Structural Analysis II
構造力学第二
(5)

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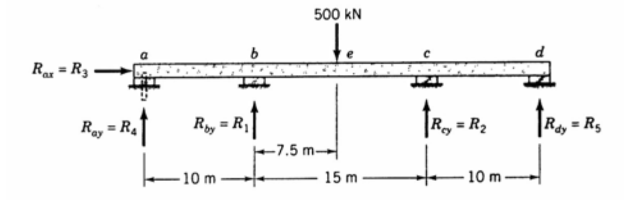
EXAMPLE 10.5

Consider again the beam and loading of example 10.2, but, in this case, assume that points b and c are supported by elastic supports that have stiffnesses of $k_{s1}=k_{s2}=0.006EI$. Determine the reaction carried by the elastic supports.

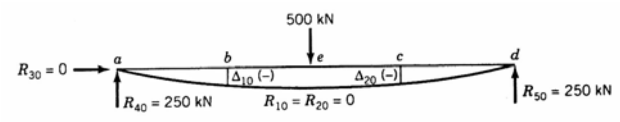


Structural clarification

Same as Example 10.2, i.e., the structure is statically indeterminate to the second order.

Primary structure

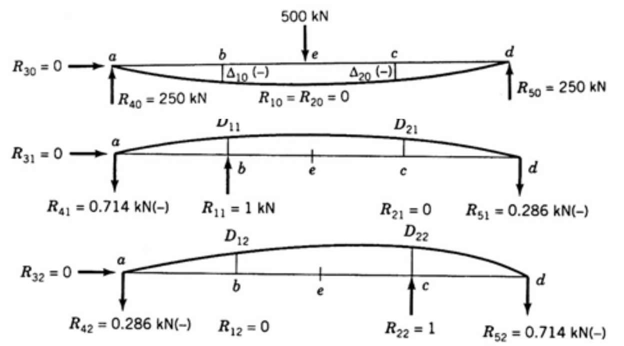
Same as Example 10.2.



Determination of displacements quantities and flexible coefficients

Same as Example 10.2.

Compatibility Equations



$$\boxed{\begin{aligned}\Delta_{10} + D_{11}R_1 + D_{12}R_2 &= \Delta_1\\ \Delta_{20} + D_{21}R_1 + D_{22}R_2 &= \Delta_2\end{aligned}}\begin{aligned}&= -\frac{R_1}{k_{s1}}\\ &= -\frac{R_2}{k_{s2}}\end{aligned} \qquad (10.7)$$

Redundant reactions

$$\begin{bmatrix} D_{11}+1/k_{s1} & D_{12} \\ D_{21} & D_{22}+1/k_{s2} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -\Delta_{10} \\ -\Delta_{20} \end{Bmatrix}$$

- Substitution leads to

$$\frac{1}{EI}\begin{bmatrix} 595.1+166.7 & 488.4 \\ 488.4 & 595.1+166.7 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} 341{,}185 \\ 341{,}185 \end{Bmatrix}$$

- From which

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 272.9 \\ 272.9 \end{Bmatrix}$$

$$\Delta_{10} + D_{11}R_1 + D_{12}R_2 = \Delta_1 = -\frac{R_1}{k_{s1}}$$

$$\Delta_{20} + D_{21}R_1 + D_{22}R_2 = \Delta_2 = -\frac{R_2}{k_{s2}}$$

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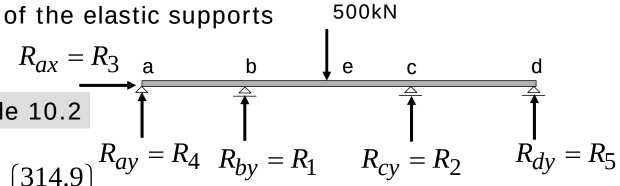
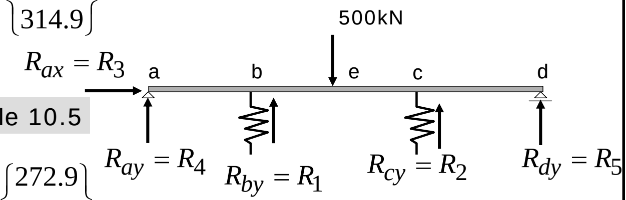
- Note that comparing the results with those of Example 10.2, the magnitudes of the interior redundant reactions are reduced by the softening effect of the elastic supports



Example 10.2

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 314.9 \\ 314.9 \end{Bmatrix}$$



Example 10.5

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 272.9 \\ 272.9 \end{Bmatrix}$$

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10.6 Selection of Internal Moments as Redundants

10.6.1 Single Redundant Moment

- Consider the propped cantilever shown in Fig. 10.6(a). This is statically indeterminate to the first degree.
- Instead of identifying an external reaction component as the redundant, we now select the internal moment at point a, M_1, as the redundant.
- The primary structure is then formed by removing the redundant moment (inserting a hinge as a moment release at point a), which essentially reduces member ab to a simply supported beam

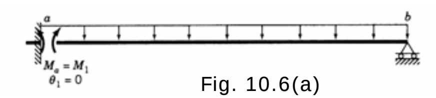

Fig. 10.6(a)

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- When the primary structure is subjected to the actual loading as shown in Fig. 10.6(b), a slope discontinuity (角度の非連続性) of θ_{10} occurs at point a that violates the internal compatibility condition (内部適合条件) for beam continuity.

- Here, the subscript 0 is again used to indicate that a response is associated with the actual loading on the primary structure.

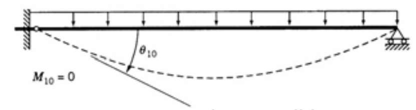

Fig. 10.6(b)

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● To restore slope compatibility at point a, a unit value of the redundant moment is introduced (M_{11}=1.0) on the primary structure as shown in Fig. 10.6(c).

● The second 1 subscript denotes that this is the value of M_1 associated with a unit value of the redundant moment.

●This loading produces a slope discontinuity at point a of D_{11}.

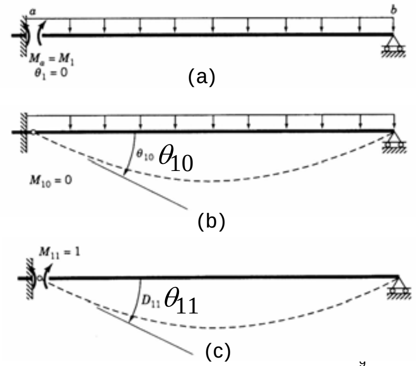


Fig. 10.6

● The final slope discontinuity θ_1 is determined by superimposing the effect of the actual loading and the effect of the unit moment amplified by the magnitude of redundant moment, i.e.,

$$\theta_{10} + D_{11}M_1 = \theta_1 \qquad (10.16)$$

● Thus, the redundan moment is

$$M_1 = \frac{\theta_1 - \theta_{10}}{D_{11}} \qquad (10.17)$$

● A clockwise rotation is taken as positive.

●Both θ_{10} and θ_{11} are positive.

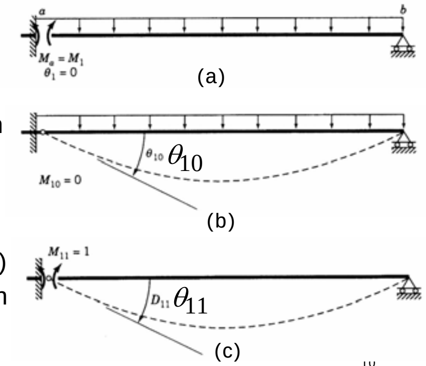


Fig. 10.6

● When there is no slope discontinuity at point a, such as the case shown in Fig. 10.6(a), θ_1 is zero.

● Having determined the magnitude of the redundant moment M_1, we can readily determine the final solution from applying statics to the primary structure or through superposition in the form

$$S = S_0 + S_1 M_1 \qquad (10.18)$$

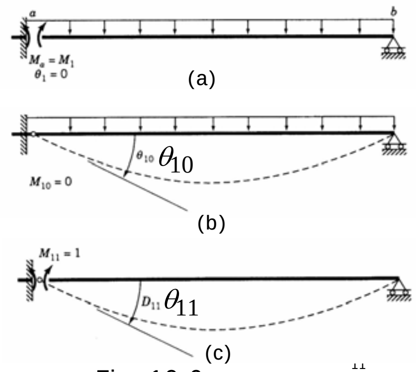


Fig. 10.6

$$\boxed{S = S_0 + S_1 M_1 \qquad (10.18)}$$

● where S is any response quantities of interest, S_0 is the value of S when the actual loading is applied to the primary structure, and, S_1 is the value of S when a unit value of M_1 is applied to the primary structure.

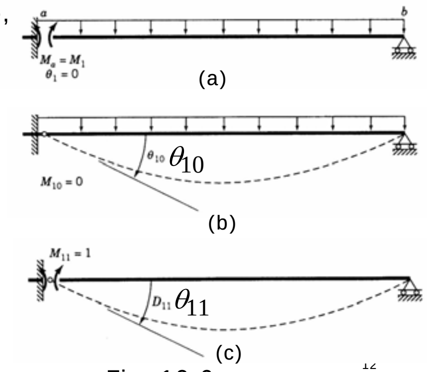
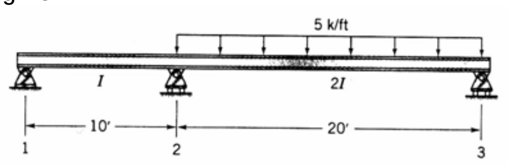


Fig. 10.6

Example 10.6

Determine the internal bending moments at the supports for the structure below. Relative I values are given.

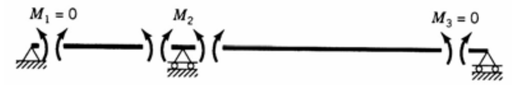


Structure classification

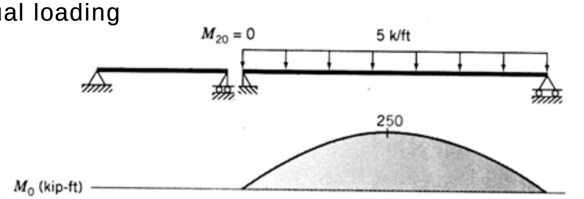
The structure is statically indeterminate to the first degree.

Primary structure and loading

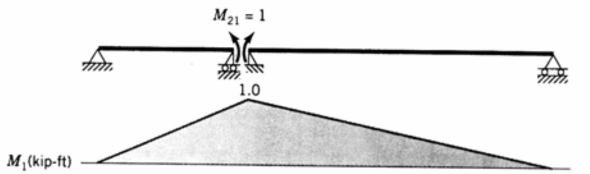
- ●The internal support moments are shown as follows.
- ●The force boundary condition at point 1 and 3 require that M_1=M_3=0, and M_2 is taken as the redundant moment.
- ● The primary structure is, therefore, two simply supported beams.



Actual loading

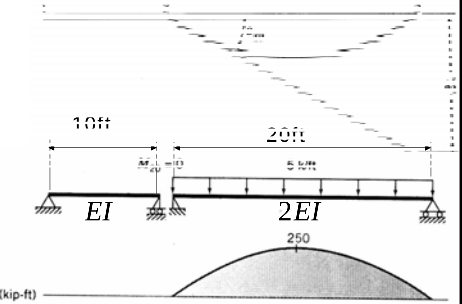


Unit value of M_2



Displacement calculation

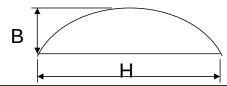
θ_{20}



$$\Delta_3^2 = \frac{2}{3} \times \frac{250}{2EI} \times 20 \times 10$$

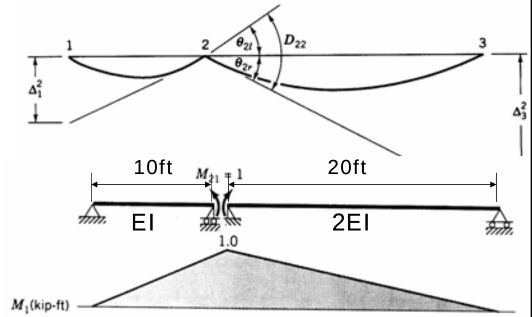
$$\theta_{20} = \frac{\Delta_3^2}{20} = 833.3(ft^2 k / EI)$$

Area of a parabola shape (放物線の面積)



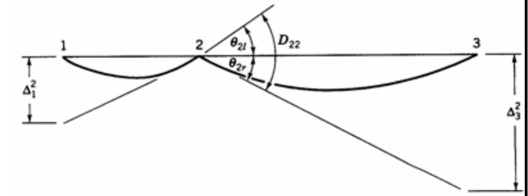
$$A = \frac{2}{3} BH$$

D$_{22}$:



$$\theta_{2l}=\frac{\Delta_1^2}{10}=\frac{\frac{1}{2}\times\frac{1}{EI}\times 10\times 6.77}{10}=3.33(ft^2k/EI)$$

$$\theta_{2r}=\frac{\Delta_3^2}{20}=\frac{\frac{1}{2}\times\frac{1}{2EI}\times 20\times 13.33}{20}=3.33(ft^2k/EI)$$

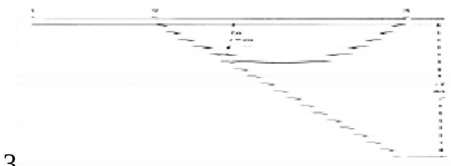


$$\theta_{2l}=\frac{\Delta_1^2}{10}=\frac{\frac{1}{2}\times\frac{1}{EI}\times 10\times 6.77}{10}=3.33(ft^2k/EI)$$

$$\theta_{2r}=\frac{\Delta_3^2}{20}=\frac{\frac{1}{2}\times\frac{1}{2EI}\times 20\times 13.33}{20}=3.33(ft^2k/EI)$$

$$D_{22}=\theta_{2l}+\theta_{2r}=6.67(ft^2k/EI)$$

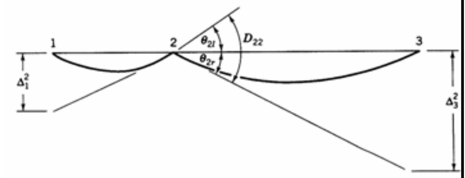
Compatibility equation and solution for redundant moment



$$\theta_{20}+D_{22}M_2=\theta_2$$

Therefore,

$$M_2=-\frac{\theta_{20}}{D_{22}}=-\frac{833.3}{6.67}=-125$$

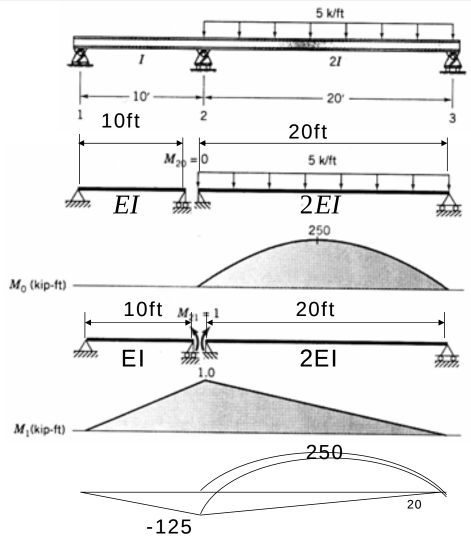


Final results

$M_1=0$

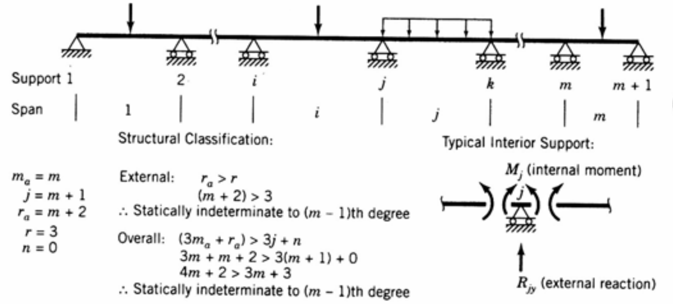
$M_2=-125$ kip ft

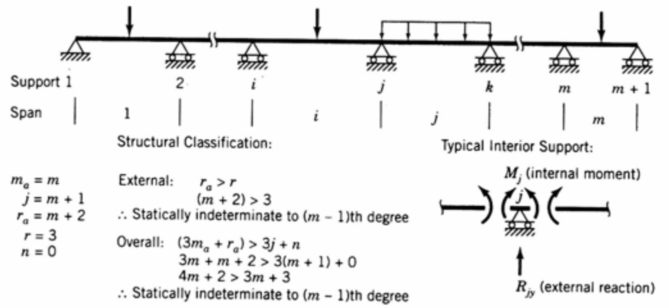
$M_3=0$



10.6.2 Multiple Redundant Moments for Continuous Beams: The three-moment equations (連続ばりに対して、余剰モーメントが複数存在する場合：3連モーメント方程式)

● Consider the m-span continuous beam (mスパン連続ばり) shown below





● The structure is externally indeterminate to the (m-1)th degree.
●The overall classification of the structure is indeterminate to the (m-1)th degree

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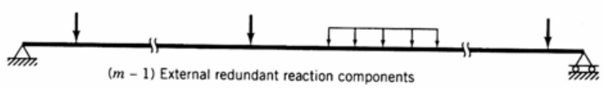
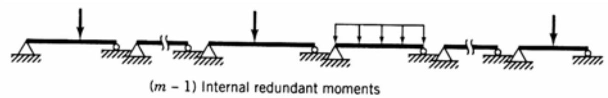
● The most obvious approach which we studied is to select the (m-1) interior reactions as the redundants as shown in Fig. 10.7(b).
●An alternative approach is to select the internal moment at the (m-1) interior support points as redundants as shown in Fig. 10.7(c).
●In the latter case, the primary structure is a series of m simply supported beams.



(b)



(c)

Fig. 10.7

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● In the approach shown in Fig. 10.7(c), the compatibility for the given structure is relaxed at each of the interior support points where continuity of slope is violated.

●The advantage of this selection of redundants is that the required displacement quantities are easily determined for the m simply supported beams.

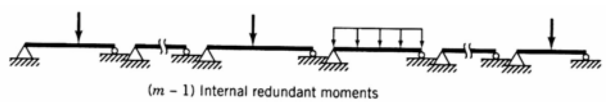

Fig. 10.7(c)

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● Attention is now focused on the two-span section that reaches over the supports i, j and k, as shown in Fig. 10.8(a).

●When the internal moments are removed and the continuous beam is transformed into a series of simply supported beams, there is a slope discontinuity θ_{j0} at support j under actual loading as shown in Fig. 10.8(b).

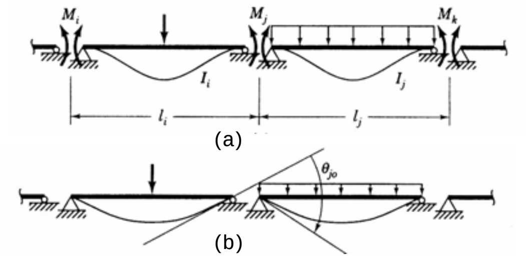

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Fig. 10.8(a)

● To correct the compatibility requirements for the original structure, introduce M_i, M_j and M_k through three separate loading conditions as shown in Fig. 10.8(c).

● The discontinuity in slope at point j corresponding to each of these unit moment cases are shown as D_{ji}, D_{jj} and D_{jk}, respectively, which are flexibility coefficients for the primary structure.

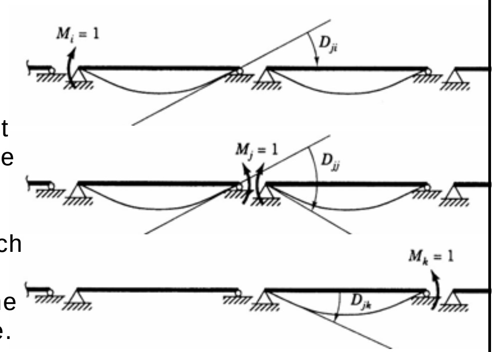


Fig. 10.8(c)

● Assuming superposition, we multiply each of these discontinuity by the actual value of the respective redundant moment and combine them with the discontinuity θ_{j0} to obtain the total discontinuity θ_j .

● Since θ_j is zero in the given structure, the final compatibility equation at point j becomes

$$D_{ji}M_i + D_{jj}M_j + D_{jk}M_k + \theta_{j0} = \theta_j = 0 \qquad (10.19)$$

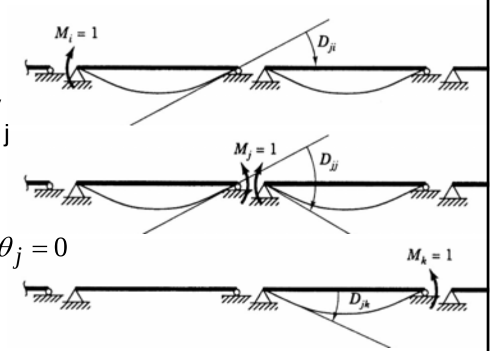


Fig. 10.8(c)

● The flexibility coefficients are

$$D_{ji} = \frac{l_i}{6EI_i} \quad D_{jj} = \frac{l_i}{3EI_i} + \frac{l_j}{3EI_j} \quad D_{jk} = \frac{l_j}{6EI_j} \qquad (10.20)$$

where, l_i and l_j are the lengths for spans i and j, and I_i and I_j are the corresponding moments of inertia.

● Substitution of Eq. (10.20) into Eq. (10.19) leads to

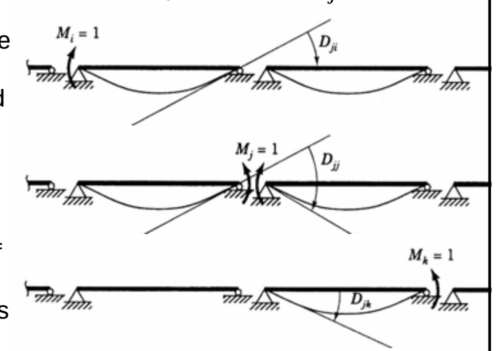


Fig. 10.8(c)

$$\frac{l_i}{I_i}M_i + \left(\frac{2l_i}{I_i}+\frac{2l_j}{I_j}\right)M_j + \frac{l_k}{I_k}M_k = -6E\theta_{j0} \qquad (10.21)$$

● For the complete structure, the slope discontinuities that occur at all of the interior supports must be corrected.
● Thus, for the structure of Fig. 10.7, Eq. (10.21) must be applied with j=2, 3, …., m.
●However, each of the (m-1) compatibility equations involves only three moments; the moment at the point where compatibility is being considered and the moments at the far ends of the spans to the left and the right.

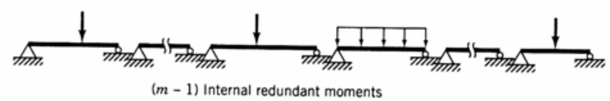
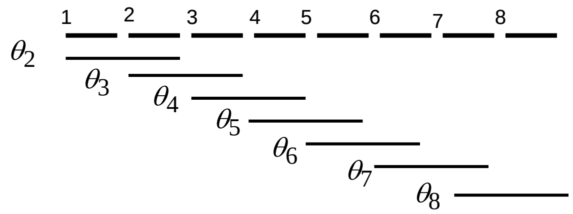


Fig. 10.7(c)

● From this reason, Eq. (10.21) is commonly referred to as the three-moment equations (3連モーメント方程式).
●The original presentation was given in 1857 by the French engineer Clapeyron (クラペイロン).



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10.6.3 Application of Three-Moment Equation

● Three moment equation is particularly useful in determining the internal support moments of a continuous beam (連続ばり).

● For the arrangement given in Fig. 10.7, the three-moment equation would be applied at each of the (m-1) interior support points. This would provide the (m-1) compatibility equations that are required for the determination of the (m-1) redundant moments.

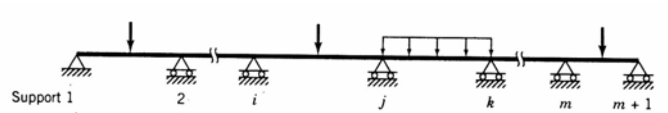

Fig. 10.7

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● For fixed-end beams, such as the one shown in Fig. 10.9, there is an additional redundant moment at each fixed end.
●Although a special form of the three-moment equation could be formulated for this case, a convenient artifice is to replace the fixed end by an imaginary end span of zero length.
●The three-moment equation is then applied at the end support points as well as the interior points.

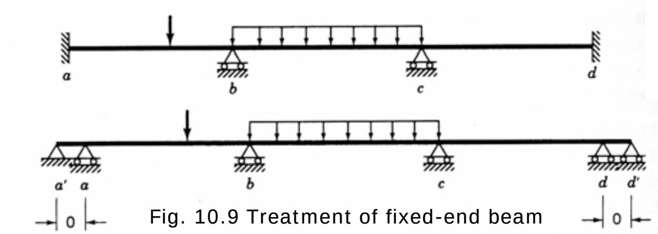
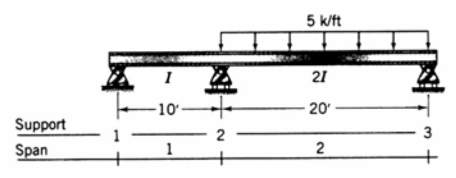


Fig. 10.9 Treatment of fixed-end beam

EXAMPLE 10.7

Determine the support moments for the structure given in Example 10.6 by applying the three-moment equation.



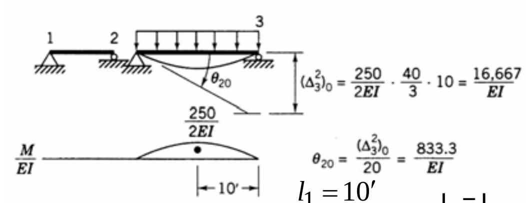
Structure classification

As already studied in Example 10.6, this structure is statically indeterminate to the first degree

Three-moment equation for support j

$$\boxed{\frac{l_i}{I_i}M_i + \left(\frac{2l_i}{I_i}+\frac{2l_j}{I_j}\right)M_j + \frac{l_k}{I_k}M_k = -6E\theta_{j0} \qquad (10.21)}$$

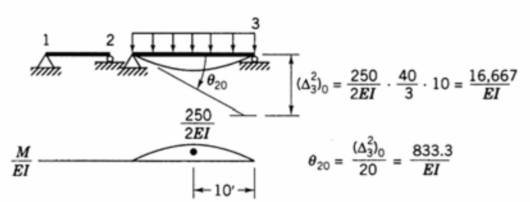
- This equation is applied at point 2 (j=2).



$l_1 = 10'$ $I_1 = I$

i=1, j=2, k=3 $l_2 = 20'$ $I_2 = 2I$

$$\frac{10}{I}M_1 + 2\left(\frac{10}{I}+\frac{20}{2I}\right)M_2 + \frac{20}{2I}M_3 = -6E\frac{833.3}{EI}$$



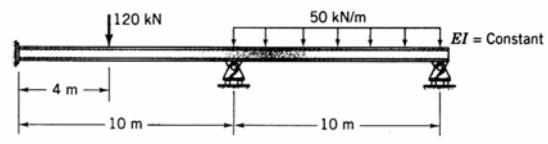
$$\boxed{\frac{10}{I}M_1 + 2\left(\frac{10}{I}+\frac{20}{2I}\right)M_2 + \frac{20}{2I}M_3 = -6E\frac{833.3}{EI}}$$

$$\frac{40}{I}M_2 = -\frac{5,000}{I}$$

$$M_2 = -125 kip\cdot ft$$

EXAMPLE 10.8

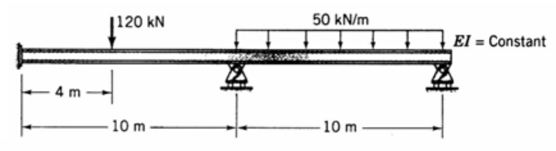
Determine the support moments for the structure shown by applying the three-moment equation.

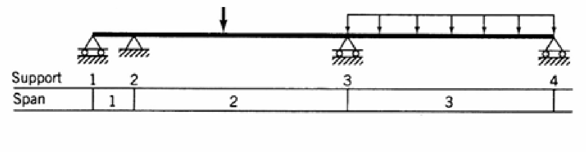


Structure classification

Structure is statically indeterminate to the second degree.

Modified structure





Three-moment equation for support j

$$l_i M_i + 2(l_i + l_j)M_j + l_k M_k = -6EI\theta_{j0} \quad (10.21)$$

- This equation must be applied at points 2 and 3.

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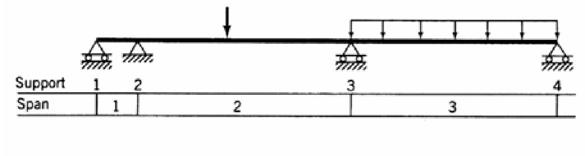
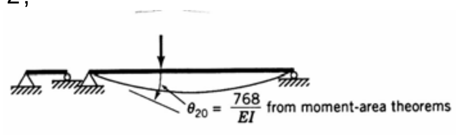


Point 2;



i=1, J=2, k=3 $\quad l_1 = 0 \quad l_2 = 10m$

$$0 \cdot M_1 + 2(0+10)M_2 + 10 \cdot M_3 = -6EI\frac{768}{EI}$$

$$20M_2 + 10M_3 = -4{,}608$$

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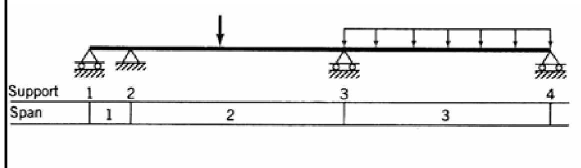
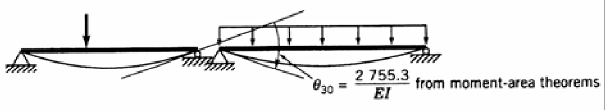


Point 3;



i=2, j=3, k=4 $\quad l_2 = 10m \quad l_3 = 10m$

$$10M_2 + 2(10+10)M_3 + 10M_4 = -6EI\frac{2{,}755.3}{EI}$$

$$10M_2 + 40M_3 = 16{,}532$$

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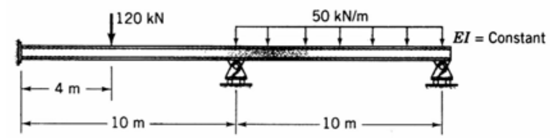
Solution for moments

$$20M_2 + 10M_3 = -4{,}608$$
$$10M_2 + 40M_3 = 16{,}532$$

Solving simultaneously,

$$M_2 = -27.2kNm$$
$$M_3 = -406.5kNm$$



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