Structural Analysis II
構造力学第二
(4)

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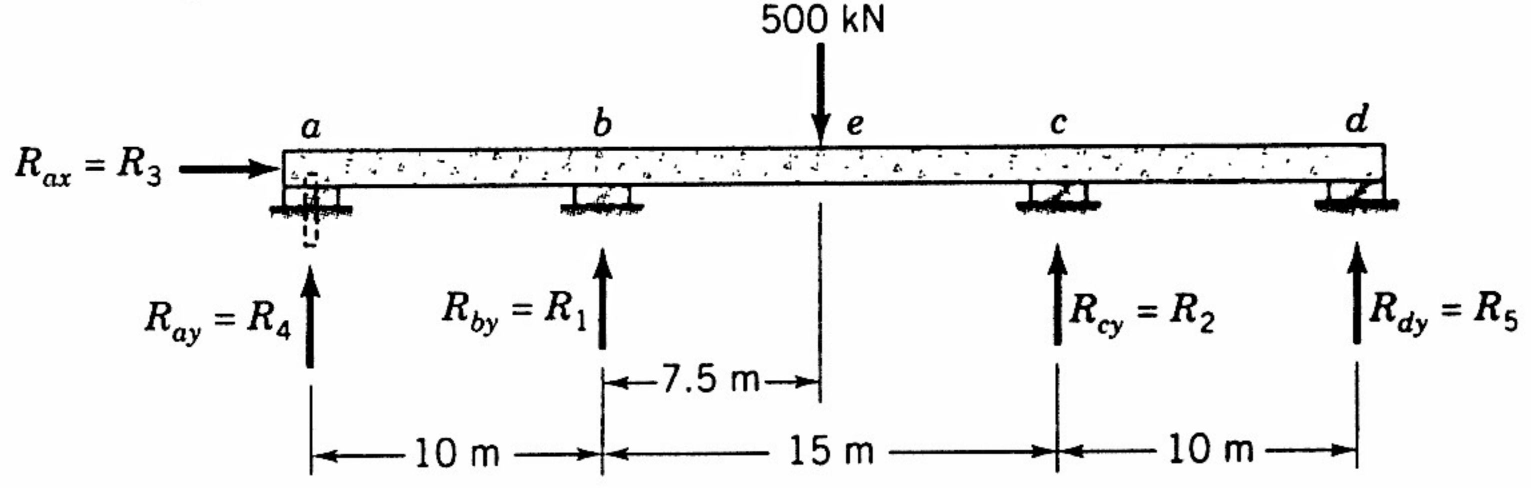
Example 10.2

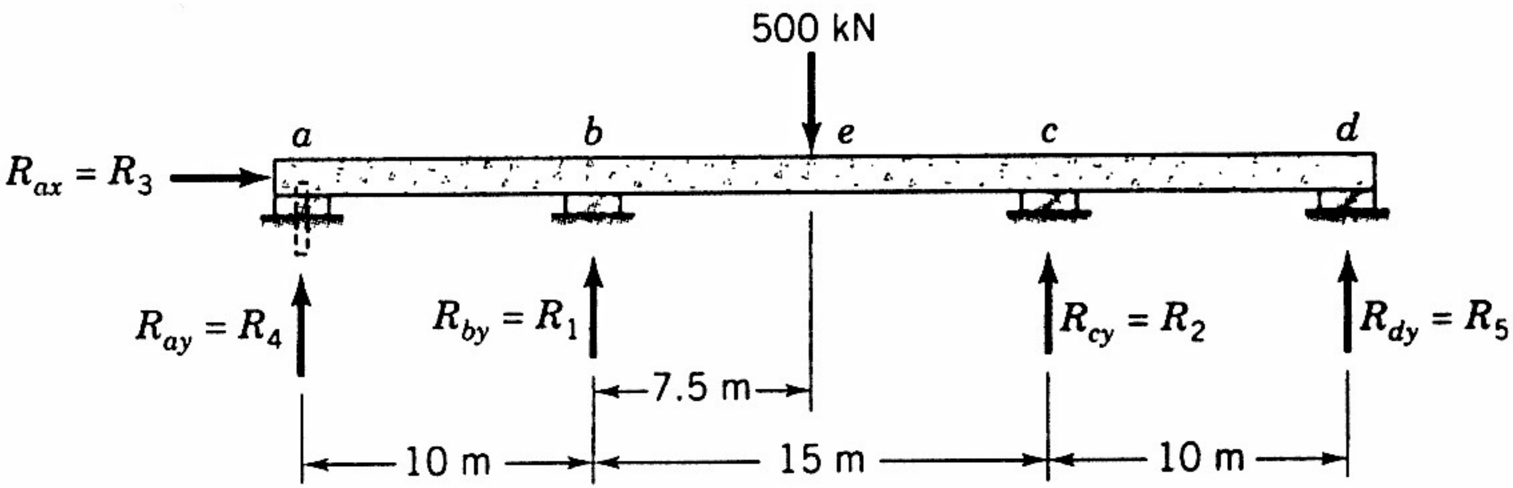
- Determine the reaction, and construct the final shear and moment diagrams for the structure and loading given. The quantity EI is the same for each span.



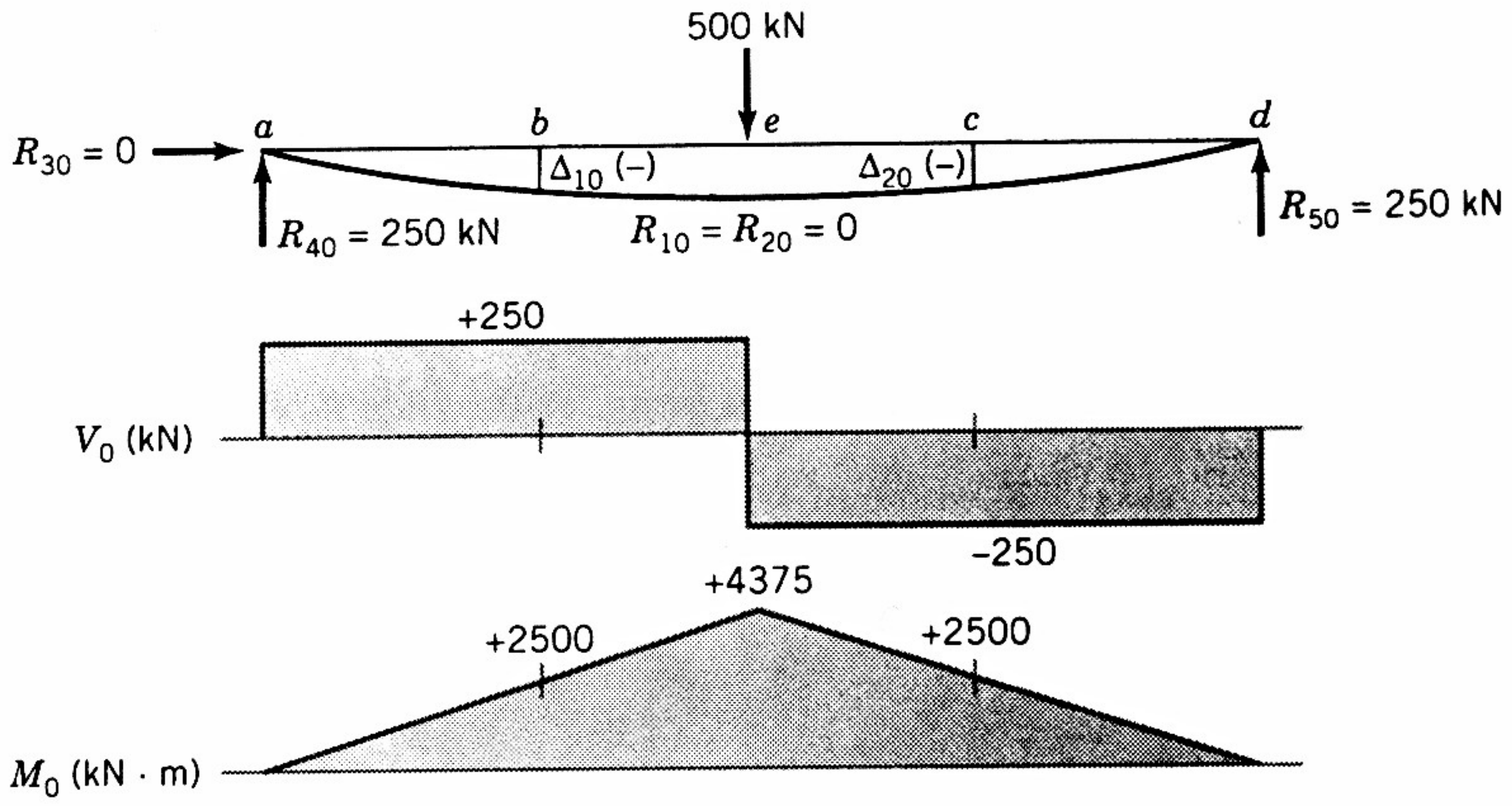
- This structure is statically indeterminate to the second order.

Primary Structure and Loading

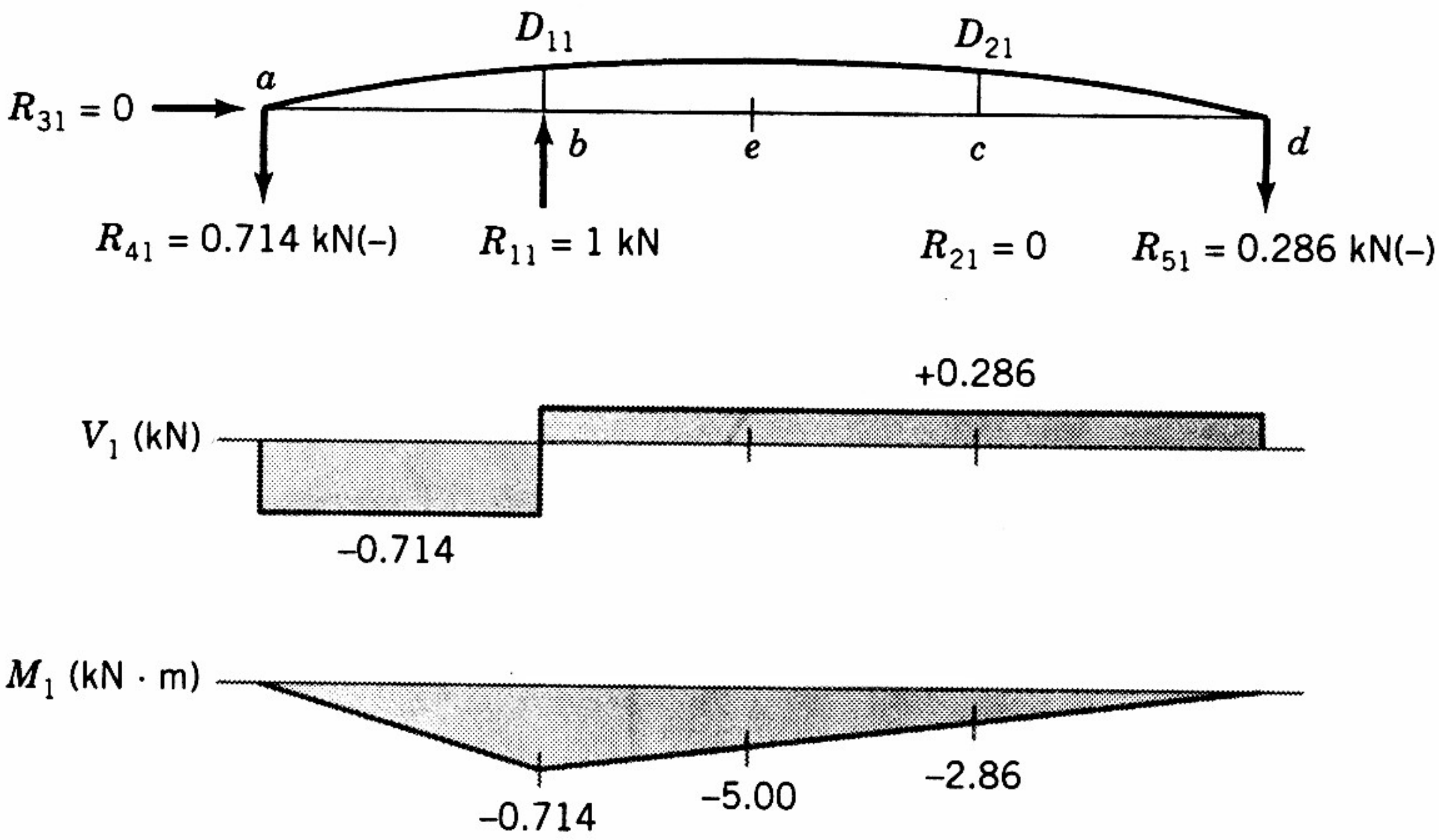
- Select R_{by} and R_{cy} as redundant reactions, which produces a simply supported beam as the primary structure.



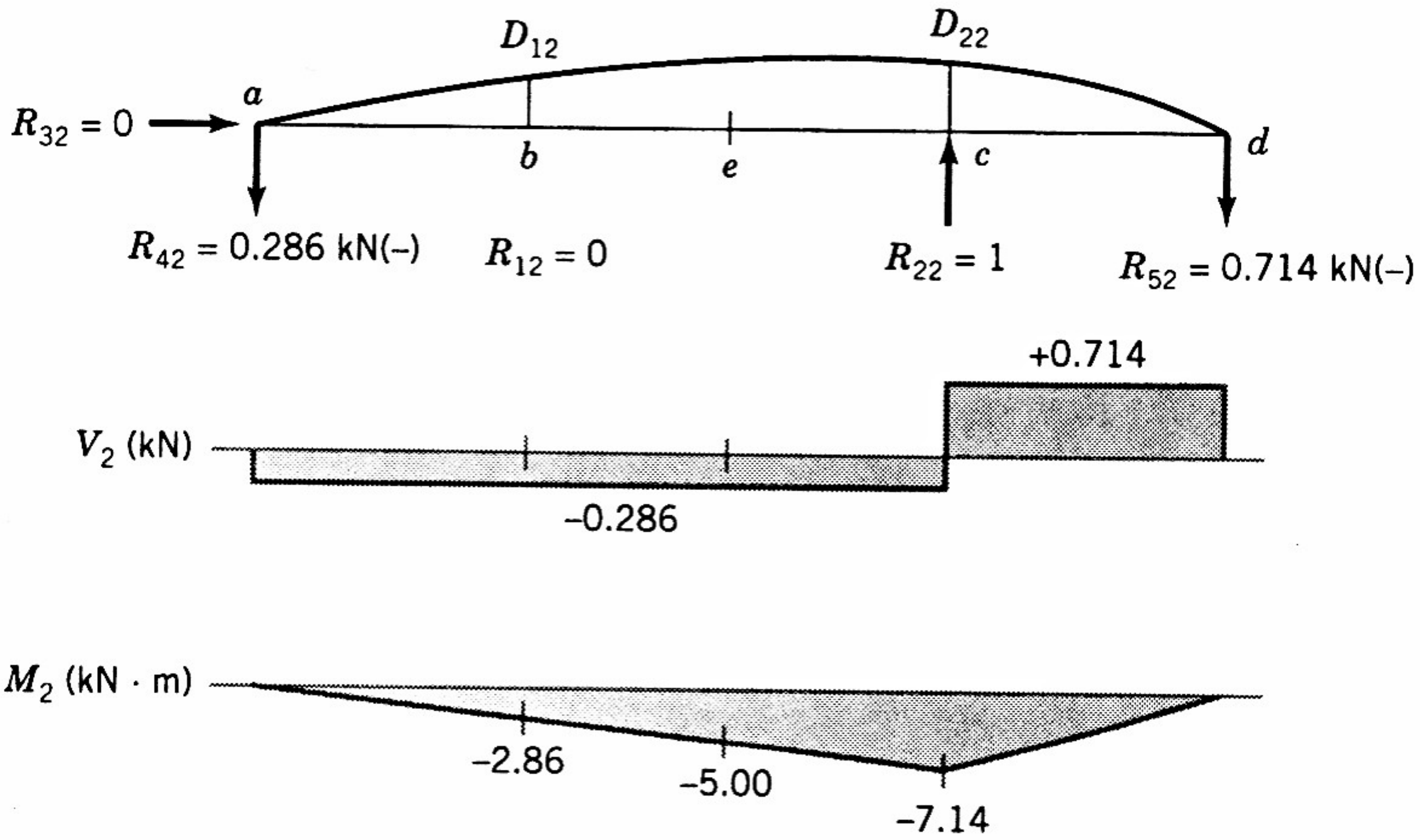
● Shear and moment of primary structure subjected to given load are obtained as



Unit value of $R_1 = R_{11}$

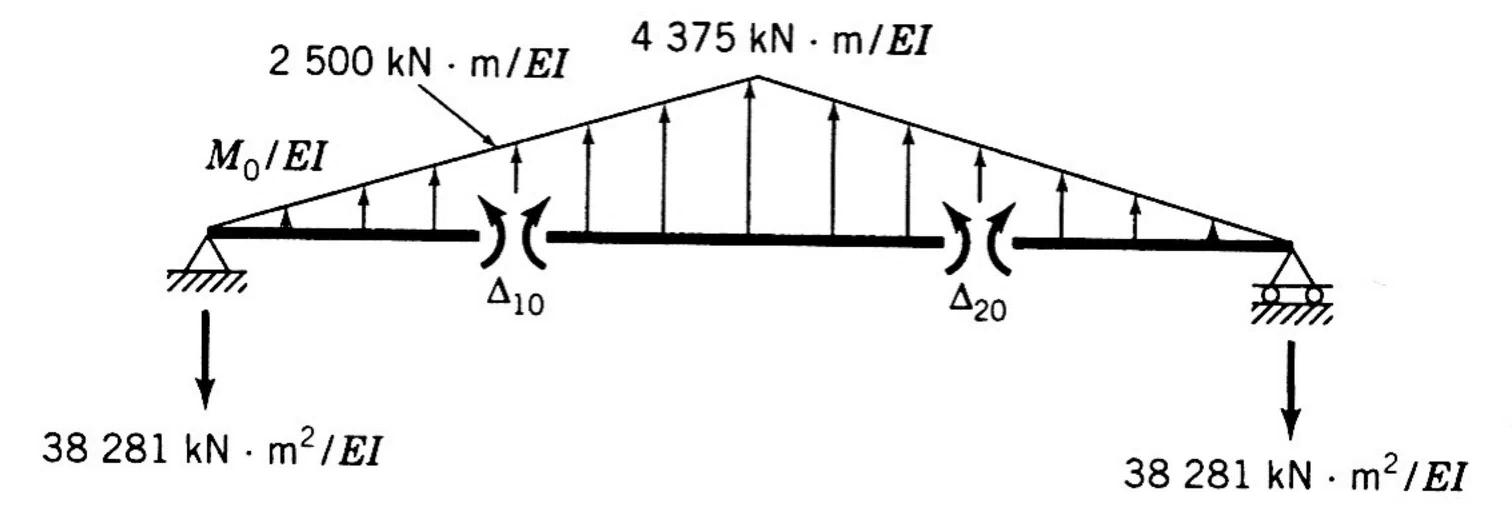


Unit value of $R_2 = R_{22}$



Displacement calculations

Conjugate beam method (共役ばり法、モールの弾性荷重法) is used because of its efficiency in determining two displacements along the length of the structure

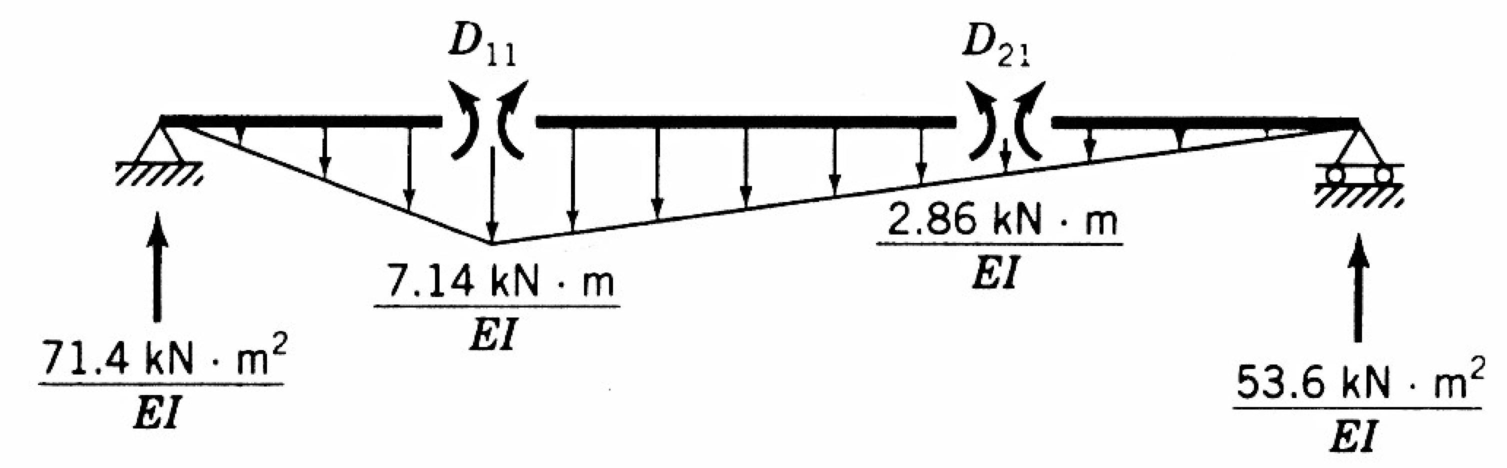


$$\Delta_{10} = -\frac{38{,}281}{EI}\times 10 + \frac{2{,}500}{2EI}\times 10\times 3.33 = -\frac{341{,}185\,kN\cdot m^3}{EI}$$

$\Delta_{20} = \Delta_{10}$ by symmetry

Both are negative, since upward displacements are positive.

The flexibility coefficients D_{11} and D_{12}



$$D_{11} = \frac{71.4}{EI}\times 10 - \frac{71.4}{2EI}\times 10 \times 3.33 = \frac{595.1 kN\cdot m^3}{EI}$$

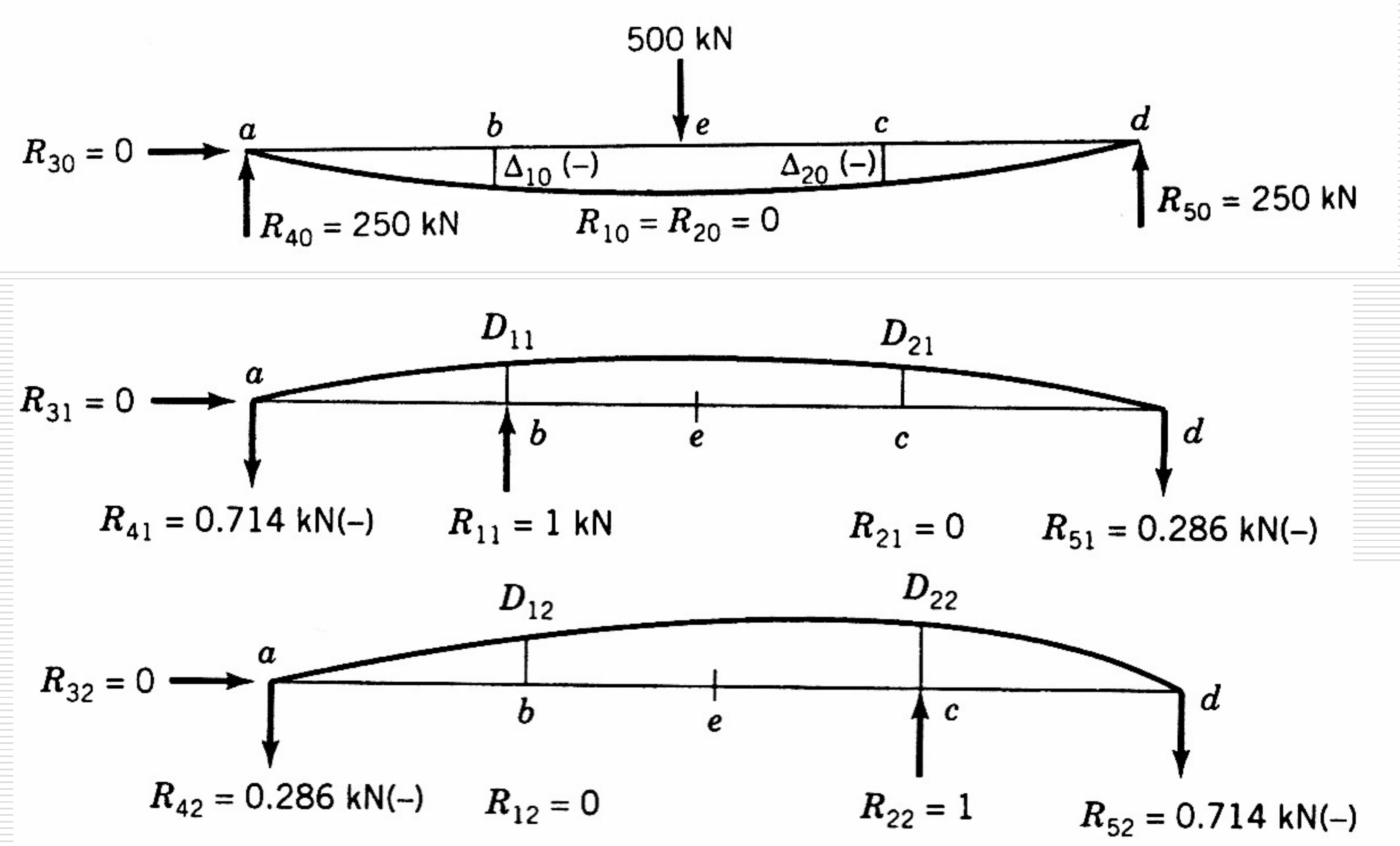
$$D_{21} = \frac{53.6}{EI}\times 10 - \frac{2.86}{2EI}\times 10 \times 3.33 = \frac{488.4 kN\cdot m^3}{EI}$$

The flexibility coefficients D_{21} and D_{22}

$D_{12} = D_{21}$ By Maxwell's law (Eq. (2.72))

$D_{22} = D_{11}$ By symmetry

Redundant reactions



$$
\begin{aligned}
\Delta_{10} + D_{11}R_1 + D_{12}R_2 &= \Delta_1 \\
\Delta_{20} + D_{21}R_1 + D_{22}R_2 &= \Delta_2
\end{aligned}
\qquad (10.7)
$$

- Since $\Delta_1 = \Delta_2 = 0$ in Eq. (10.7), we have

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -\Delta_{10} \\ -\Delta_{20} \end{Bmatrix}$$

$$\boxed{\begin{aligned} \Delta_{10} + D_{11}R_1 + D_{12}R_2 &= \Delta_1 \\ \Delta_{20} + D_{21}R_1 + D_{22}R_2 &= \Delta_2 \end{aligned} \qquad (10.7)}$$

- Substituting for displacement quantities and, noting that EI cancels, leads to

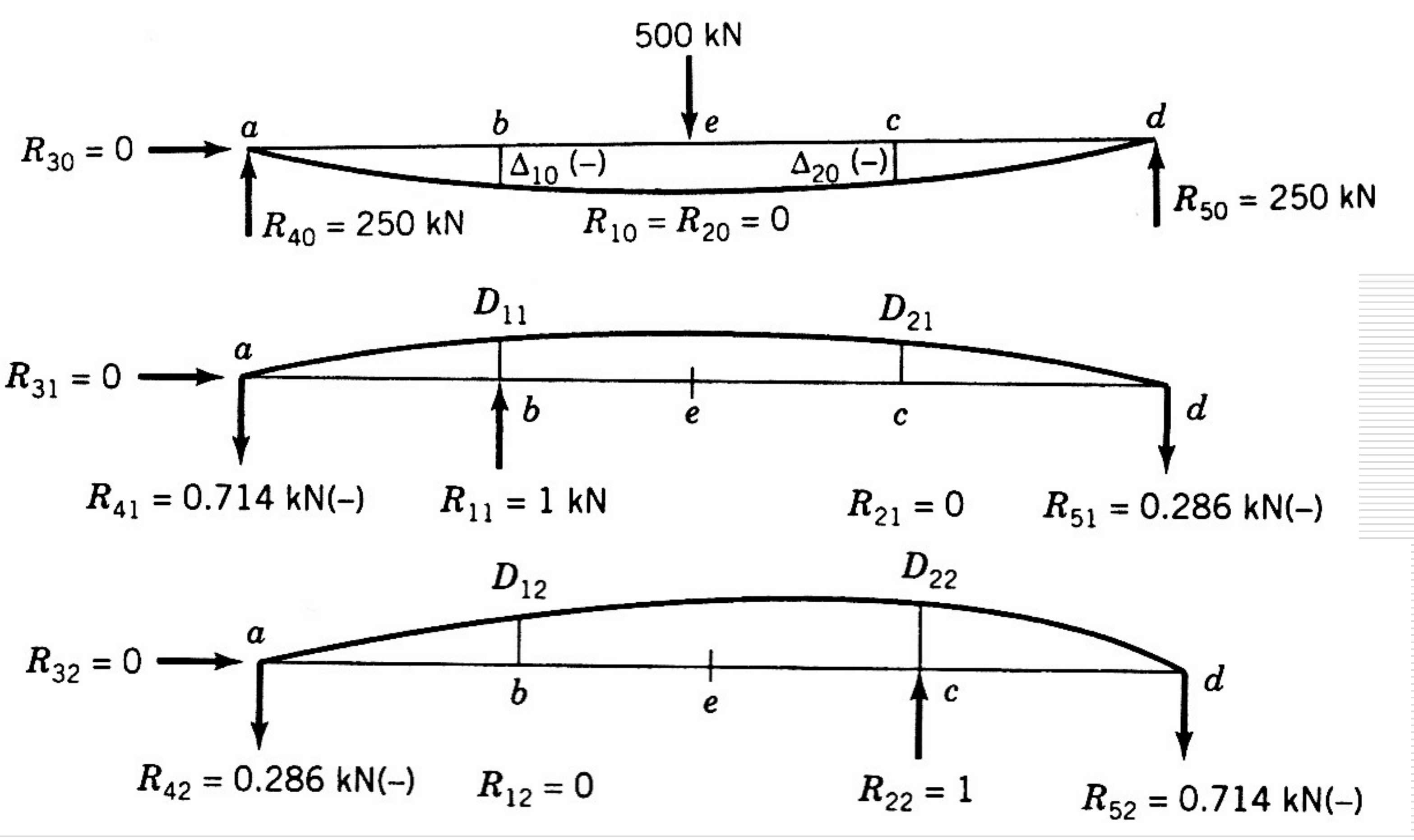
$$\begin{bmatrix} 595.1 & 488.4 \\ 488.1 & 595.1 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 341{,}185 \\ 341{,}185 \end{Bmatrix}$$

Solution for the redundant reactions gives

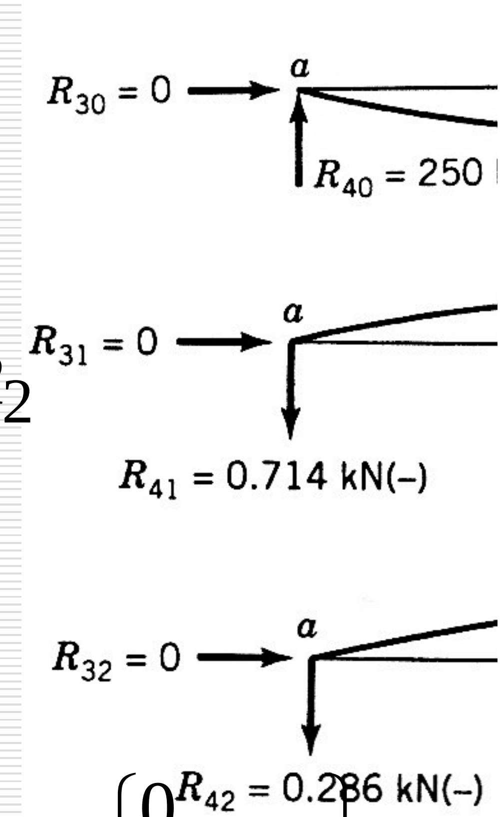
$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 314.9 \\ 314.9 \end{Bmatrix}$$

●The remaining reactions are readily determined from superposition

$$S = S_0 + S_1R_1 + S_2R_2$$



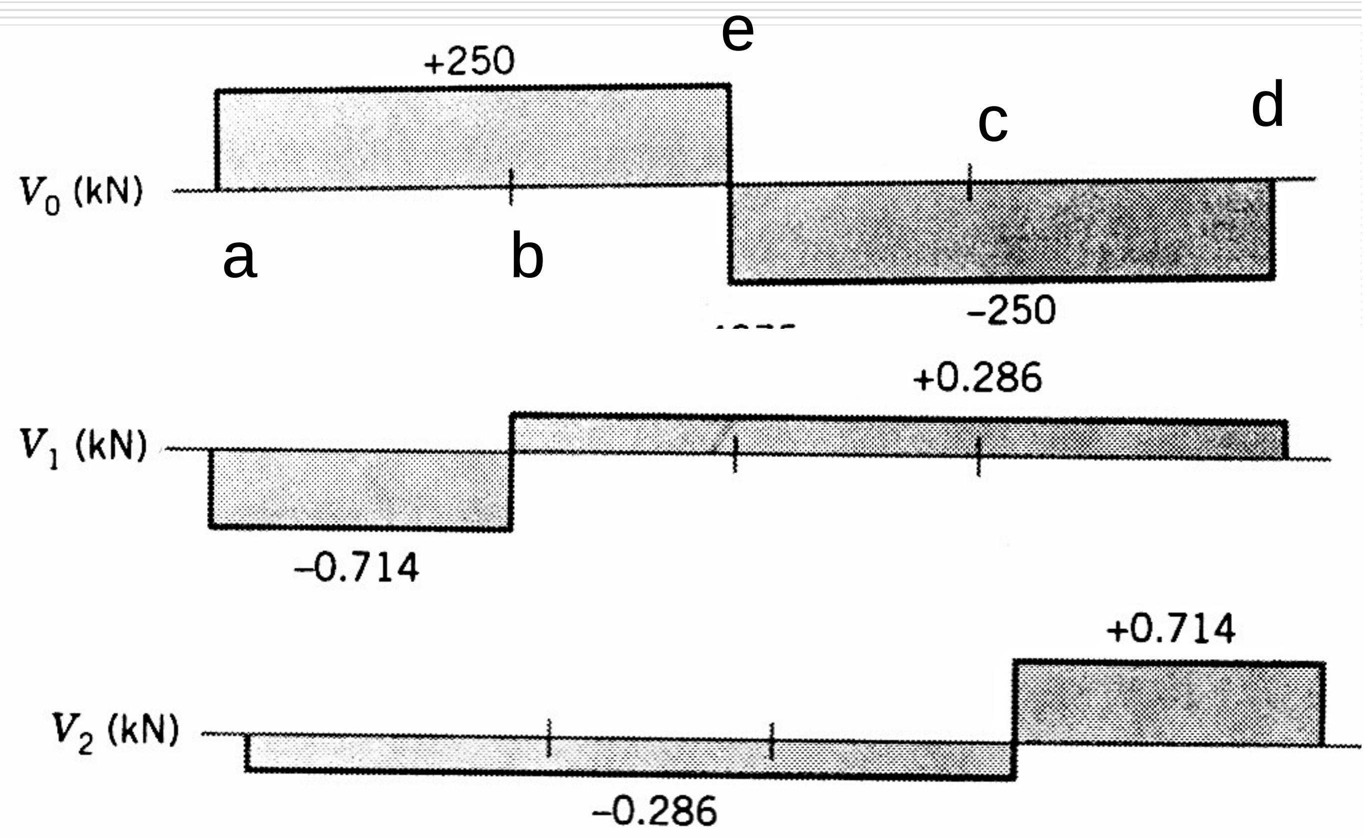
$$
\begin{Bmatrix} R_{ax} \\ R_{ay} \\ R_{by} \\ R_{cy} \\ R_{dy} \end{Bmatrix} = \begin{Bmatrix} R_{30} \\ R_{40} \\ R_{10} \\ R_{20} \\ R_{50} \end{Bmatrix} + \begin{Bmatrix} R_{31} \\ R_{41} \\ R_{11} \\ R_{21} \\ R_{51} \end{Bmatrix} \times R_1 + \begin{Bmatrix} R_{32} \\ R_{42} \\ R_{12} \\ R_{22} \\ R_{52} \end{Bmatrix} \times R_2
$$



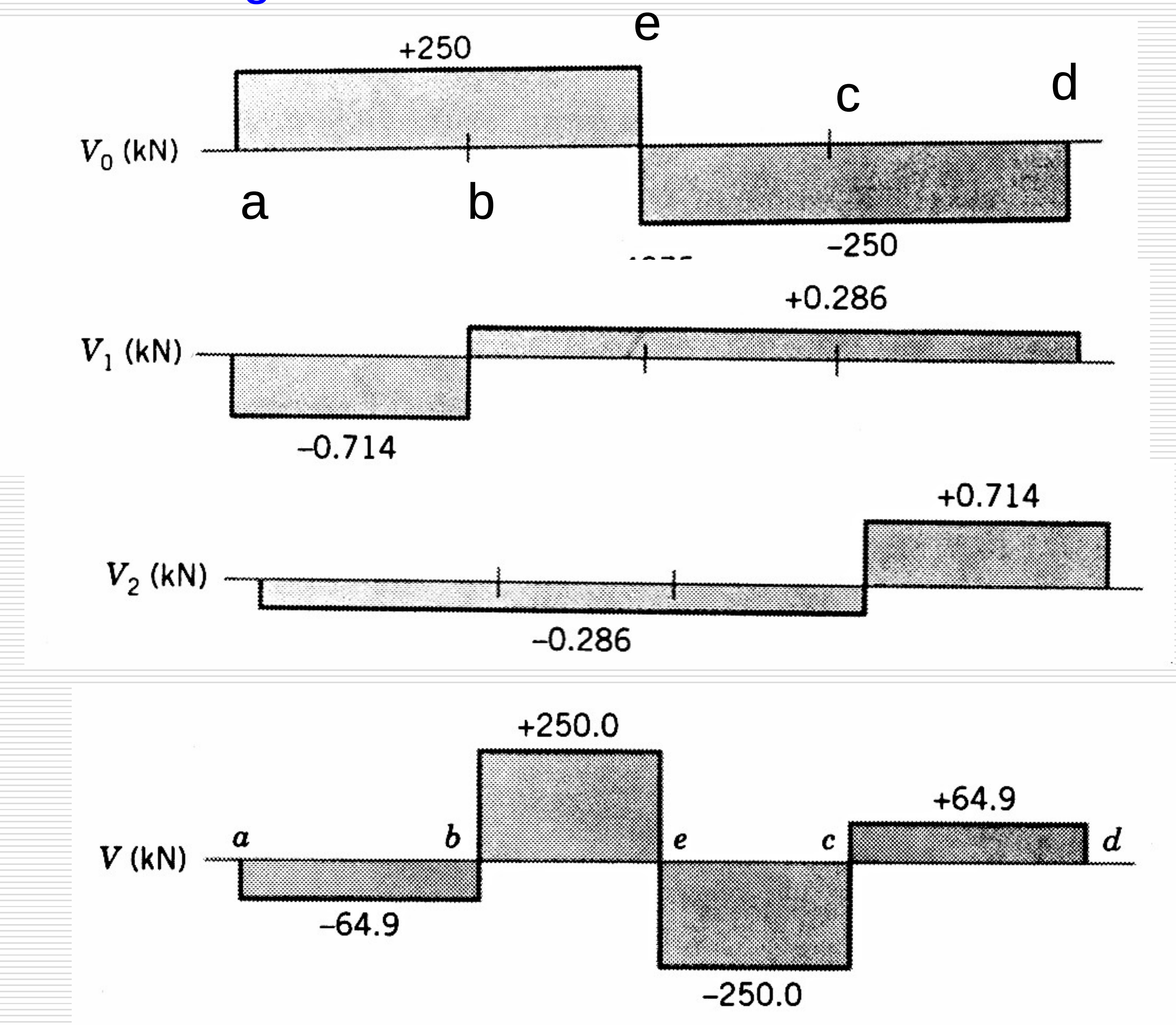
$$
= \begin{Bmatrix} 0 \\ 250 \\ 0 \\ 0 \\ 250 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -0.714 \\ 1 \\ 0 \\ -0.286 \end{Bmatrix} \times 314.9 + \begin{Bmatrix} 0 \\ -0.286 \\ 0 \\ 1 \\ -0.714 \end{Bmatrix} \times 314.9 = \begin{Bmatrix} 0 \\ -64.9 \\ 314.9 \\ 314.9 \\ -64.9 \end{Bmatrix} kN
$$

$$\begin{Bmatrix} V_{a-b} \\ V_{b-e} \\ V_{e-c} \\ V_{c-d} \end{Bmatrix} = \begin{Bmatrix} 250 \\ 250 \\ -250 \\ -250 \end{Bmatrix} + \begin{Bmatrix} -0.714 \\ 0.286 \\ 0.286 \\ 0.286 \end{Bmatrix} \times 314.9 + \begin{Bmatrix} -0.286 \\ -0.286 \\ -0.286 \\ 0.714 \end{Bmatrix} \times 314.9$$

$$= \begin{Bmatrix} -64.9 \\ 250 \\ -250 \\ 64.9 \end{Bmatrix} kN$$

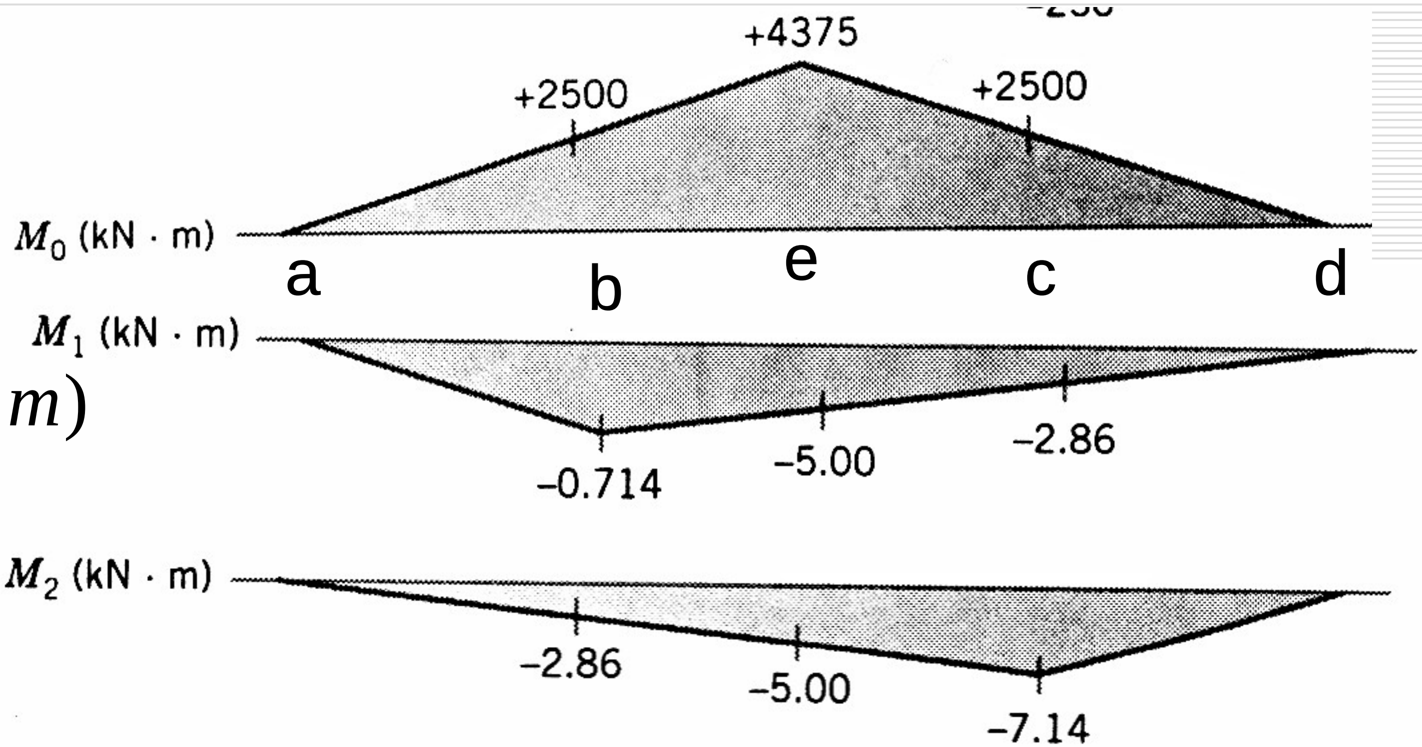


Shear Diagram

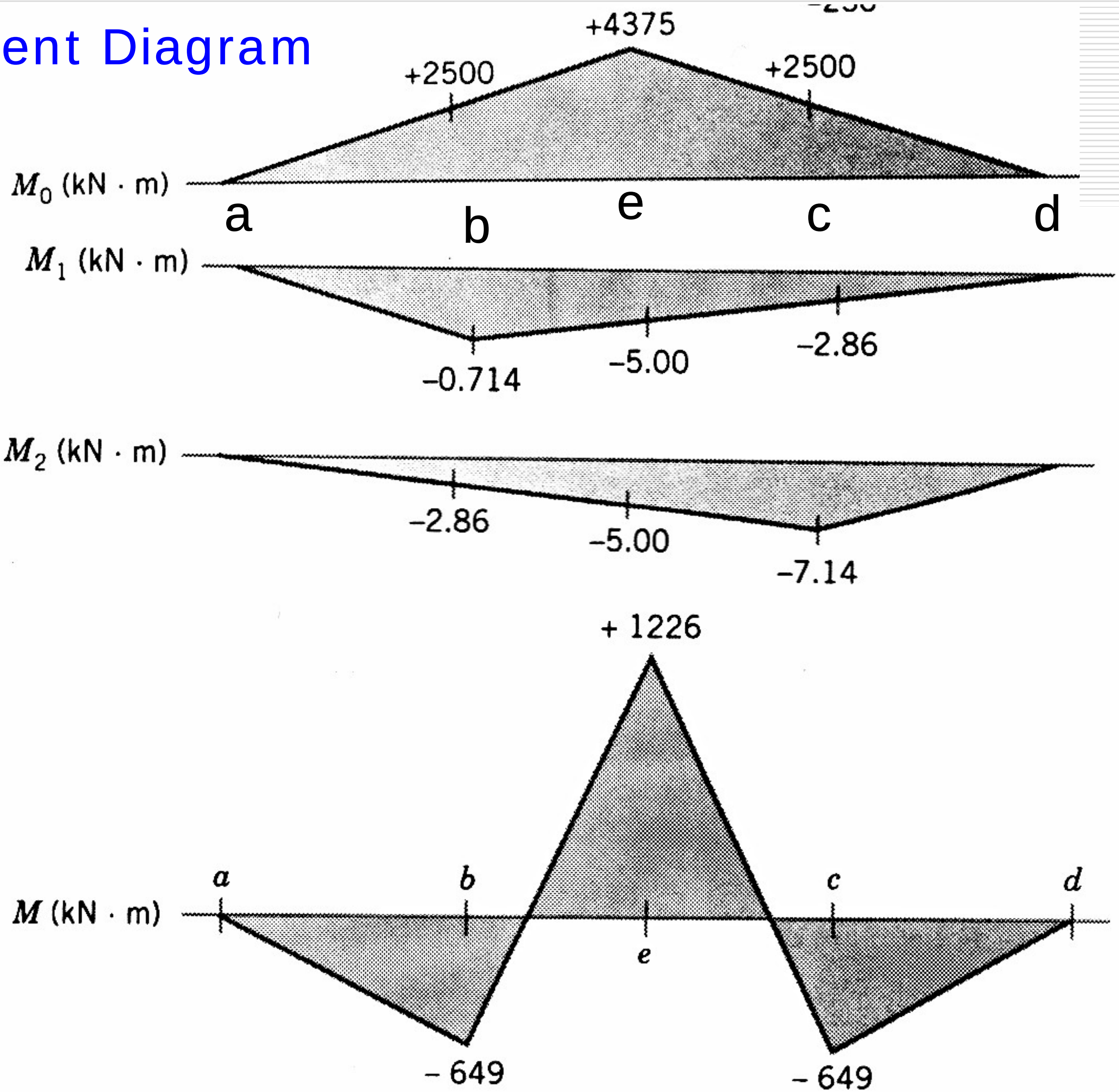


$$
\begin{Bmatrix} M_a \\ M_b \\ M_e \\ M_c \\ M_e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2{,}500 \\ 4{,}375 \\ 2{,}500 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -7.14 \\ -5.00 \\ -2.86 \\ 0 \end{Bmatrix} \times 314.9 + \begin{Bmatrix} 0 \\ -2.86 \\ -5.00 \\ -7.14 \\ 0 \end{Bmatrix} \times 314.9
$$

$$
= \begin{Bmatrix} 0 \\ -649 \\ 1{,}226 \\ -649 \\ 0 \end{Bmatrix} (kN \cdot m)
$$

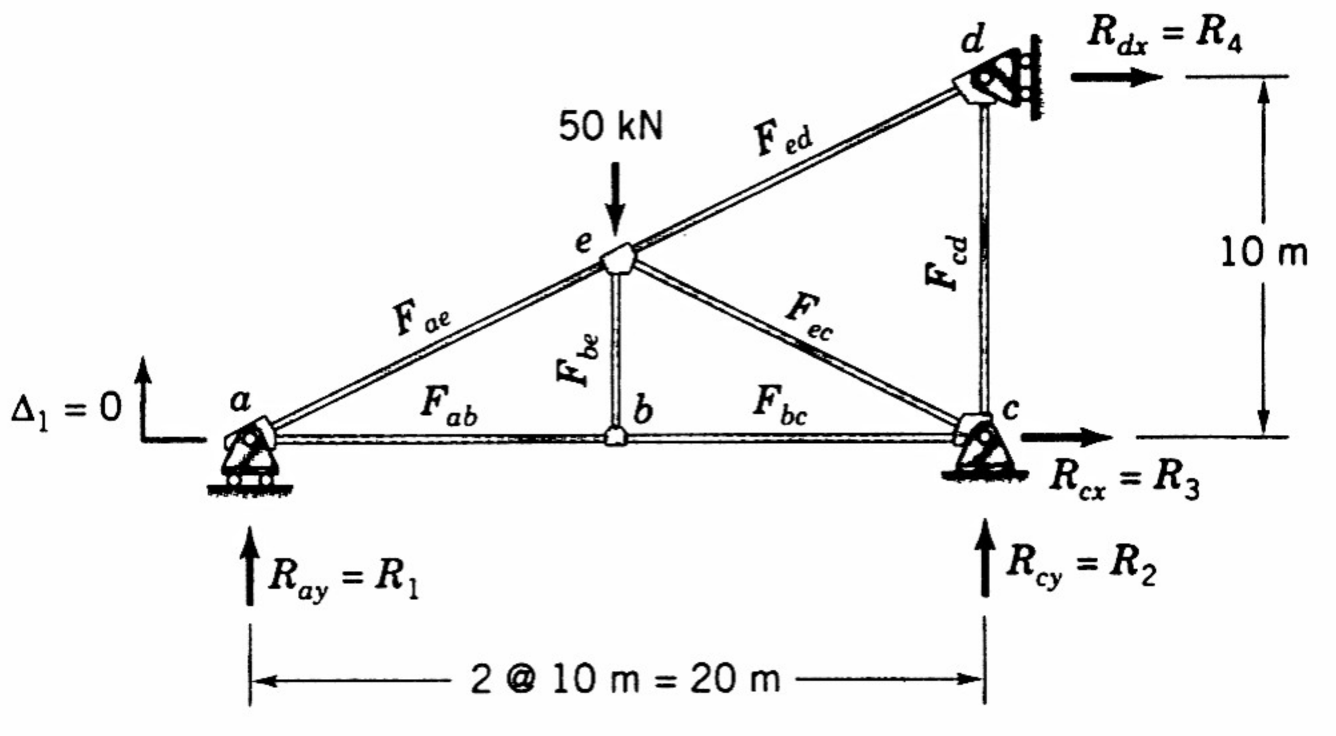


Moment Diagram



EXAMPLE 10.3

Determine the reactions and bar forces for the statically indeterminate truss shown. The quantity EA is the same for each member.



$$r_a = 4$$

$$r = 3$$

$$m_a = 7$$

$$j = 5$$

$$m = 2j - r = 7$$

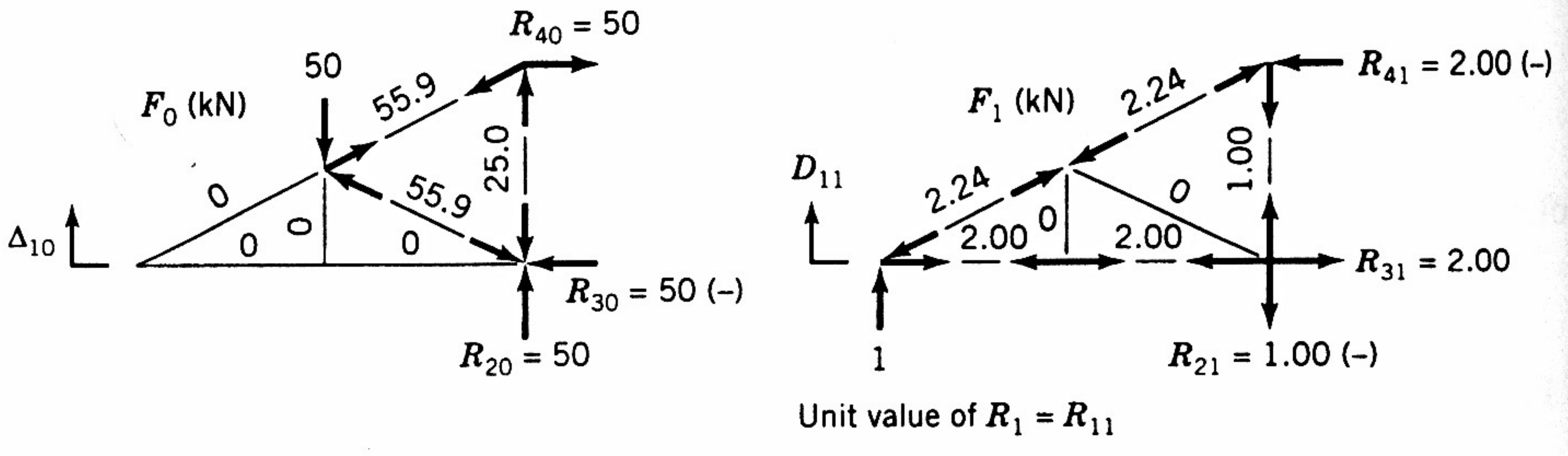
$$m_a = m$$

$$r_a = r + 1$$

- The structure is statically determinate internally (静的内的静定) and statically indeterminate externally to the first order (1次の静的外的不静定).

Primary structure and loadings

Select R_1 as the redundant reaction

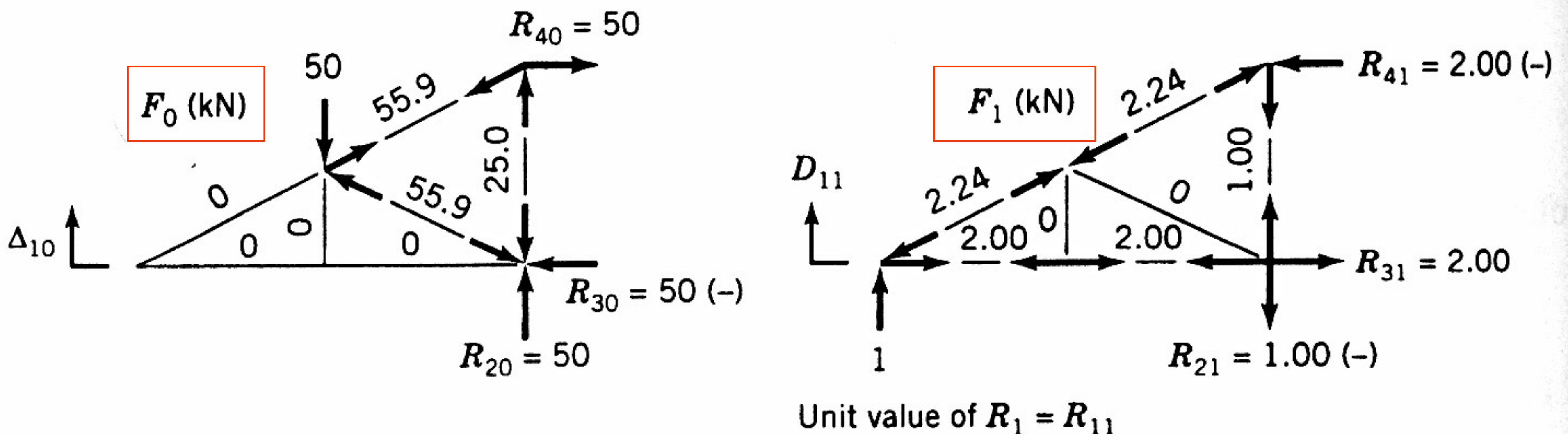


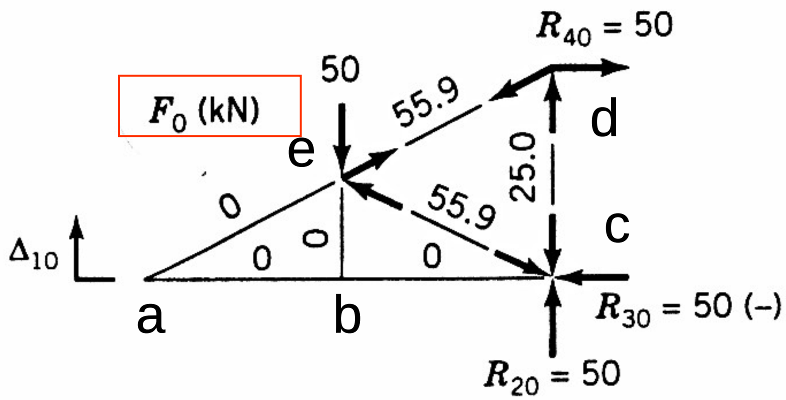
Displacement calculations

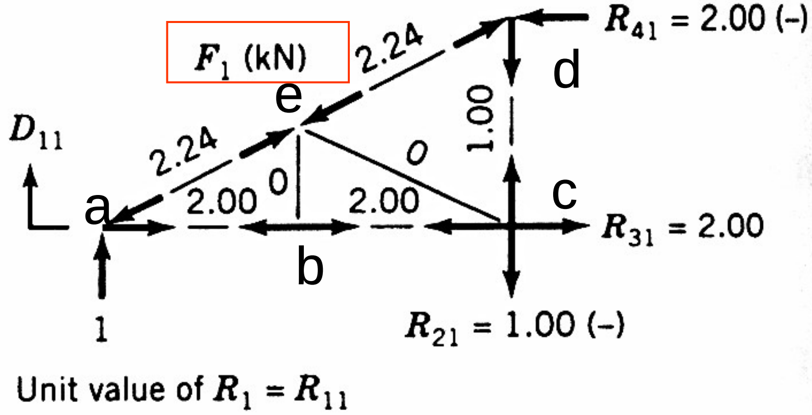
The complementary virtual work method (**補仮想仕事の原理**) is used to obtain the required displacement quantities.

$$\sum_{i=1}^{n} \delta P_i \Delta_i = \sum_{j=1}^{m} \frac{\delta F_p \cdot F_\Delta}{EA} l \qquad (7.18)$$

$$1 \times \Delta_{10} = \frac{\sum F_1 F_0}{EA} l \qquad 1 \times D_{11} = \frac{\sum F_1{}^2}{EA} l$$







$$1\times\Delta_{10}=\frac{\sum F_1F_0}{EA}l \qquad 1\times D_{11}=\frac{\sum F_1{}^2}{EA}l$$

Member	l m	F_0 kN	F_1 kN	F_1F_0l $(\mathrm{kN})^2\cdot\mathrm{m}$	F_1^2l $(\mathrm{kN})^2\cdot\mathrm{m}$
ab	10.0	0	+2.00	0	40.0
bc	10.0	0	+2.00	0	40.0
ae	11.2	0	–2.24	0	56.2
ed	11.2	+55.9	–2.24	–1 402.4	56.2
eb	5.0	0	0	0	0
ec	11.2	–55.9	0	0	0
cd	10.0	–25.0	+1.00	–250.0	10.0
Σ				–1 652.4	+202.4

$$1\times\Delta_{10}=\frac{\sum F_1F_0}{EA}l$$

$$=\frac{-1{,}652.4}{EA}(kN\cdot m/EA)$$

$$1\times D_{11}=\frac{\sum F_1{}^2}{EA}l$$

$$=\frac{202.4}{EA}(kN\cdot m/EA)$$

Member	l m	F_0 kN	F_1 kN	F_1F_0l $(\text{kN})^2\cdot\text{m}$	F_1^2l $(\text{kN})^2\cdot\text{m}$
ab	10.0	0	+2.00	0	40.0
bc	10.0	0	+2.00	0	40.0
ae	11.2	0	–2.24	0	56.2
ed	11.2	+55.9	–2.24	–1 402.4	56.2
eb	5.0	0	0	0	0
ec	11.2	–55.9	0	0	0
cd	10.0	–25.0	+1.00	–250.0	10.0
Σ				–1 652.4	+202.4

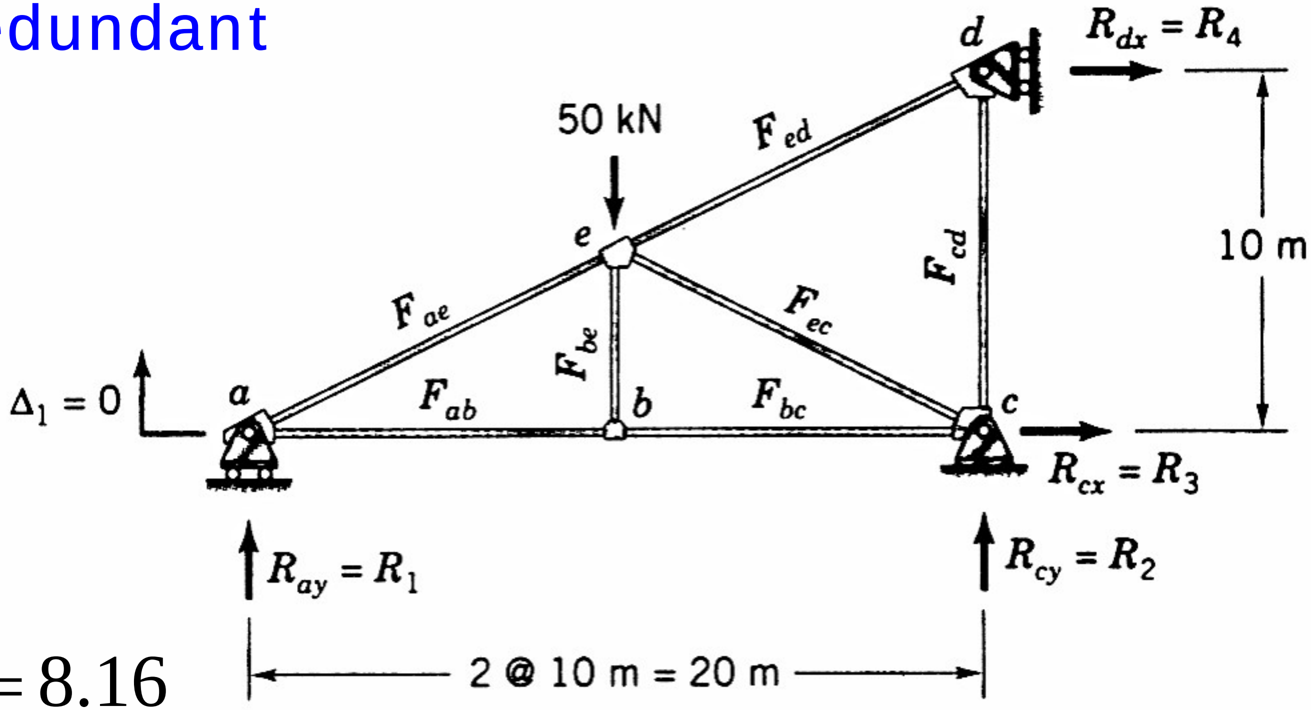
Calculation of redundant reaction

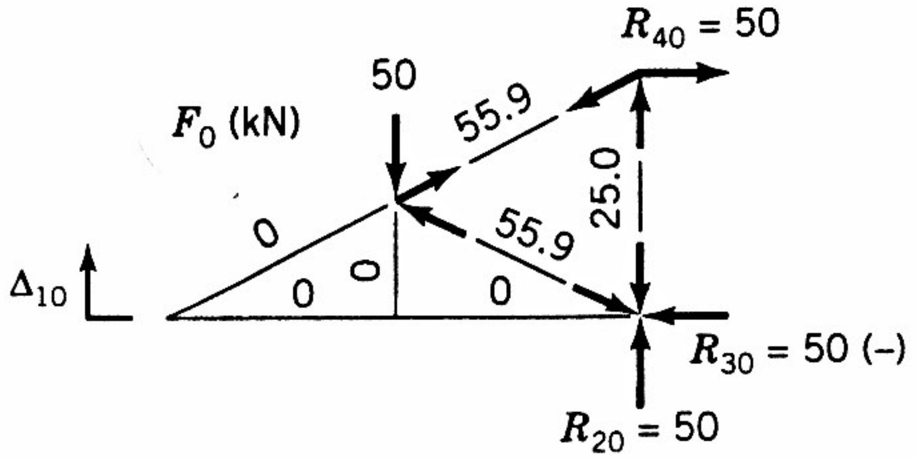
$$\Delta_{10} + D_{11}R_1 = \Delta_1$$

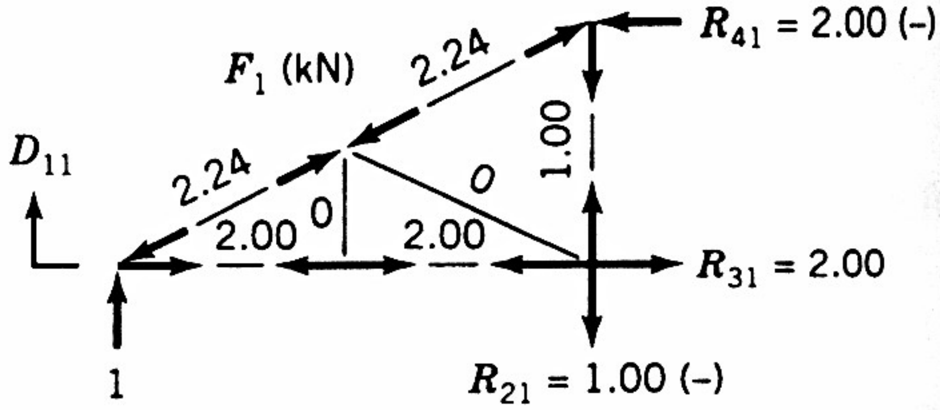
Since $\Delta_1 = 0$

$$R_1 = -\frac{\Delta_{10}}{D_{11}}$$

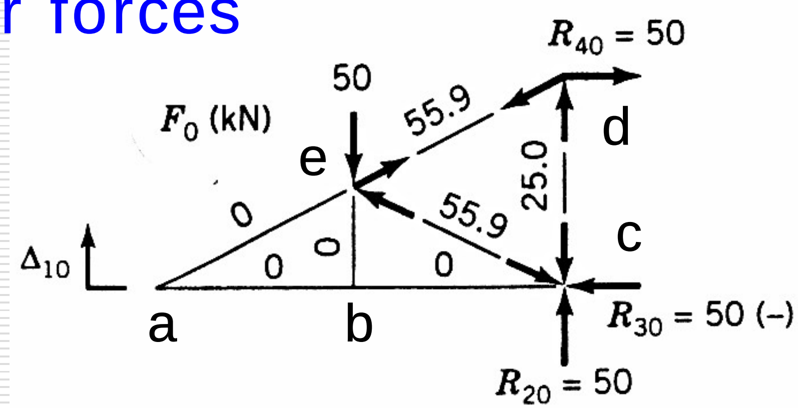
$$= -\frac{-1{,}652.4}{202.4} = 8.16$$

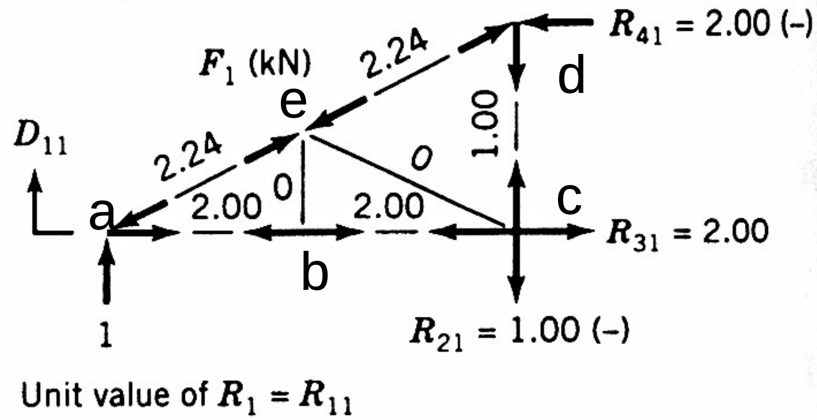






Member forces



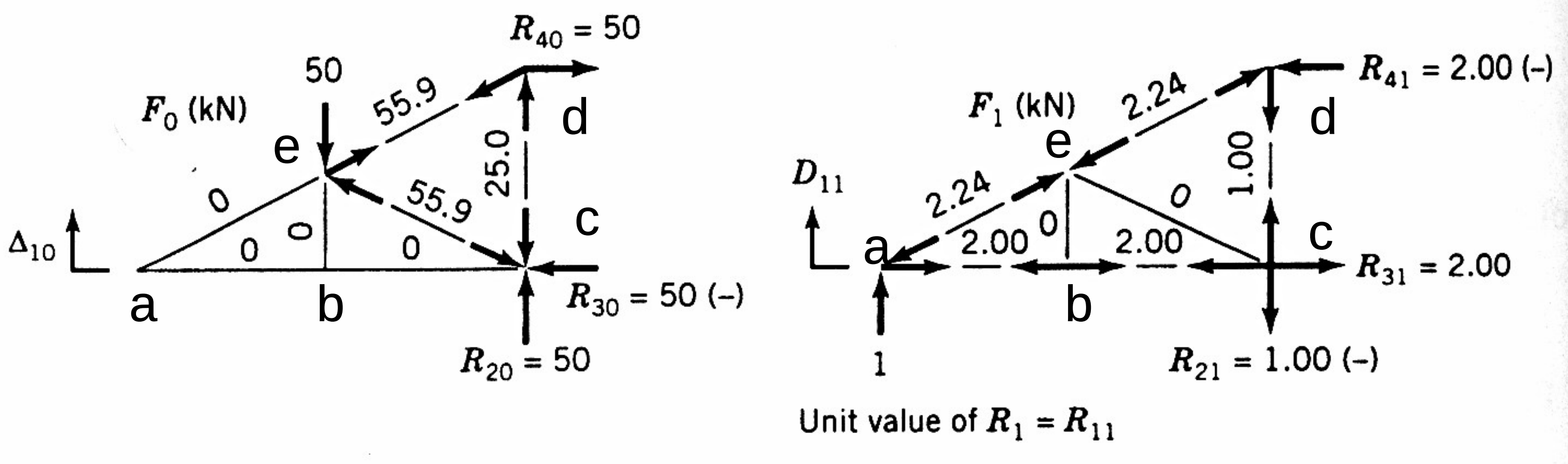


Member rs	$(F_{rs})_1 \cdot R_1$ kN	$F_{rs} = (F_{rs})_0 + (F_{rs})_1 \cdot R_1$ kN
ab	+16.32	+16.32
bc	+16.32	+16.32
ae	–18.28	–18.28
ed	–18.28	+37.62
eb	0	0
ec	0	–55.90
cd	+8.16	–16.84

$$R_1 = 8.16$$

where $(F_{rs})_1 = F_1$ force in member rs, and $(F_{rs})_0 = F_0$ force in member rs.

Reactions

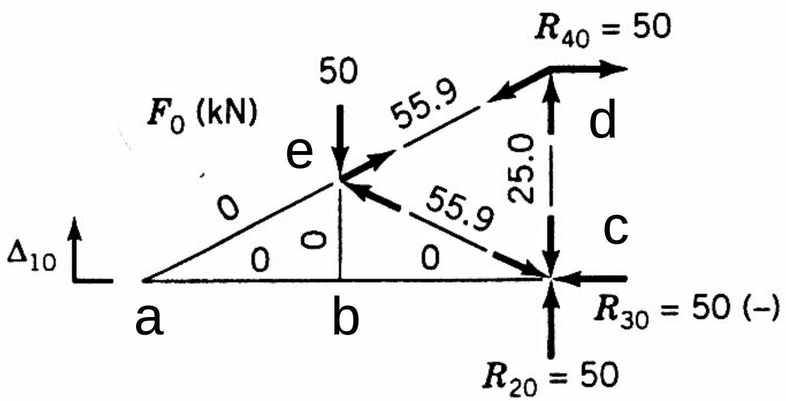


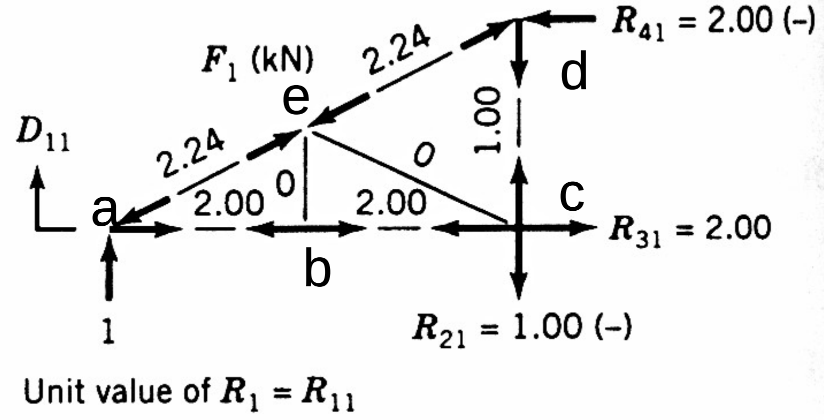
Reaction q	R_{q0} kN	R_{q1} kN	$R_{q1} \cdot R_1$ kN	$R_q = R_{q0} + R_{q1} \cdot R_1$ kN
1	—	1.00	+8.16	+8.16
2	50	–1.00	–8.16	+41.84
3	–50	2.00	+16.32	–33.68
4	50	–2.00	–16.32	+33.68

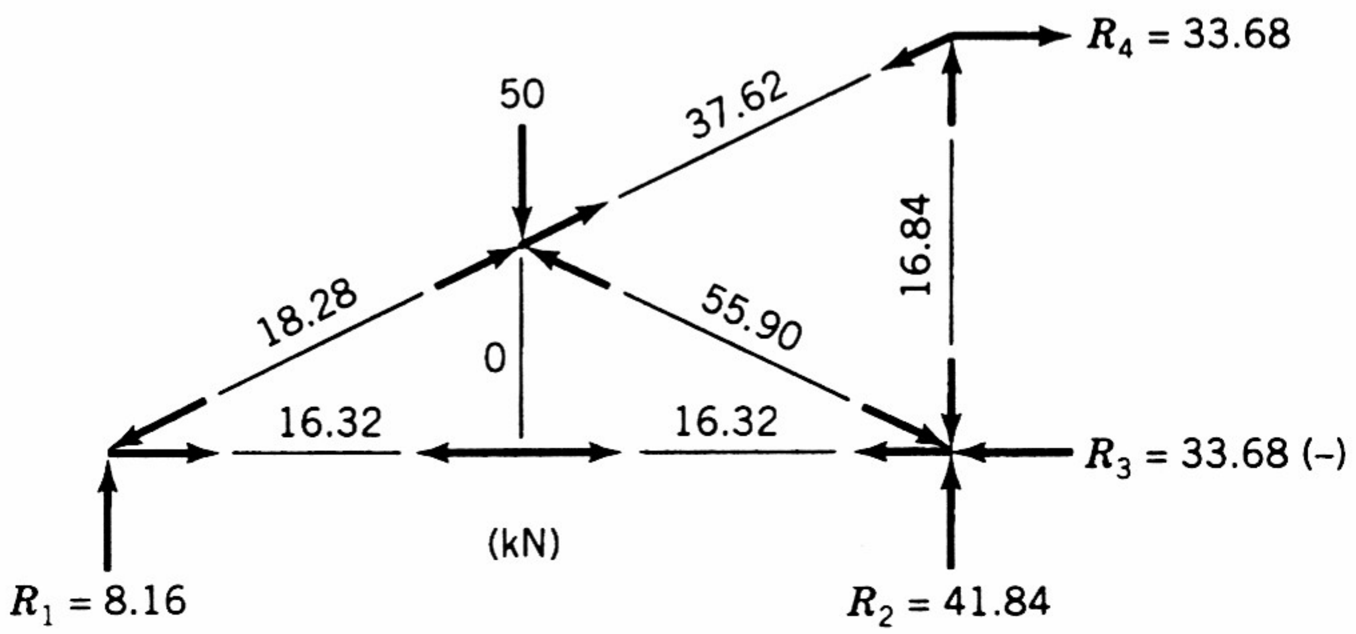
where $R_{q1} = R_q$ for F_1 system, and $R_{q0} = R_q$ for F_0 system.

$$R_1 = 8.16$$

Final results for reactions



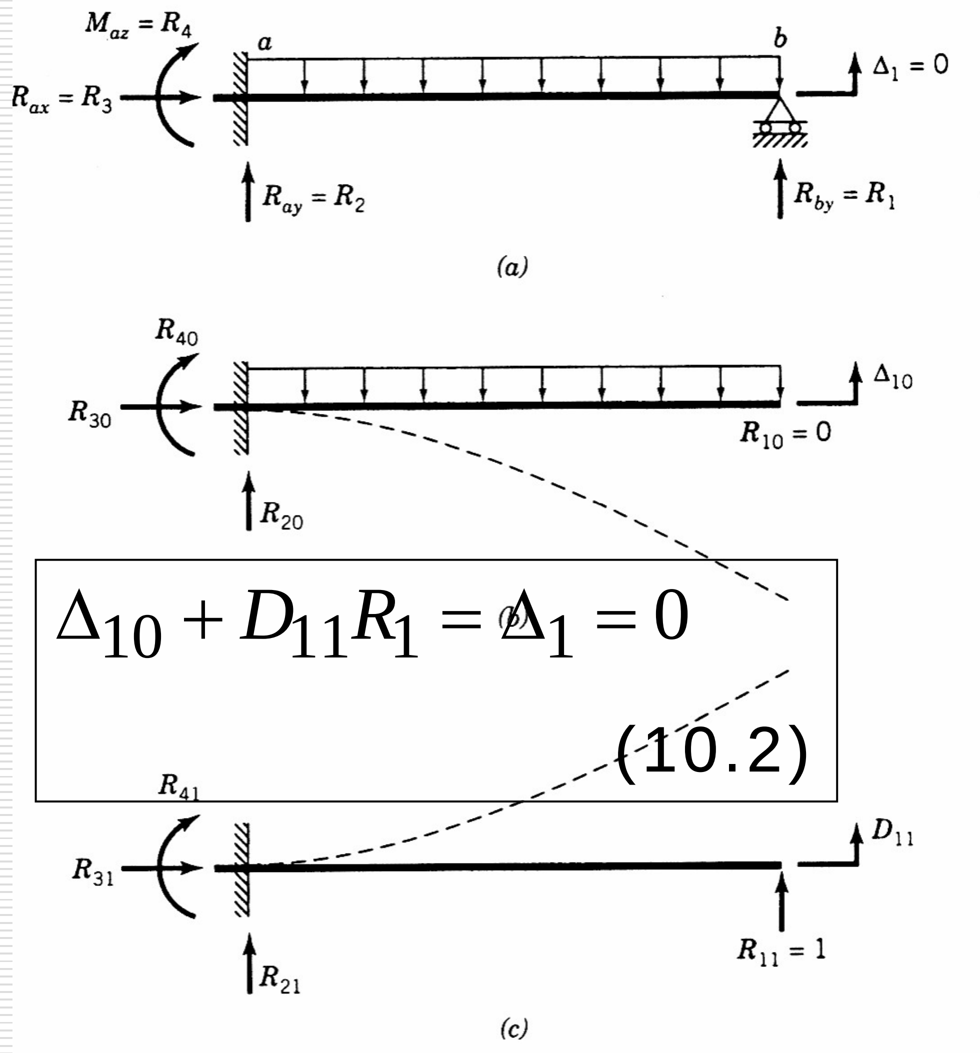




10.5 Support Settlements (支点沈下) and Elastic Supports (弹性支点)

- Support settlement in statically indeterminate structures induces forces throughout the structure.

- Consider the structure shown in Fig. 10.1 The compatibility equation given as Eq. (10.2) includes the displacement Δ_1, which is the upward vertical displacement at point b for the actual structure.



$$\Delta_{10} + D_{11}R_1 = \Delta_1 = 0 \tag{10.2}$$

Fig. 10.1

- If thee is a downward settlement of Δ_{1s} , then

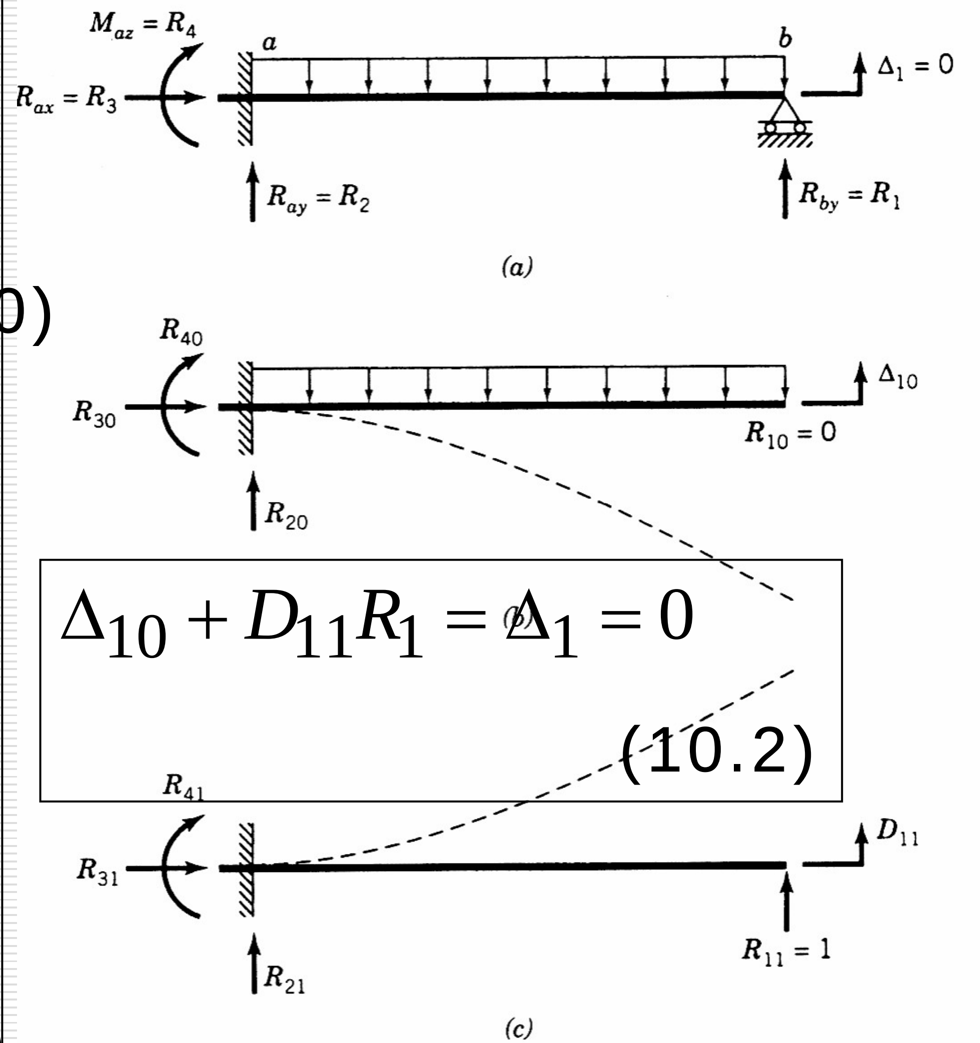
$$\Delta_1 = -\Delta_{1s}$$

In such a case, R_1 becomes from Eq. (10.2) as

$$R_1 = \frac{-\Delta_{1s} - \Delta_{10}}{D_{11}} \qquad (10.10)$$

- Thus, it is clear that the magnitude of redundant reaction is affected by the support settlement

- Once R_1 has been obtained, the final response quantities are evaluated by superposition



$$\Delta_{10} + D_{11} R_1 = \Delta_1 = 0 \qquad (10.2)$$

Fig. 10.1

● In a similar manner, if downward settlements of Δ_{1s} and Δ_{2s} are prescribed for the structure of Fig. 10.2, Eq. (10.8) becomes

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -\Delta_{s1} - \Delta_{10} \\ -\Delta_{2s} - \Delta_{20} \end{Bmatrix} \quad (10.11)$$

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 - \Delta_{10} \\ \Delta_2 - \Delta_{20} \end{Bmatrix} \quad (10.8)$$

● The solution of Eq. (10.11) gives the magnitudes of the redundant reactions R_1 and R_2 and other quantities.

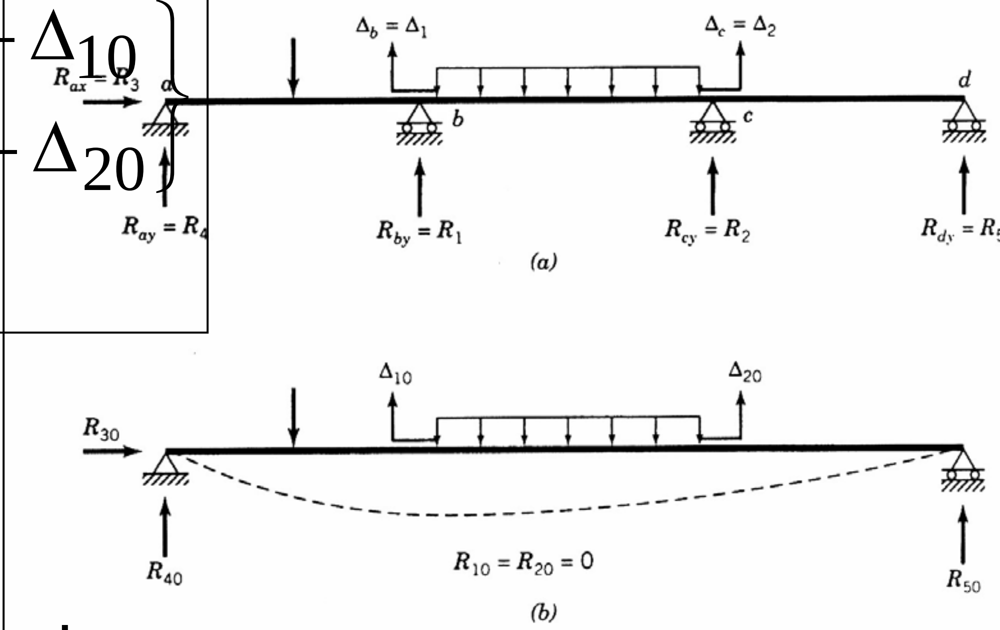


Fig. 10.2

● If there are prescribed settlements for the end supports (**はりの両端で支点沈下が与えられる場合**), then and must be interpreted as the displacement relative to the chord connecting the ends of the beam (**はりの両端を結ぶ直線からの距離**) as shown in Fig. 10.3.

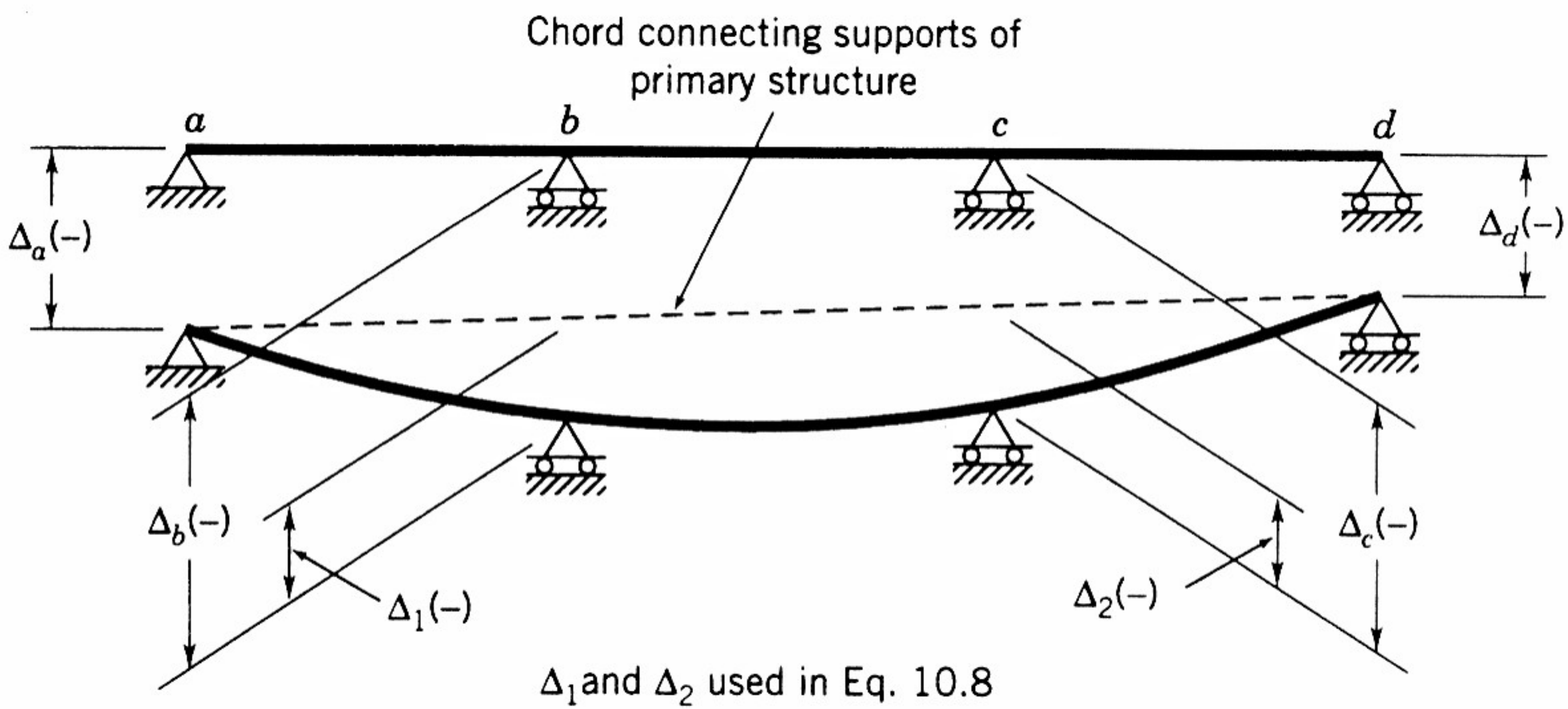


Figure 10.3 Settlement of continuous beam. Note that upward displacements are positive.

Elastic Supports (弹性支持)

- Frequently. Structures are mounted on supports in which the support movement depends on the magnitude of the reaction force.

- The simplest representation of this kind of support is shown in Fig. 10.4 and it is called elastic support in which settlement is a linear function of the reaction force.

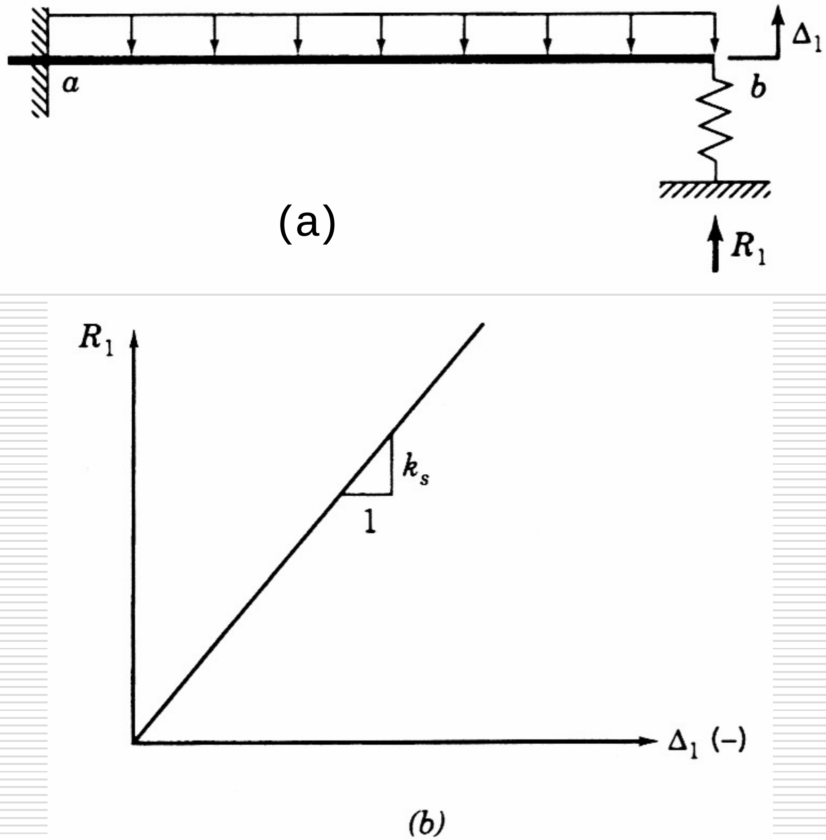


Fig. 10.4 Elastic support

- It is clear in this case that

$$R_1 = k_s(-\Delta_1) \qquad (10.12)$$

where k_s is the elastic spring constant.

- Substituting Eq. (10.12) into Eq. (10.2), we obtain

$$\Delta_{10} + D_{11}R_1 = -\frac{R_1}{k_s} \qquad (10.13)$$

$$\boxed{\Delta_{10} + D_{11}R_1 = \Delta_1 = 0 \qquad (10.2)}$$

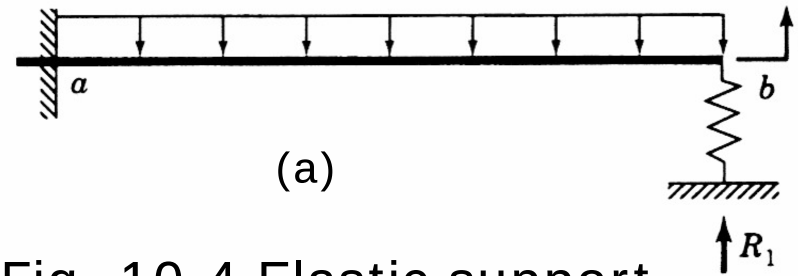


Fig. 10.4 Elastic support

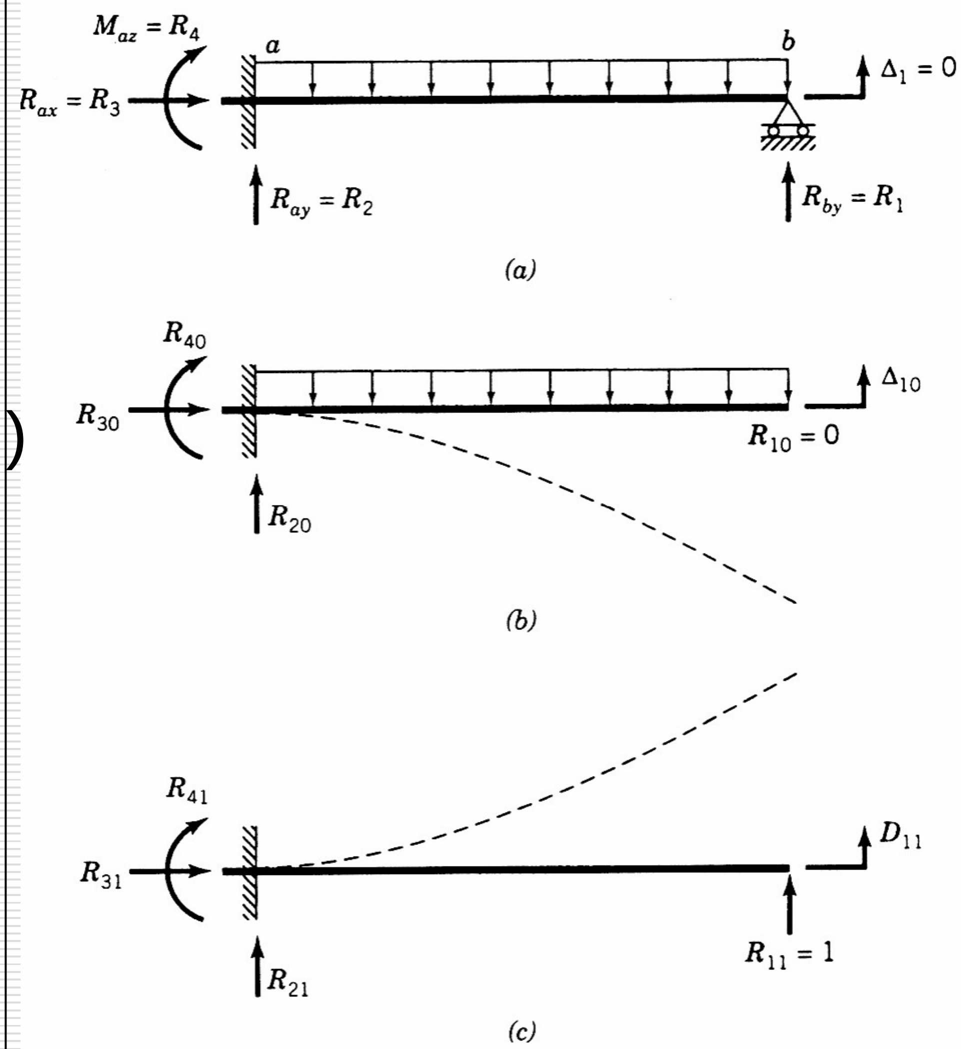


Fig. 10.1

- From Eq. (10.13), we obtain

$$R_1 = \frac{-\Delta_{10}}{D_{11} + 1/k_s} \quad (10.14)$$

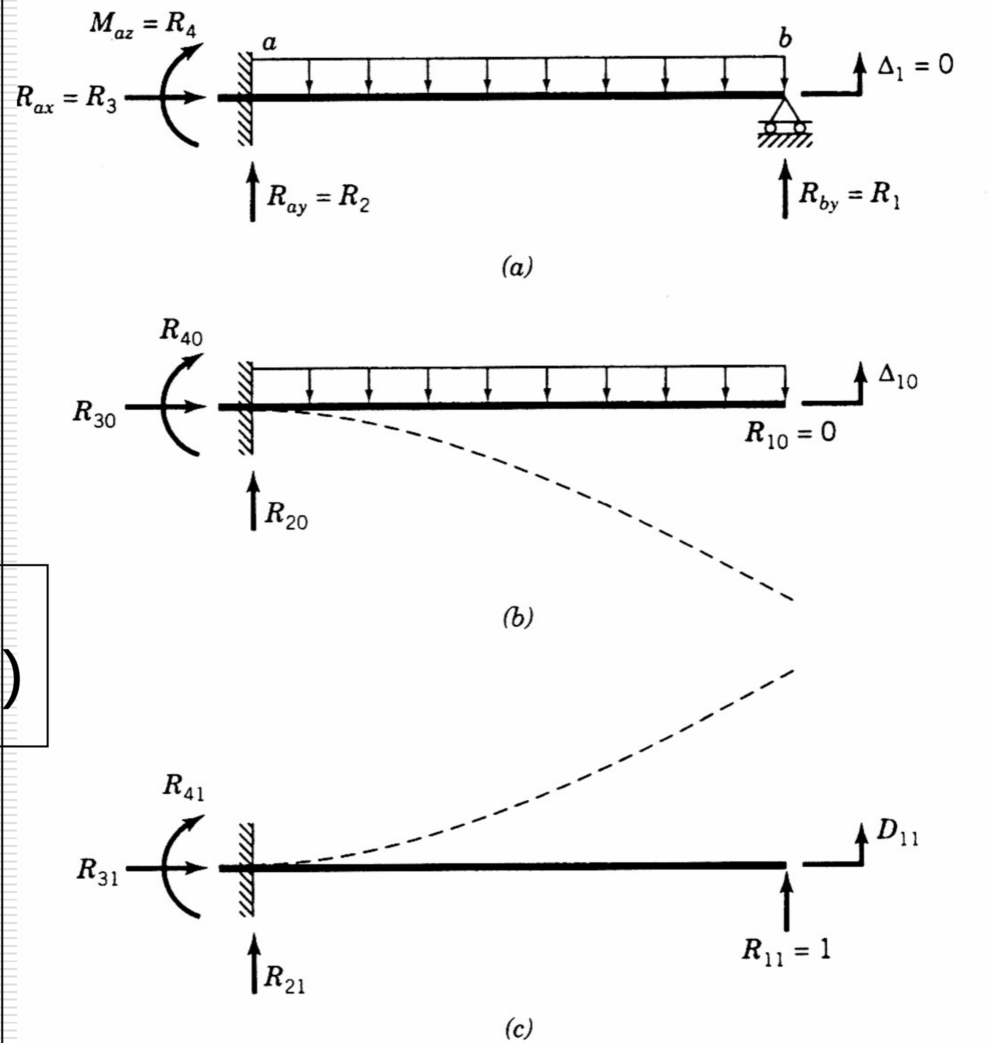
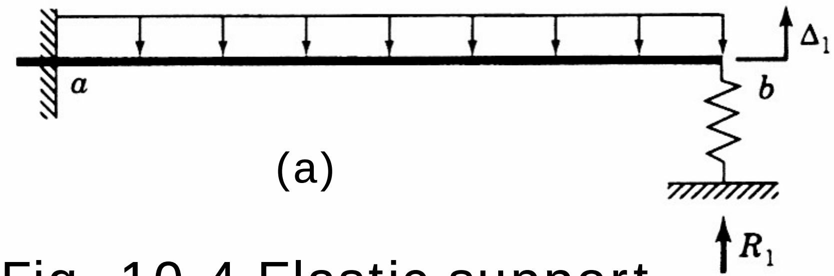


Fig. 10.1

$$\boxed{\Delta_{10} + D_{11} R_1 = -\frac{R_1}{k_s} \quad (10.13)}$$



(a)

Fig. 10.4 Elastic support

● If the supports at points b and c of Fig. 10.2 (a) are provided by elastic supports as shown in Fig. 10.5,

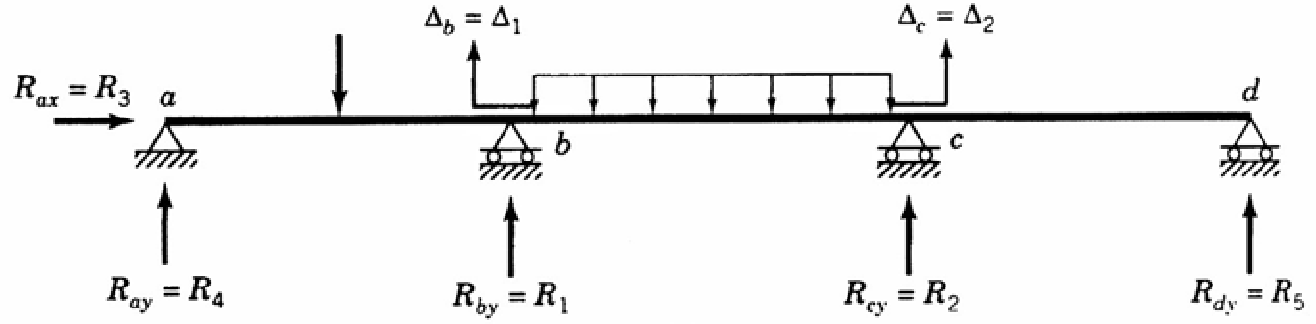


Fig. 10.2 (a)

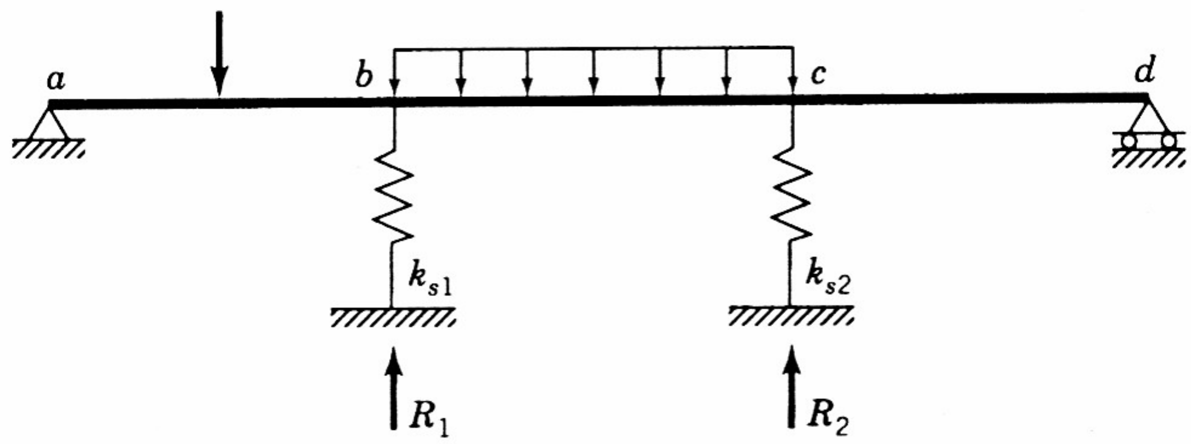


Figure 10.5 Continuous beam on elastic supports.

Eq. (10.8) takes the form

$$\begin{bmatrix} D_{11}+1/k_s & D_{12} \\ D_{21} & D_{22}+1/k_s \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -\Delta_{10} \\ -\Delta_{20} \end{Bmatrix} \quad (10.15)$$

$$\boxed{\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1-\Delta_{10} \\ \Delta_2-\Delta_{20} \end{Bmatrix} \quad (10.8)}$$

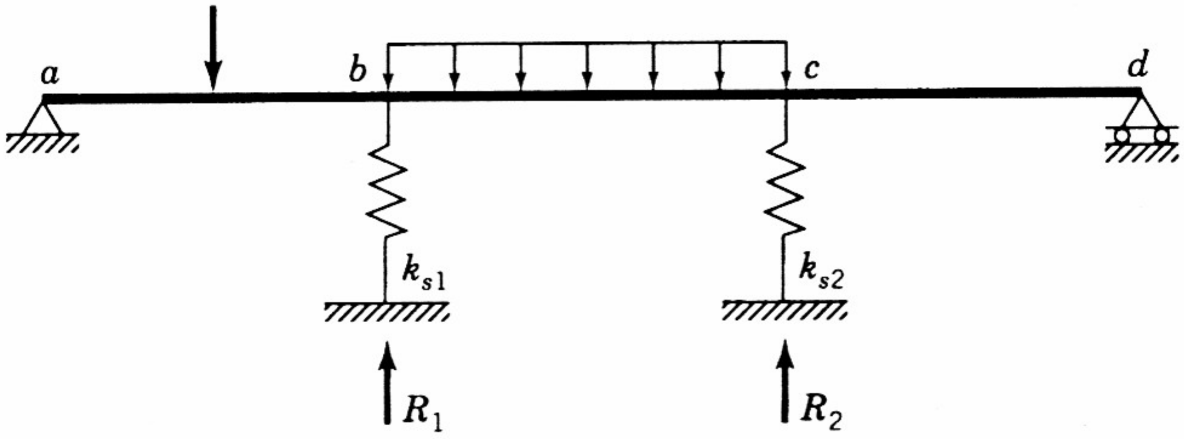


Figure 10.5 Continuous beam on elastic supports.

- This arrangement would correspond to the case in which R_1 and R_2 are provided by columns. Here, the elastic constants k_{s1} and k_{s2} would be a function of the properties of the columns

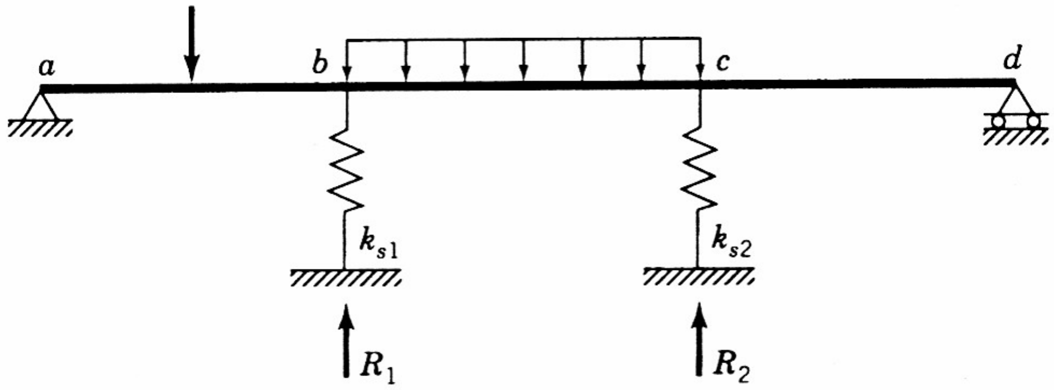


Figure 10.5 Continuous beam on elastic supports.

- Redundant reactions can be determined by solving Eq. (10.15), and any other response quantities are determined from Eq. (10.9).

$$\begin{bmatrix} D_{11}+1/k_s & D_{12} \\ D_{21} & D_{22}+1/k_s \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -\Delta_{10} \\ -\Delta_{20} \end{Bmatrix} \qquad (10.15)$$

$$S = S_0 + S_1 R_1 + S_2 R_2 \qquad (10.9)$$

EXAMPLE 10.4

Consider the continuous beam of Example 10.2, and determine the moment diagram for the following set of support settlements:

$$\Delta_a = -27.5mm \qquad \Delta_b = -47.5mm$$

$$\Delta_c = -22mm \qquad \Delta_d = -10mm$$

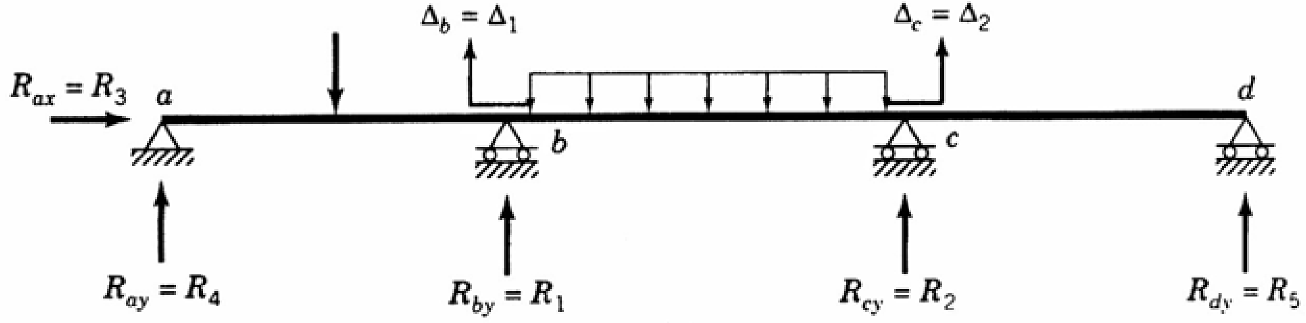
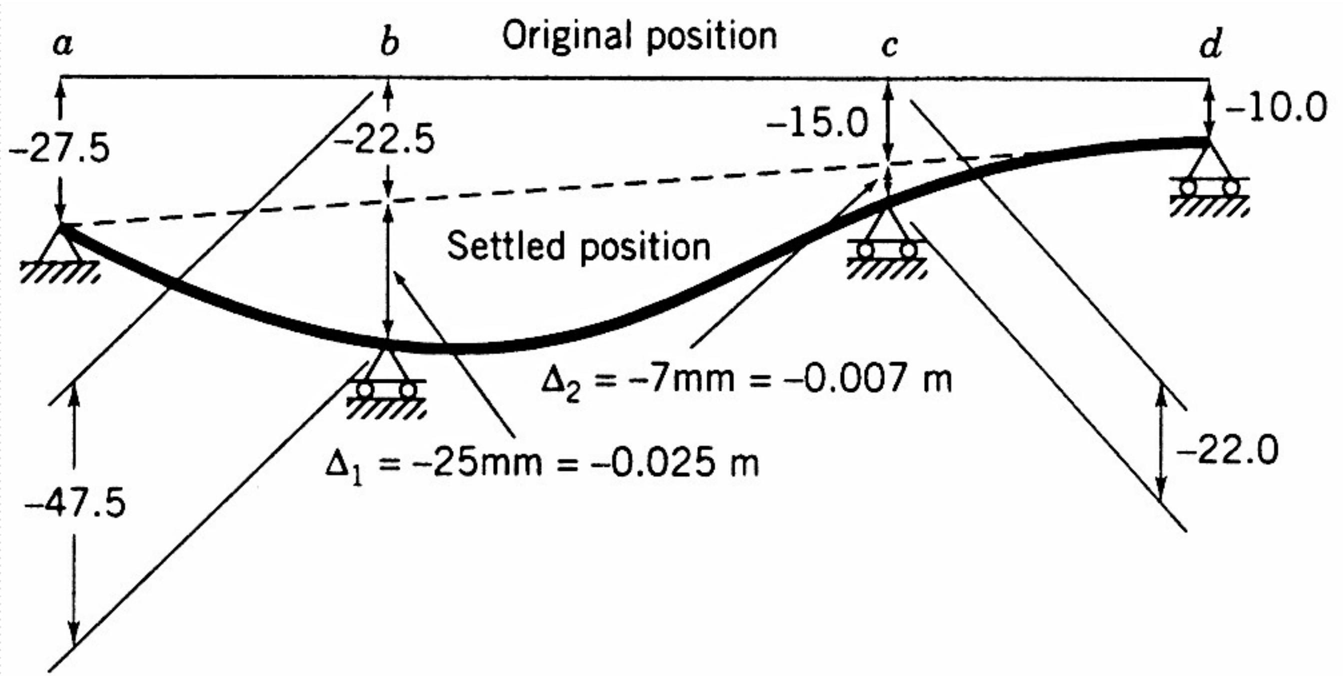


Fig. 10.2 (a)

Settlement pattern

$$\Delta_a = -27.5mm \quad \Delta_b = -47.5mm$$

$$\Delta_c = -22mm \quad \Delta_d = -10mm$$



- The displacements of points b and c relative to the chord connecting points a and d are needed, thus

$$\Delta_1 = -0.025m \quad \text{and} \quad \Delta_2 = -0.007m$$

Structural Classification

- This structure is statically indeterminate to the second order. (same as Example 10.2)

Compatibility equations

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 - \Delta_{10} \\ \Delta_2 - \Delta_{20} \end{Bmatrix} \qquad (10.8)$$

Redundant reactions

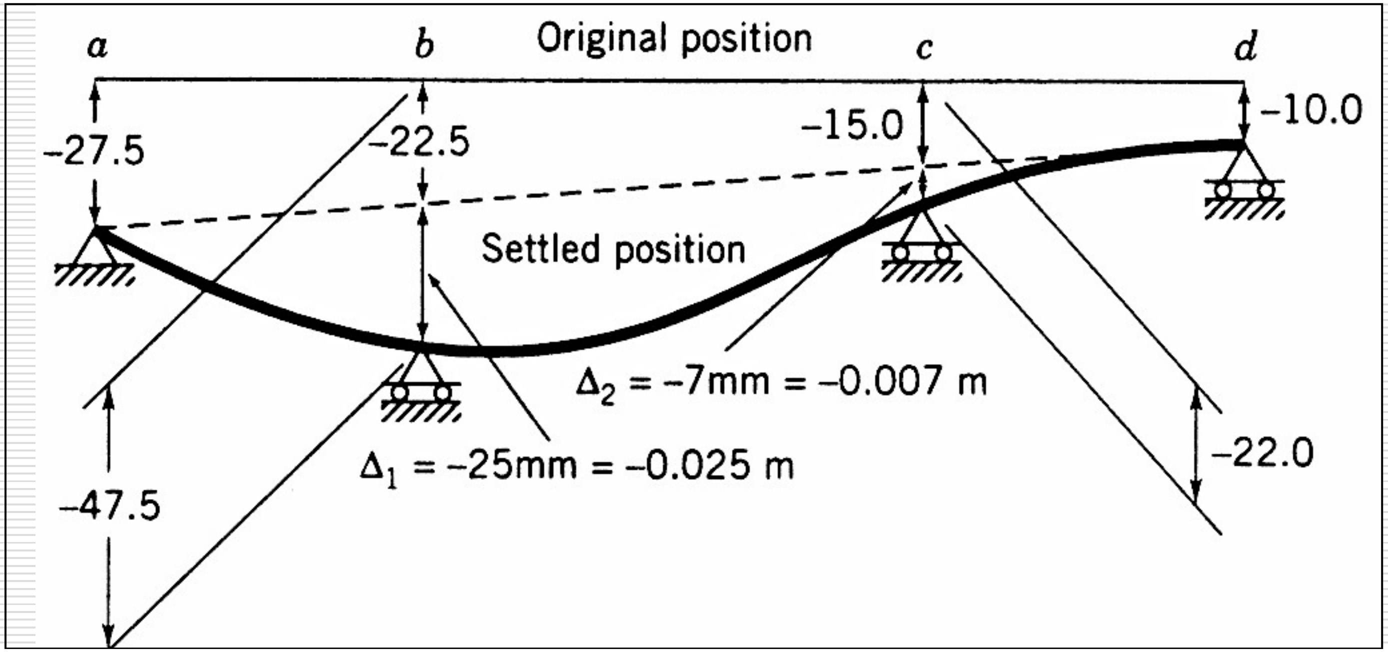
In this case, $\Delta_{10} = \Delta_{20} = 0$, and thus

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} \qquad \text{(a)}$$

- Because $\Delta_1 = -0.025m$ and $\Delta_2 = -0.007m$ substituting the appropriate displacement quantities, Eq. (a) becomes

$$\frac{1}{EI}\begin{bmatrix} 595.1 & 488.4 \\ 488.1 & 595.1 \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -0.025 \\ -0.007 \end{Bmatrix} \quad \text{(b)}$$

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} \quad \text{(a)}$$



●Solving Eq. (b) for the redundant reactions, we obtain

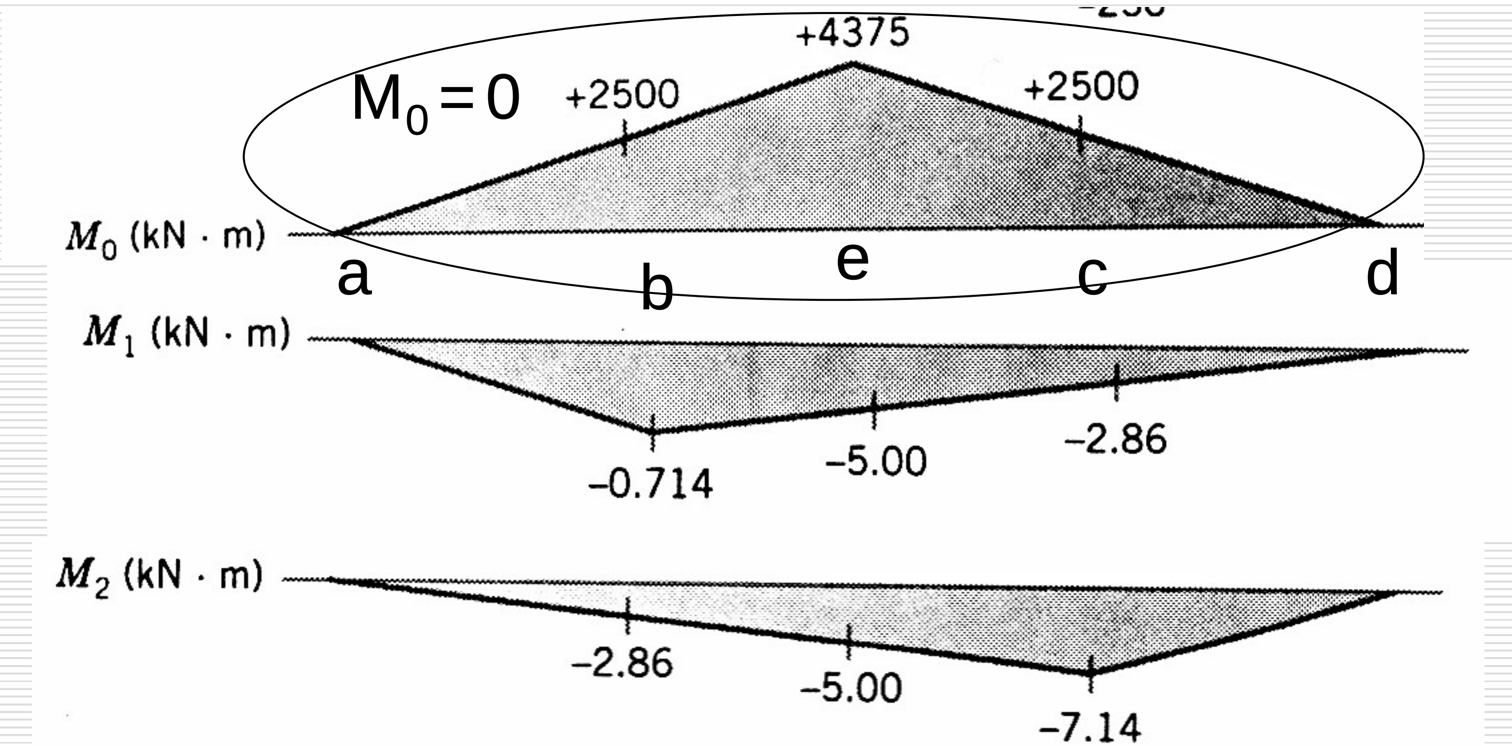
$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = EI \begin{Bmatrix} -99.1\times10^{-6} \\ 69.5\times10^{-6} \end{Bmatrix} \left(\frac{EI}{kN\cdot m^2}\right)$$

$$\boxed{\frac{1}{EI}\begin{bmatrix} 595.1 & 488.4 \\ 488.1 & 595.1 \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} -0.025 \\ -0.007 \end{Bmatrix} \qquad \text{(b)}}$$

Final moments

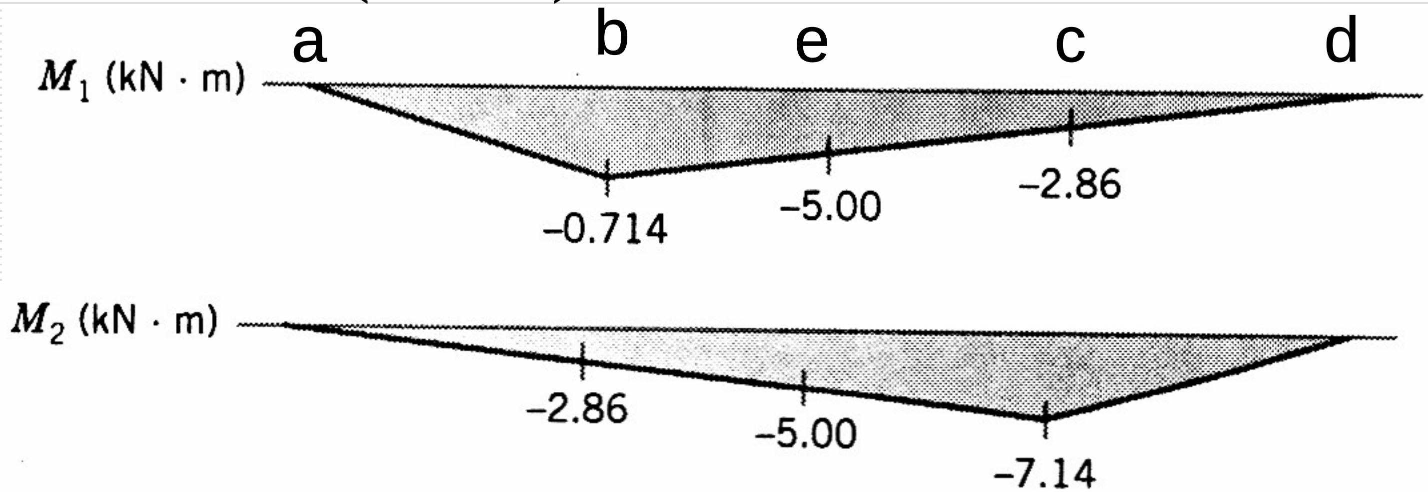
$$S = S_0 + S_1 R_1 + S_2 R_2 \qquad (10.9)$$

- In this case, $M_0=0$ (because no load is applied to the structure), and M_1 and M_2 are the moment diagram ordinates given in Example 10.2.



$$
\begin{Bmatrix} M_a \\ M_b \\ M_e \\ M_c \\ M_d \end{Bmatrix} = \begin{Bmatrix} 0 \\ -7.14 \\ -5.00 \\ -2.86 \\ 0 \end{Bmatrix} \times -99.1EI \times 10^{-6} + \begin{Bmatrix} 0 \\ -2.86 \\ -5.00 \\ -7.14 \\ 0 \end{Bmatrix} \times 69.5EI \times 10^{-6}
$$

$$
= \begin{Bmatrix} 0 \\ 508.8 \\ 148.0 \\ -212.8 \\ 0 \end{Bmatrix} \times EI \times 10^{-6} (EI / m)
$$



● Note that reaction forces and moments are dependent on the flexural stiffness (**曲げ剛性**) EI.

●For a specific case of $EI=3000\times10^{-6}m^4$ and $E=200\times10^9Pa$, we obtain