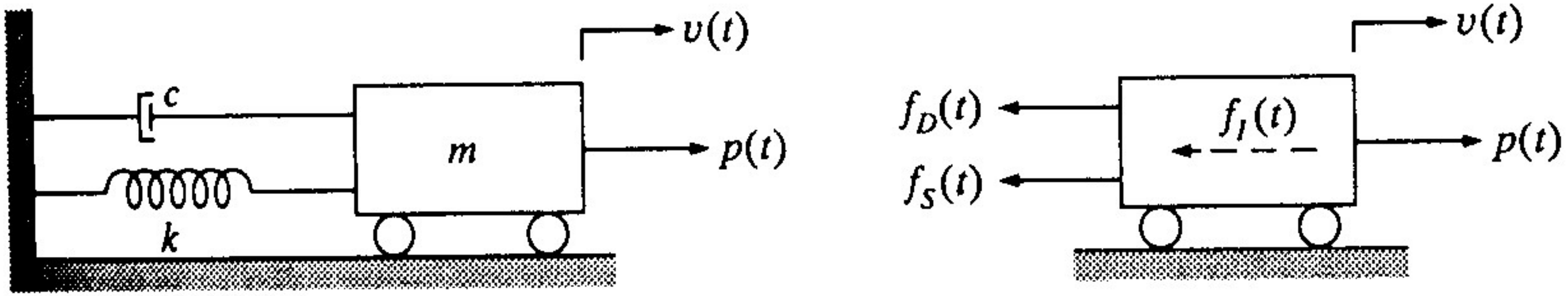
$$f_I(t) + f_D(t) + f_S(t) = p(t) \qquad (2.1)$$

●In accordance with d'Alembert's principle, the inertia force is the product of the mass and acceleration

$$f_I(t) = m\ddot{v}(t) \qquad (2.2a)$$

●Assuming a viscous damping mechanism, the damping force (**減衰力**) is the product of the damping constant c (**減衰係数**) and the velocity

$$f_D(t) = c\dot{v}(t) \qquad (2.2b)$$



(a) Basic components (b) Forces in equilibrium

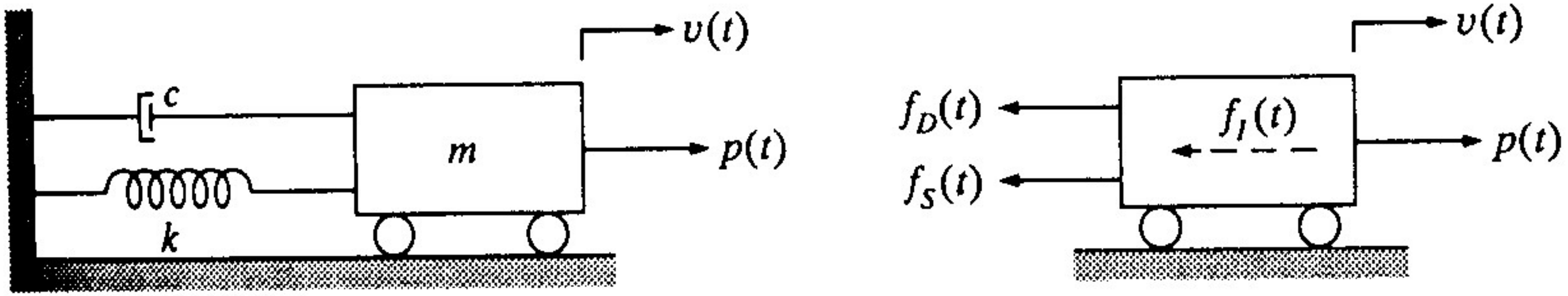
Fig. 2.1 Idealized SDOF system

●Finally, the elastic restoring force (**復元力**) is the product of the spring stiffness (**ばね定数、ばね剛性**)and the displacement

$$f_S(t) = kv(t) \qquad (2.2c)$$

●When Eqs. (2.2) are introduced into Eq. (2.1), the equation of motion for this single-degree-of-freedom system (SDOF, 1**自由度系**) is found to be

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = p(t) \qquad (2.3)$$



(a) Basic components (b) Forces in equilibrium

Fig. 2.1 Idealized SDOF system

2.3 INFLUENCE OF GRAVITATIONAL FORCES (重力)

- Consider the system shown in Fig. 2.2(a), which is the system of Fig. 2.1(a) rotated through 90 degree so that the forces of gravity acts in the direction of the displacement.
- The system of forces acting in the direction of the displacement degree of freedom is that set shown in Fig. 2.2(b).

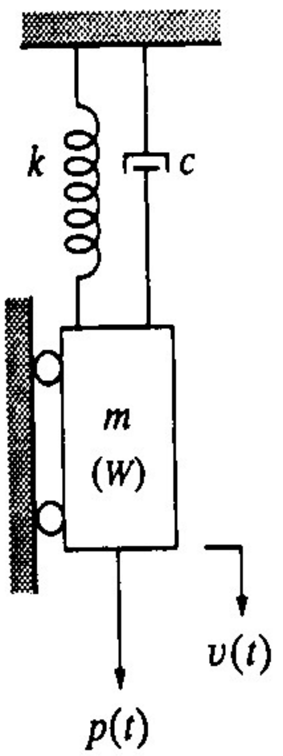


Fig. 2.2 (a)

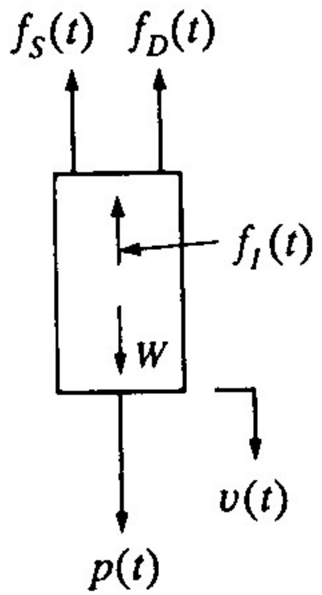


Fig. 2.2(b)

● Using Eqs. (2.2), the equilibrium of these forces is given by

$$m\ddot{v}(t)+c\dot{v}(t)+kv(t)=p(t)+W \qquad (2.6)$$

where W is the weight of the rigid block.

● However, if the total displacement v(t) is expressed the sum of the static displacement Δ_{st} caused by the weight W plus the additional dynamic displacement (**動的変位**) $\bar{v}(t)$ as shown in Fig. 2.2(c), i.e.,

$$v(t)=\Delta_{st}+\bar{v}(t) \qquad (2.7)$$

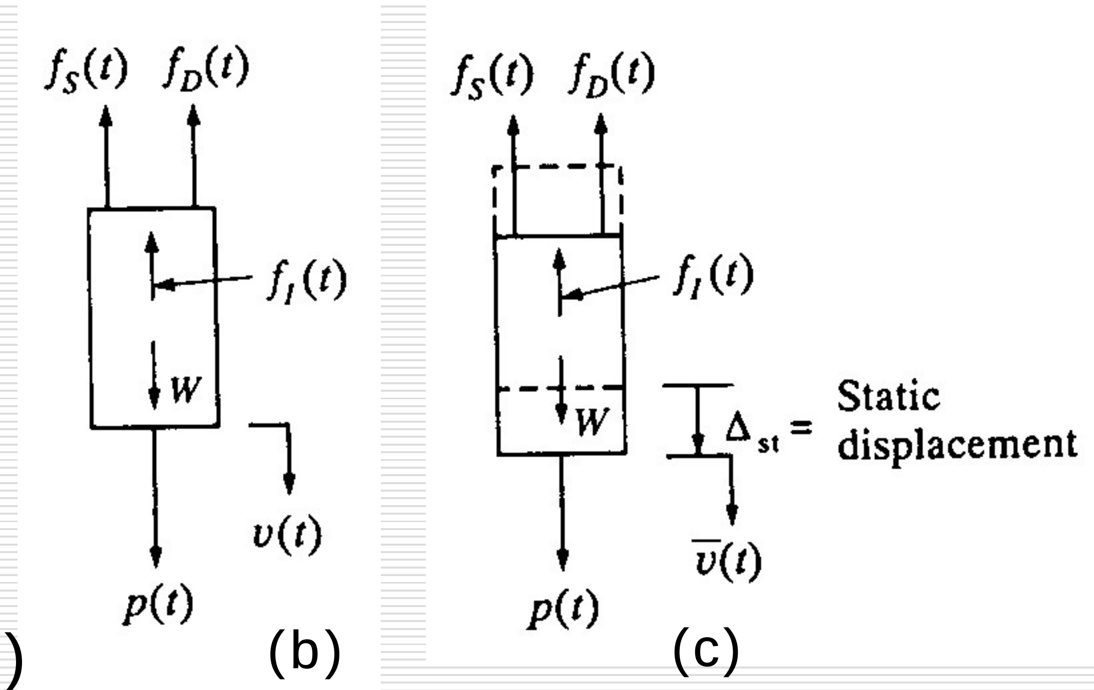


Fig. 2.2 (b)　Fig. 2.2 (c)

- The spring restoring force is given by

$$f_S(t) = kv(t) = k\Delta_{st} + k\bar{v}(t) \qquad (2.8)$$

- Introducing Eq. (2.8) into Eq. (2.6) yields

$$m\ddot{v}(t) + c\dot{v}(t) + k\Delta_{st} + k\bar{v}(t) = p(t) + W \qquad (2.9)$$

- Noting that $k\Delta_{st} = W$ leads to

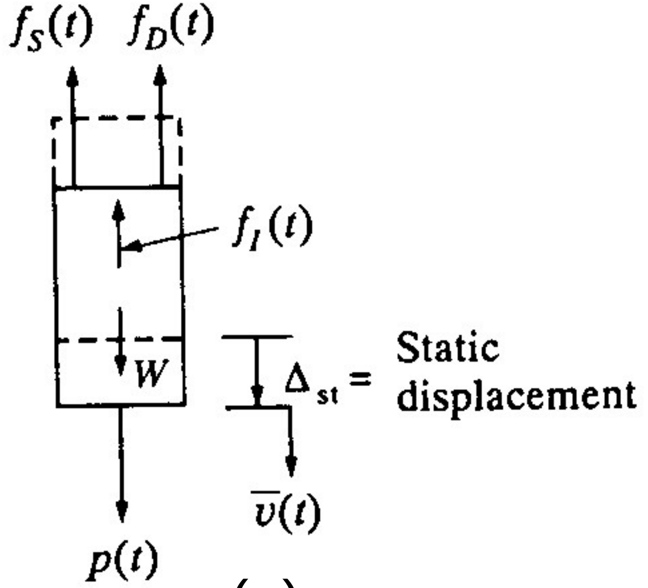
$$m\ddot{v}(t) + c\dot{v}(t) + k\bar{v}(t) = p(t)$$

(2.10)

- Because Δ_{st} does not vary in time

$$\dot{v}(t) = \frac{d}{dt}(\Delta_{st} + \bar{v}(t)) = \dot{\bar{v}}(t)$$

$$\ddot{v}(t) = \ddot{\bar{v}}(t)$$



(c)

Fig. 2.2 (c)

- So Eq. (2.10) can be written as Eq. (2.11)

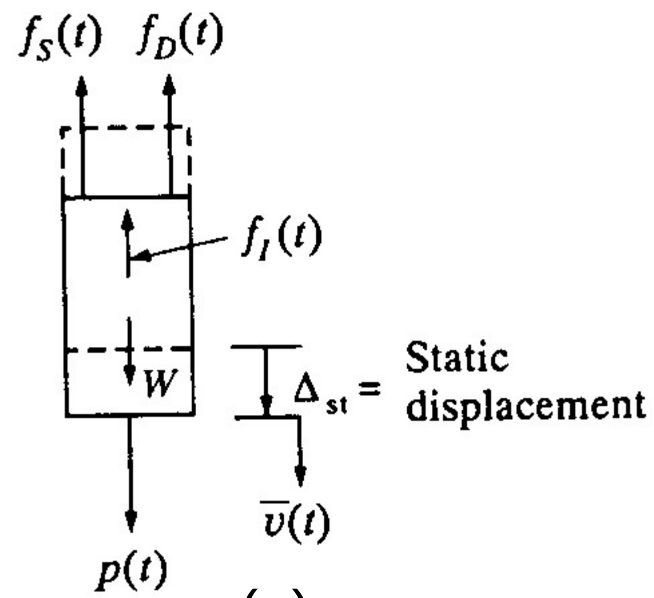
$$m\ddot{v}(t)+c\dot{v}(t)+k\bar{v}(t)=p(t) \qquad (2.10)$$

$$m\ddot{\bar{v}}(t)+c\dot{\bar{v}}(t)+k\bar{v}(t)=p(t) \qquad (2.11)$$

$$m\ddot{v}(t)+c\dot{v}(t)+kv(t)=p(t) \qquad (2.3)$$

- Comparison of Eqs. (2.11) and (2.3) shows that the equation of motion expressed with reference to the static equilibrium position of the dynamic system is not affected by gravity force.

- For this reason, displacements in all future discussion will be referenced from the static equilibrium position and will be denoted v(t) without the overbar.



(c)

Fig. 2.2 (c)

2.4 INFLUENCE OF SUPPORT EXCITATION

- Dynamic stresses and deflection can be induced in a structure not only by a time-varying applied load, but also by motions of its support points.
- A simplified model of the earthquake-excitation problem is shown in Fig. 2.3, in which the horizontal ground motion caused by the earthquake is indicated by the displacement $v_g(t)$ of the structure's base relative to the fixed reference axis.

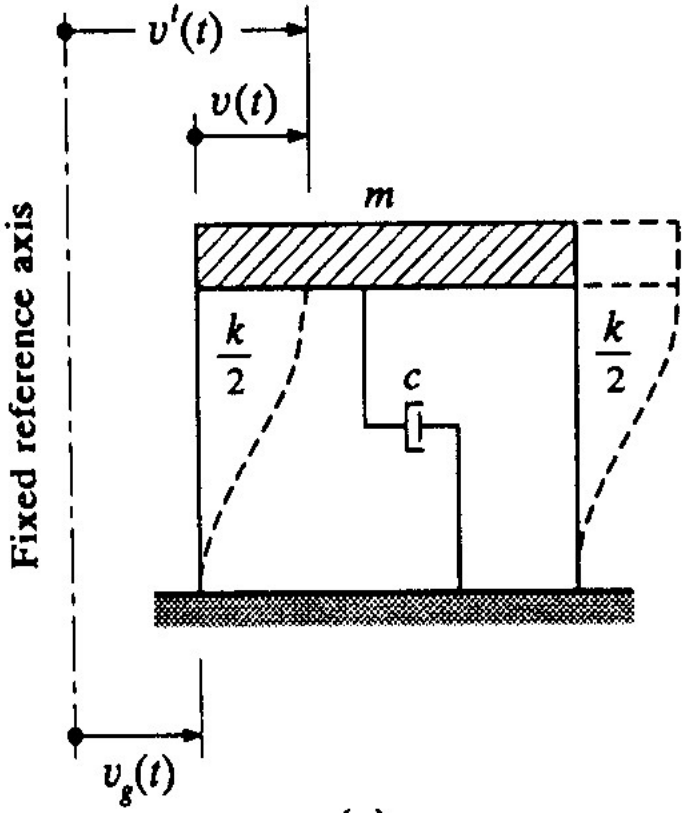


Fig. 2.3 Influence of support excitation on SDOF equilibrium

- The horizontal girder in this frame is assumed to be rigid and to include all the moving mass of the structure.
- The vertical columns are assumed to be weightless and inextensible in the vertical (axial) direction, and the resistance to girder displacement provided by each column is represented by its spring constant k/2.
- The mass thus has a single degree of freedom, v(t), which is associated with column flexure.

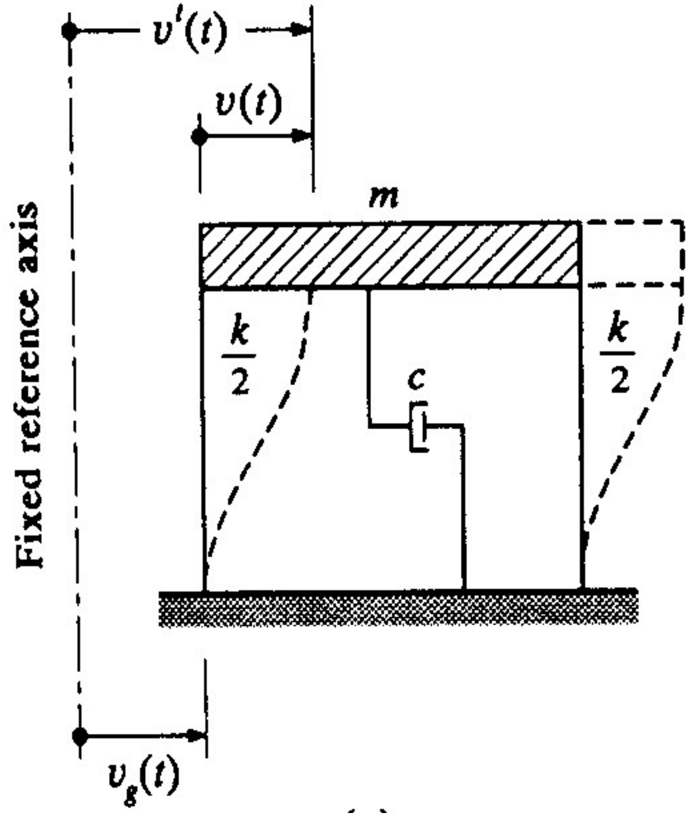


Fig. 2.3 Influence of support excitation on SDOF equilibrium

- The damper c provides a velocity-proportional resistance (**速度比例型抵抗**) to the motion in this coordinate.

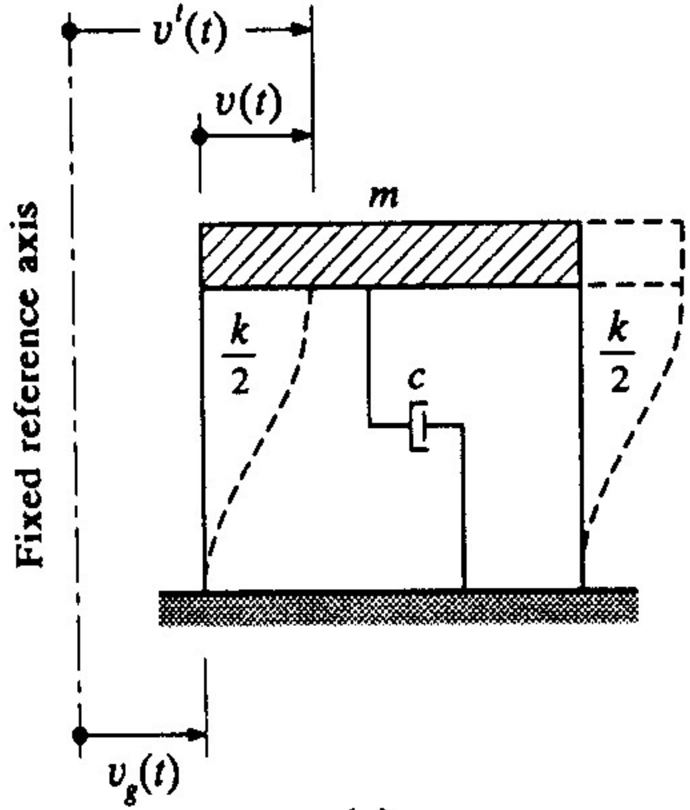


Fig. 2.3 Influence of support excitation on SDOF equilibrium

●As shown in Fig. 2.3(b), the equilibrium of forces for this system can be written as

$$f_I(t) + f_D(t) + f_S(t) = 0 \qquad (2.12)$$

where,

$$f_I(t) = m\ddot{v}^t(t) \qquad (2.13)$$

$$f_D(t) = c\dot{v}(t)$$

$$f_S(t) = kv(t)$$

●Note that $v^t(t)$ represents the total displacement of the mass from the fixed reference axis as

$$v^t(t) = v_g(t) + v(t) \qquad (2.15)$$

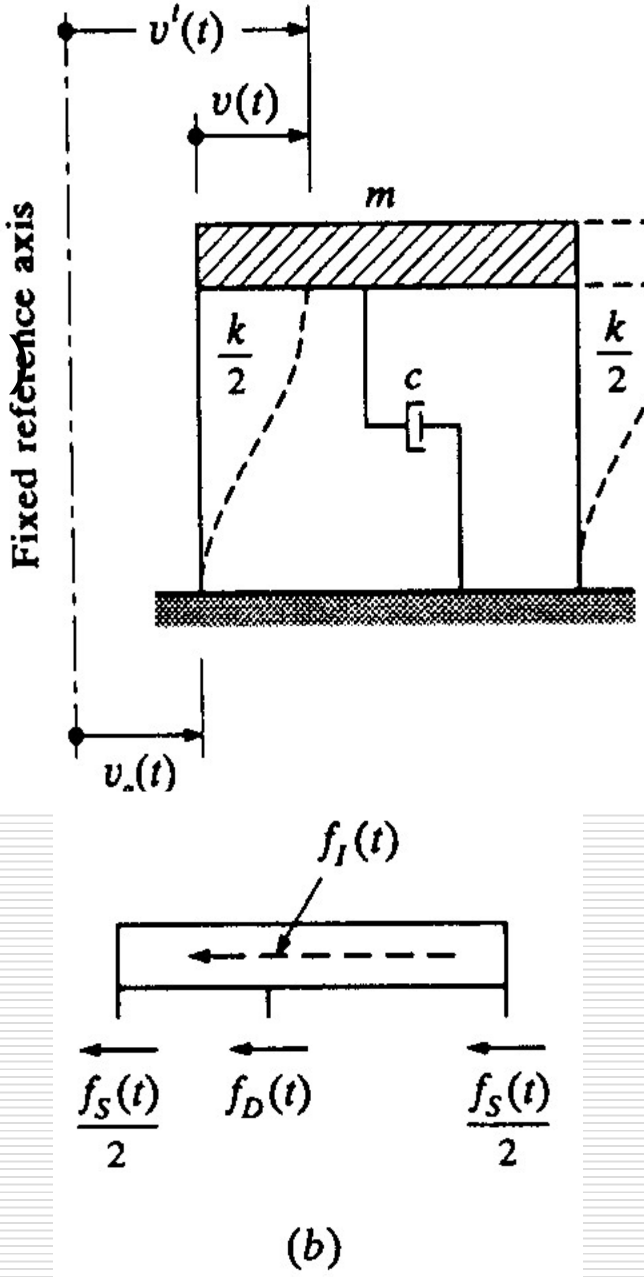
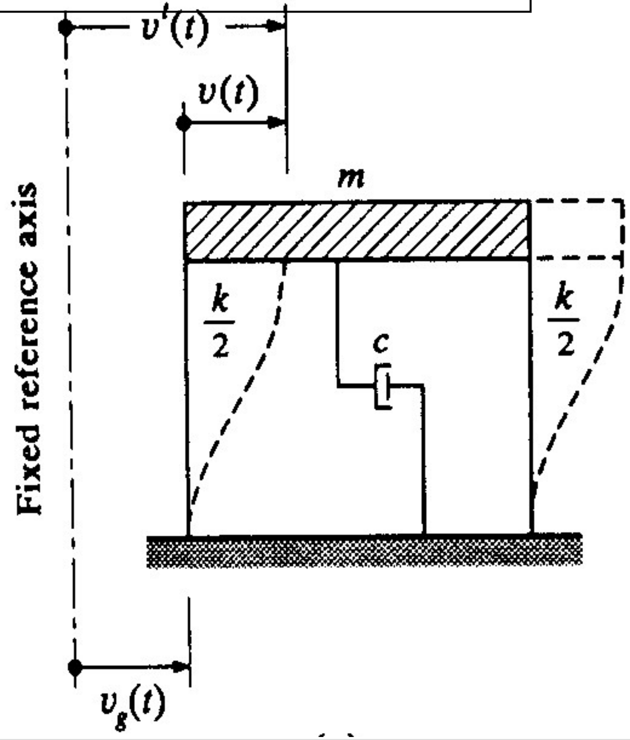


Fig. 2.3 Influence of support excitation on SDOF equilibrium

Relative displacement, Absolute displacement, and Ground displacement



$v_g(t)$: ground displacement (地震動変位)

$\dot{v}_g(t)$: ground velocity (地震動速度)

$\ddot{v}_g(t)$: ground acceleration (地震動加速度)

$v(t)$: structural displacement relative to the support (relative displacement of the structure) (構造物の相対変位)

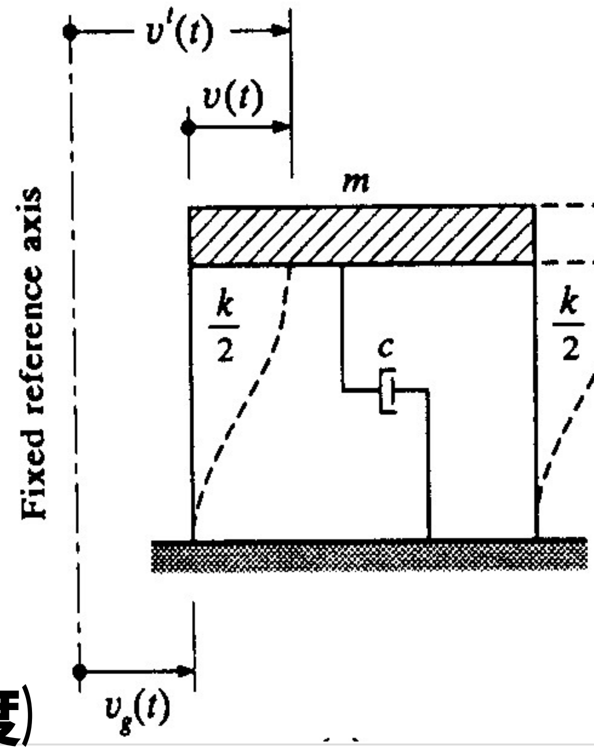
$\dot{v}(t)$: relative velocity of the structure (構造物の相対速度)

$\ddot{v}(t)$: relative acceleration of the structure (構造物の相対加速度)

$v^t(t)$: total (absolute) displacement of the structure (**構造物の絶対変位**)

$\dot{v}^t(t)$: total (absolute) velocity of the structure （**構造物の絶対速度**）

$\ddot{v}^t(t)$: total (absolute) acceleration of the structure （**構造物の絶対加速度**）



●Note that $f_I(t)$ and $f_S(t)$ are expressed as in Eq. (2.2), however $f_I(t)$ is not

$$f_I(t) = m\ddot{v}(t)$$

●Substituting for the inertial, damping and elastic forces in Eq. (2.12) yields

$$m\ddot{v}^t(t) + c\dot{v}(t) + kv(t) = 0 \quad (2.14)$$

●Substituting Eq. (2.15) into Eq. (2.14) yields

$$m\ddot{v}(t) + m\ddot{v}_g(t) + c\dot{v}(t) + kv(t) = 0 \quad (2.16)$$

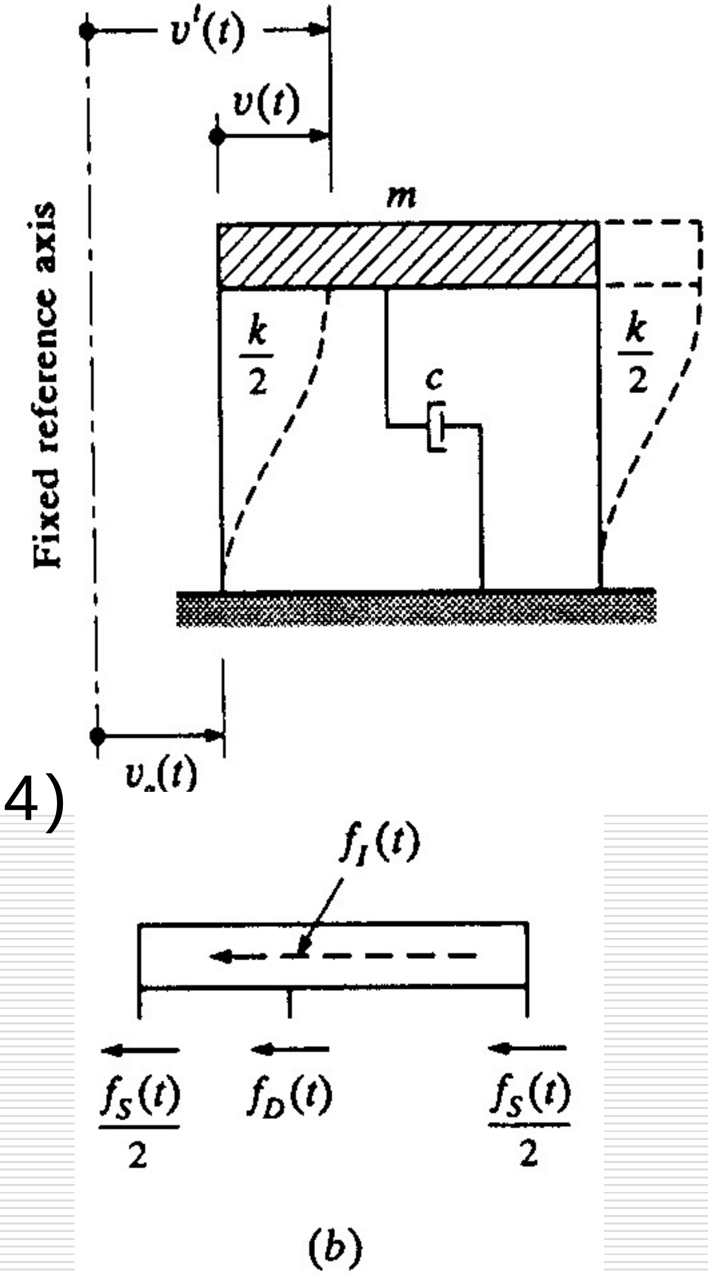


Fig. 2.3 Influence of support excitation on SDOF equilibrium

●Since the ground acceleration represents the specified dynamic input (**構造物に対する動的入力**) to the structure, the equation of motion can more conveniently be written

$$m\ddot{v}(t)+c\dot{v}(t)+kv(t)=-m\ddot{v}_g(t)\equiv p_{eff}(t) \qquad (2.17)$$

where, $p_{eff}(t)$ denotes the effective support excitation loading (**有効支点載荷荷重**).

●The structural deformation caused by ground acceleration $\ddot{v}_g(t)$ are exactly the same as those which would be produced by an external load p(t) equal to $-m\ddot{v}_g(t)$.

● An alternative form of the equation of motion can be obtained by using Eq. (2.15) and expressing Eq. (2.14) in terms of $v^t(t)$ and its derivatives, rather than in terms of v(t) and its derivatives, giving

$$m\ddot{v}^t(t) + c\dot{v}^t(t) + kv^t(t) = c\dot{v}_g(t) + kv_g(t) \qquad (2.18)$$

$$v^t(t) = v_g(t) + v(t) \qquad (2.15)$$

$$m\ddot{v}^t(t) + c\dot{v}(t) + kv(t) = 0 \qquad (2.14)$$

● In Eq. (2.18), the effective loading shown on the right hand side depends on the ground velocity and ground displacement, and the response obtained by solving Eq. (2.18) is the total (absolute) displacement of the structure from a fixed reference rather than the relative displacement

- Solutions are seldom obtained in this manner, however, because the ground motion generally is measured in terms of accelerations and the seismic record would have to be integrated once and twice to evaluate the effective loading contributions due to the ground velocity and ground displacement.

2.5 ANALYSIS OF UNDAMPED FREE VIBRATIONS

●It has been shown in the preceding sections that the equation of motion of a single spring-mass system with damping (**減衰のある1自由度マスーばね系システム**) can be expressed as

$$m\ddot{v}(t)+c\dot{v}(t)+kv(t)=p(t) \qquad (2.19)$$

●in which v(t) represents the dynamic response (**動的応答変位**) , that is, the displacement from the static-equilibrium position (**静的つり合い状態からの変位**), and p(t) represents the effective load (**有効荷重**) acting on the system, either applied directly or resulting from support motions (**構造物に直接作用するか、支点変位(地震動)の結果、作用するか**) .

- The solution of Eq. (2.19) will be obtained by considering first the homogeneous form with the right side set equal to zero, i.e.,

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = 0 \qquad (2.20)$$

- Motions taking place with no applied force are called free vibration (**自由振動**).

- The free-vibration response may be expressed in the following form:

$$v(t) = Ge^{st} \qquad (2.21)$$

- G is an arbitrary complex constant.

● Consider first the complex constant G, this may be represented as a vector plotted in the complex plane as shown in Fig. 2.4. This sketch demonstrates that the vector may be expressed in terms of its real and imaginary Cartesian components:

$$G = G_R + iG_I \qquad (2.22a)$$

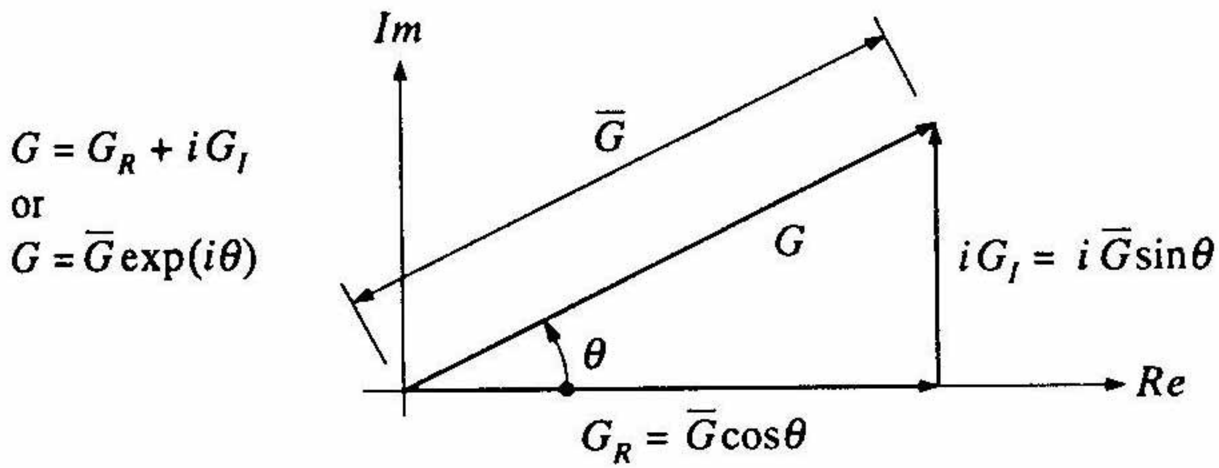


FIGURE 2-4
Complex constant representation in complex plane.

- or, alternatively, that it may be expressed in polar coordinates using its absolute value $\overline{G}$ (the length of the vector) and its angle θ, measured counterclockwise from the real axis:

$$G = \overline{G} e^{i\theta} \qquad (2.22b)$$

In addition, from the trigonometric relations shown in the sketch, it is clear that Eq. (2.22a) also may be written

$$G = \overline{G}\cos\theta + i\overline{G}\sin\theta \qquad (2.22c)$$

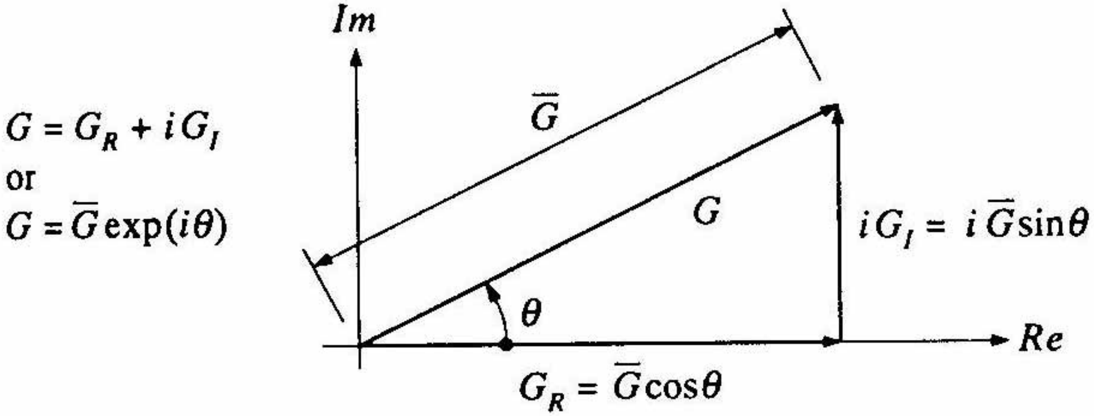


FIGURE 2-4
Complex constant representation in complex plane.

$$G = \overline{G}\cos\theta + i\overline{G}\sin\theta$$

$$\cos\theta = \sin(\theta + \frac{\pi}{2}) \qquad \sin\theta = -\cos(\theta + \frac{\pi}{2})$$

- It is easy to show that multiplying a vector by *i* has the effect of rotating it counterclockwise in the complex plane through an angle of $\pi/2$
- Similarly it may be seen that multiplying by –i rotates 90 degree clockwise.

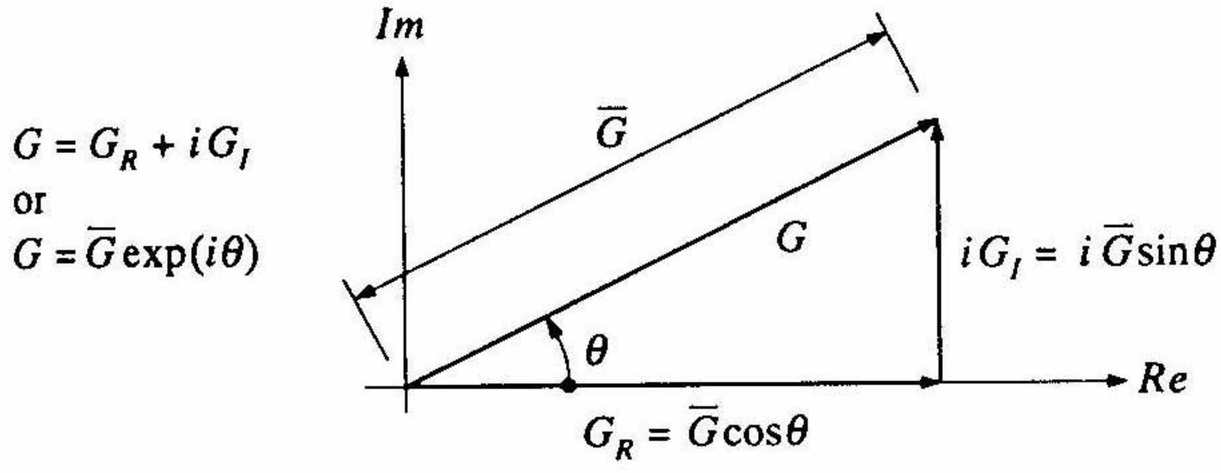


FIGURE 2-4
Complex constant representation in complex plane.

●Eqs. (2.22b) and (2.22c) lead to Euler's pair of equations that serve to transform from trigonometric to exponential function.

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2.23a)$$

$$G = \overline{G}e^{i\theta} \qquad (2.22b)$$

$$G = \overline{G}\cos\theta + i\overline{G}\sin\theta \qquad (2.22c)$$

●Furthermore, Eqs. (2.23a) may be solved simultaneously to obtain the inverse form of Euler's equations:

$$\cos\theta = \frac{1}{2}\left\{e^{i\theta} + e^{-i\theta}\right\}$$
$$\sin\theta = -\frac{i}{2}\left\{e^{i\theta} - e^{-i\theta}\right\} \qquad (2.23b)$$

●Eq. (2.21) is substituted into Eq. (2.20) to derive a free-vibration response, one obtains

$$(ms^2 + cs + k)Ge^{st} = 0$$

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = 0 \qquad (2.20)$$

$$v(t) = Ge^{st} \qquad (2.21)$$

●and after dividing by mGexp(st) and introducing the notation

$$\omega^2 \equiv \frac{k}{m} \qquad (2.24)$$

Eq. (2.20) becomes

$$s^2 + \frac{c}{m}s + \omega^2 = 0 \qquad (2.25)$$

1) Undamped System

- Consider first the undamped system for which c=0, it is evident that the two values of s in Eq. (2.25) are

$$s_{1,2} = \pm i\omega \qquad (2.26)$$

- Thus the total response includes two terms of the form of Eq. (2.21) as follows:

$$v(t) = G_1 e^{i\omega t} + G_2 e^{-i\omega t} \qquad (2.27)$$

- The complex constants G1 and G2 represent the arbitrary amplitudes of the corresponding vibration terms.

●By representing

$$G_1 = G_{1R} + iG_{1I} \qquad\qquad G_2 = G_{2R} + iG_{2I}$$

and substituting Eq. (2.23a) into Eq. (2.27), Eq. (2.27) becomes

$$v(t) = (G_{1R} + iG_{1I})(\cos\omega t + i\sin\omega t)$$
$$+ (G_{2R} + iG_{2I})(\cos\omega t - i\sin\omega t)$$

After simplifying

$$v(t) = (G_{1R} + G_{2R})\cos\omega t - (G_{1I} - G_{2I})\sin\omega t$$
$$+ i\left[(G_{1I} + G_{2I})\cos\omega t + (G_{1R} - G_{2R})\sin\omega t\right] \quad (2.28)$$

●However the free-vibration response must be real, so the imaginary term must be zero for all values of t, and this condition requires that

$$G_{1i} = -G_{2I} \equiv G_I \qquad\qquad G_{1R} = G_{2R} \equiv G_R$$

● From this it is seen that G_1 and G_2 are a complex conjugate pair(共役複素数):

$$G_1 = G_R + iG_I \qquad G_2 = G_R - iG_I$$

● With these, Eq. (2.27) becomes finally

$$v(t) = (G_R + iG_I)e^{i\omega t} + (G_R - iG_I)e^{-i\omega t} \qquad (2.29)$$

$$\boxed{v(t) = G_1 e^{i\omega t} + G_2 e^{-i\omega t} \qquad (2.27)}$$

● The first term of Eq. (2.29) is shown in Fig. 2.5

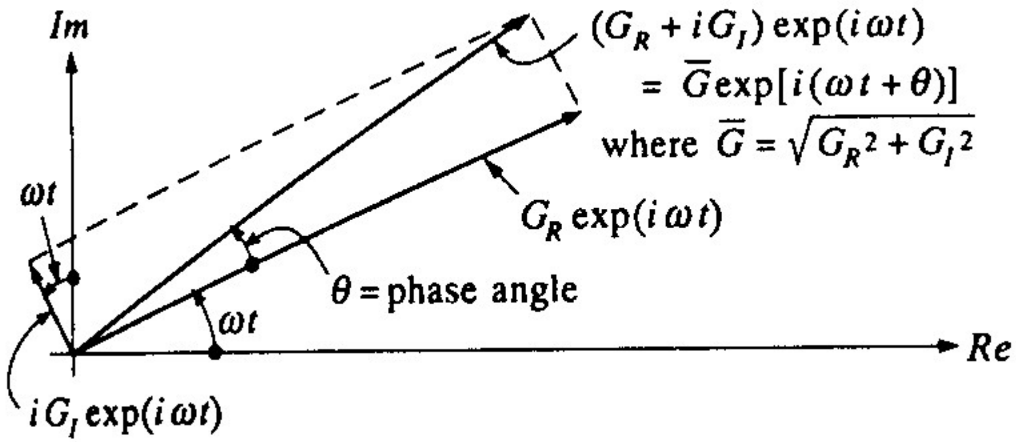
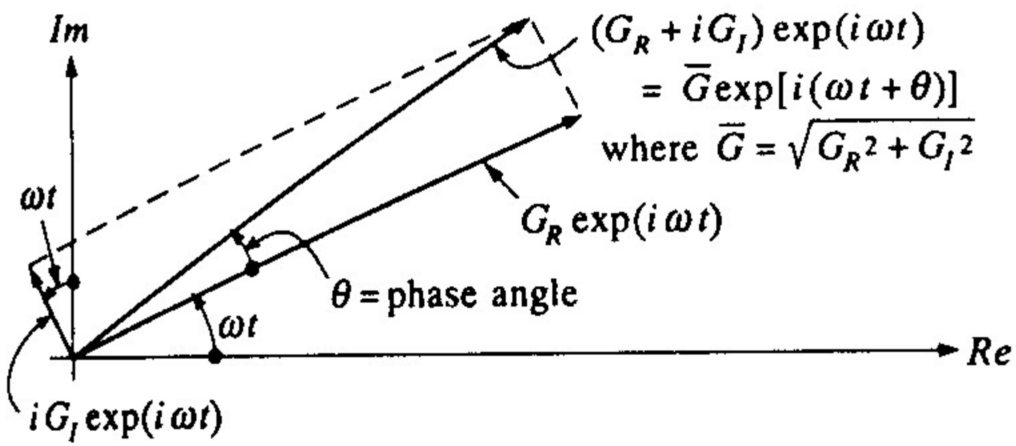


FIGURE 2-5
Portrayal of first term of Eq. (2-29).

- A vector representing the complex constant G_1 is rotating in the counterclockwise direction with the angular velocity ω .

- Note that the resultant response vector $(G_R + iG_I)\exp(i\omega t)$ leads vector $G_R \exp(i\omega t)$ by the phase angle θ .

- Note also that the response can be expressed in terms of the absolute value $\overline{G}$ and the combined angle $(\omega t + \theta)$



$$\overline{G} = \sqrt{G_R{}^2 + G_I{}^2}$$

FIGURE 2-5
Portrayal of first term of Eq. (2-29).

●The second term of Eq. (2.29) shows that the response of the second term is entirely equivalent to that shown in Fig. 2.5 except that the resultant vector $\bar{G}\exp\{-i(\omega t+\theta)\}$ is rotating in the clockwise direction, and the phase angle by which it leads the components $G_R\exp(-i\omega t)$ also is in the clockwise direction.

$$v(t)=(G_R+iG_I)e^{i\omega t}+(G_R-iG_I)e^{-i\omega t} \qquad (2.29)$$

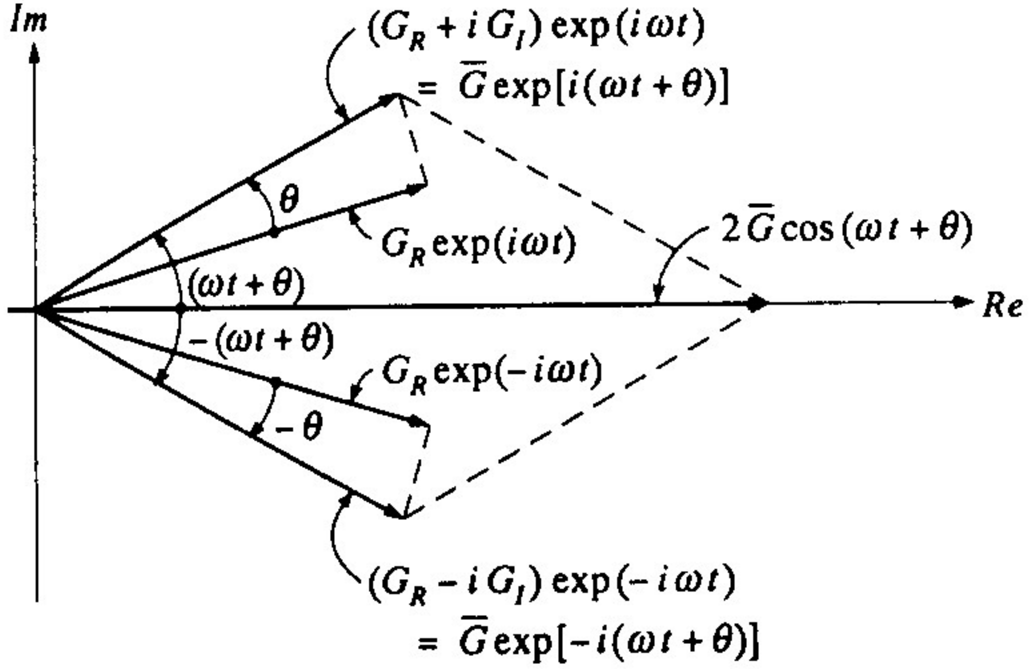


FIGURE 2-6
Total free-vibration response.

- The two counter-rotating vectors $\overline{G}\exp\{i(\omega t+\theta)\}$ and $\overline{G}\exp\{-i(\omega t+\theta)\}$ that represent the total free-vibration response by Eq. (2.29) are shown in Fig. 2.6.

$$v(t)=(G_R+iG_I)e^{i\omega t}+(G_R-iG_I)e^{-i\omega t} \qquad (2.29)$$

- It is evident here that the imaginary components of the two vectors cancel each other leaving only the vibratory motion

$$v(t)=2\overline{G}\cos(\omega t+\theta) \qquad (2.30)$$

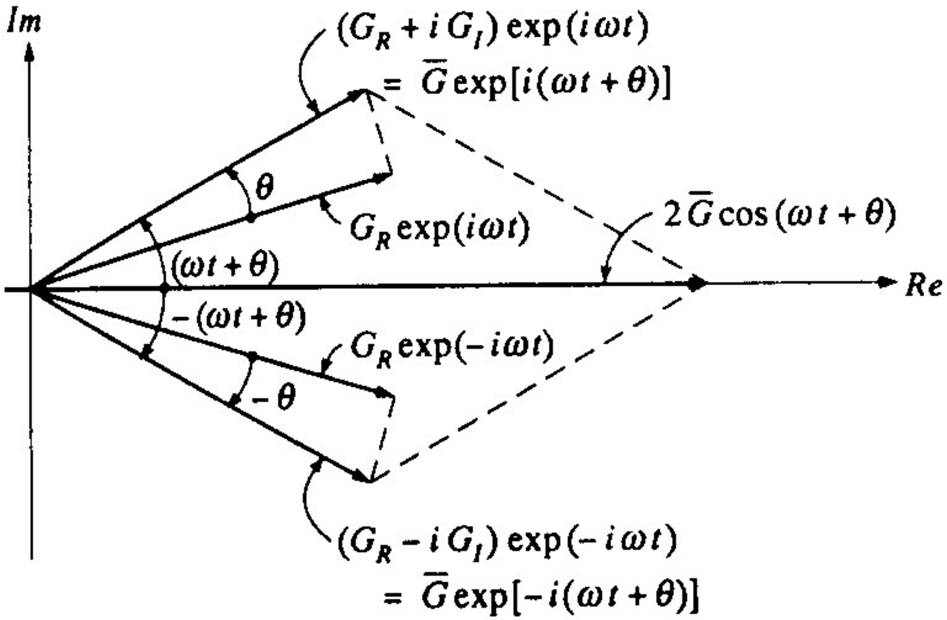


FIGURE 2-6
Total free-vibration response.

●An alternative for this real motion expression may be derived by applying the Euler transformation Eq. (2.23a) to Eq. (2.29) as

$$v(t) = A\cos\omega t + B\sin\omega t \qquad (2.31)$$

in which A=2G_R and B=-2G_I.

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2.23a)$$

$$v(t) = (G_R + iG_I)e^{i\omega t} + (G_R - iG_I)e^{-i\omega t} \qquad (2.29)$$

- The values of two constants A and B may be determined from the initial condition (**初期条件**), that is, the displacement v(0) and velocity $\dot{v}(0)$ at time 0 when the free vibration was set in motion.

$$\boxed{v(t) = A\cos\omega t + B\sin\omega t \qquad (2.31)}$$

- Substituting the initial conditions

$$v(0) = A = 2G_R \qquad \frac{\dot{v}(0)}{\omega} = B = -2G_I \qquad (2.32)$$

- Thus, Eq. (2.31) becomes

$$v(t) = v(0)\cos\omega t + \frac{\dot{v}(0)}{\omega}\sin\omega t \qquad (2.33)$$

●This solution represents a simple harmonic motion as shown in Fig. 2.7. The quantity ω, which we have identified previously as the angular velocity (**角速度** measured in radian per unit of time) of the vectors rotating in the complex plane, is known as the circular frequency (**角振動数**).

$$v(t) = v(0)\cos\omega t + \frac{\dot{v}(0)}{\omega}\sin\omega t \qquad (2.33)$$

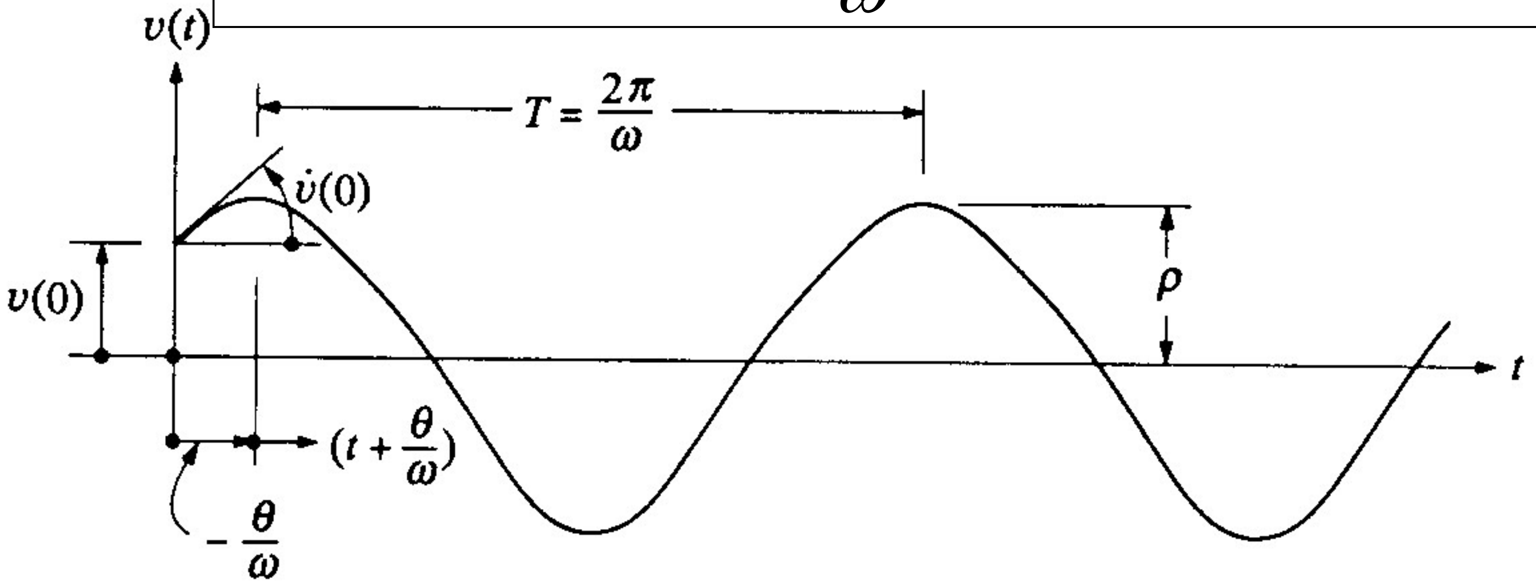


FIGURE 2-7
Undamped free-vibration response

● The cyclic frequency, usually referred to as natural frequency (**固有振動数**) is given as.

$$f = \frac{\omega}{2\pi} \qquad (2.34a)$$

Frequency *f* is measured in cycles per second, commonly referred to as Hertz (Hz).

● Its reciprocal is the time required to complete one cycle and is called the natural period (**固有周期**)

$$\frac{1}{f} = \frac{2\pi}{\omega} = T \qquad (2.34b)$$

The period *T* is measured in seconds.