Structural Analysis II

(2)

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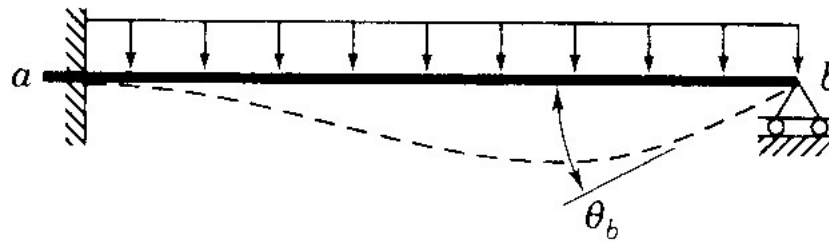# 9.4 KINEMATIC INDETERMINACIES ; REDUNDANCIES

1) What is kinematic indeterminacies?

● static indeterminacy refer to number of force quantities that must be determined in order to render the equilibrium solution complex.

● On the other hand, kinematic indeterminacy refers to the number of displacement quantities (kinematic degree of freedom) that are necessary to define the deformation response of the structure.
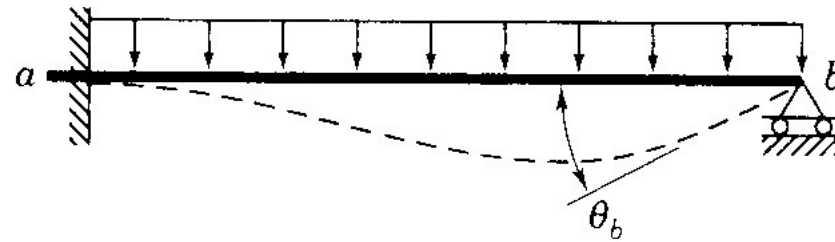
2) Example of kinematic indeterminacy

(a) Kinematically indeterminate structure with redundant displacement

Fig. 9.5 Selection of redundant displacement

●Since the structure is fixed at point a and vertically restrained at point b and the axial deformation is zero, there is only one kinematic degree of freedom, θ_b

●We identify the rotation θ_b as a kinematic redundant.

●If the rotation θ_b is removed from the structure $(\theta_b = 0)$, the resulting primary structure is called kinematically determinate (   ).

(a) Kinematically indeterminate structure with redundant displacement

(b) Kinematically determinate primary structure

●Fig. 9.5(b) shows the kinematically determinate primary structure.
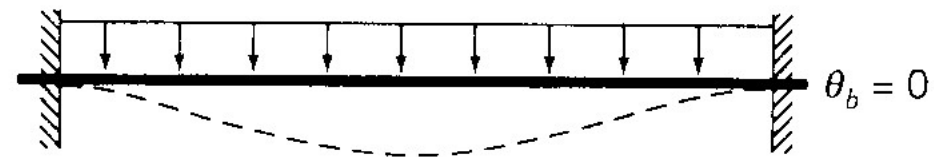
Fig. 9.5 Selection of redundant displacement

9.5 ALTERNATIVE FORM OF ANALYSIS

●Consider again the structure of Fig. 9.1 as shown in Fig. 9.6(a).

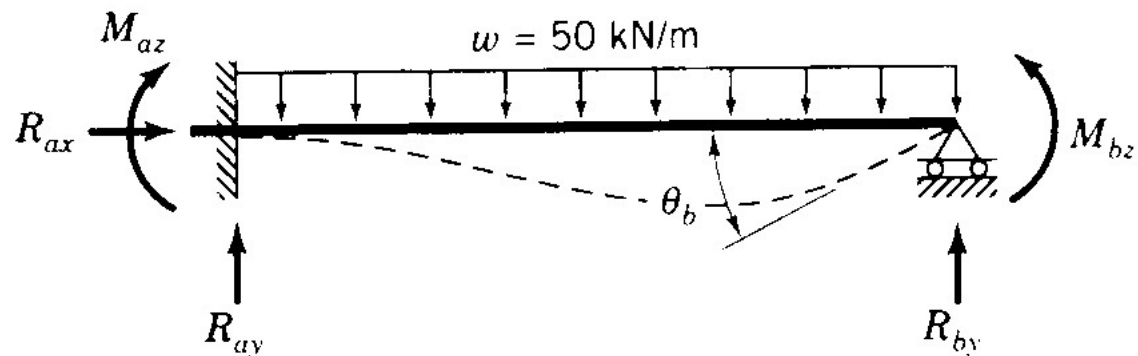●An end moment M_{bz} has been added which must be eventually be set equal to zero.

Fig. 9.6(a) Loading & deformation of a statically indeterminate beam

●Next, the statically determinate primary structure of Fig. 6.3(c) is separately loaded with w, R_{by} and M_{bz} as shown in Fig. 9.6(b)
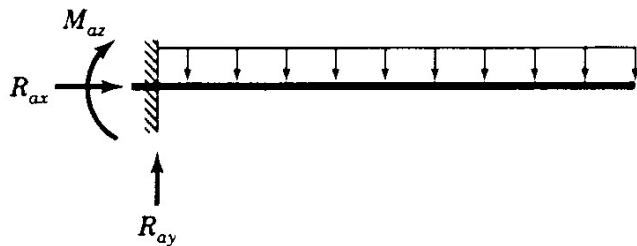
w

Δ_{b1} $\Delta_{b1} = wl^4/8EI(-)$

θ_{b1} $\theta_{b1} = wl^3/6EI(-)$

θ_{b2} $\theta_{b2} = R_{by}l^2/2EI$

Δ_{b2} $\Delta_{b2} = R_{by}l^3/3EI$

R_{by} R_{by}

θ_{b3} $\theta_{b3} = M_{bz}l/EI$

Δ_{b3} $\Delta_{b3} = M_{bz}l^2/2EI$

M_{bz} M_{bz}

Fig. 9.6 (b)

M_{az}

R_{ax}

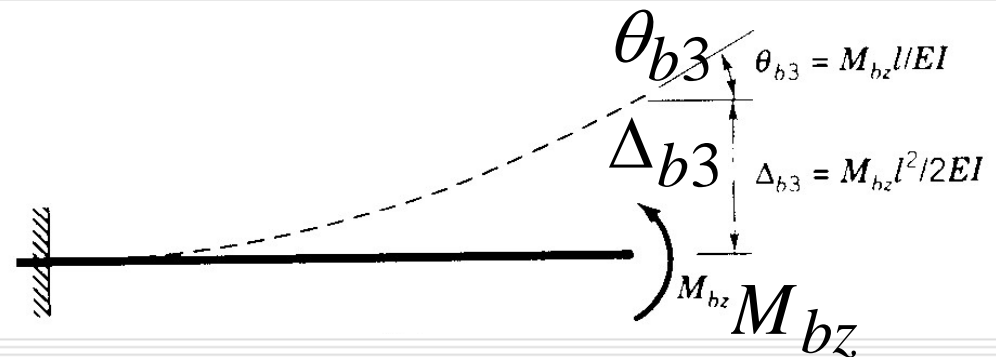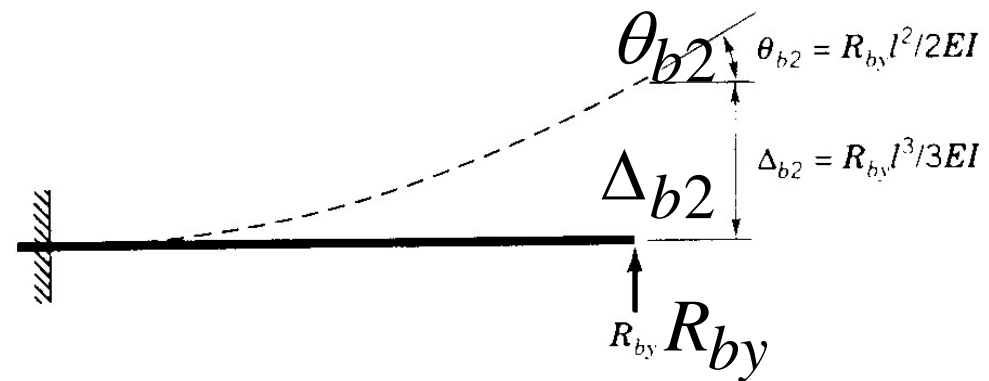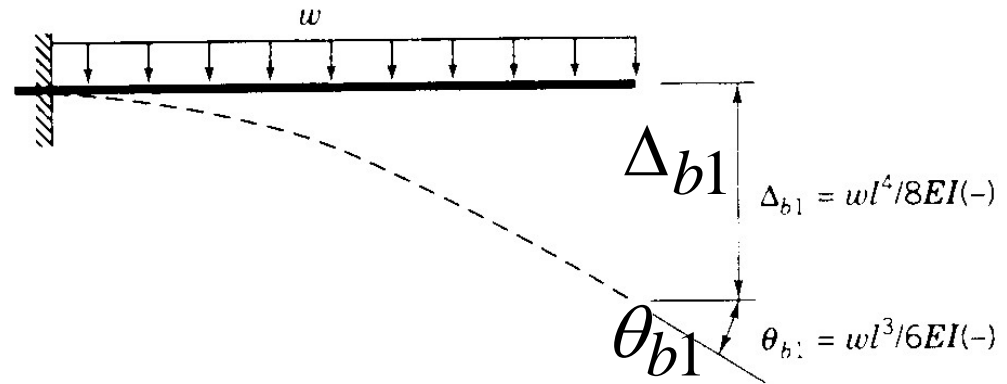R_{ay}

Fig. 9.3(c) Stable and statically determinate primary structure

●The separate
solutions of Fig. 9.6(b)
must be superimposed
to obtain the correct
boundary conditions
for the given structure
of Fig. 9.6(a) as

$$\Delta_{b1} + \Delta_{b2} + \Delta_{b3} = \Delta_b = 0$$

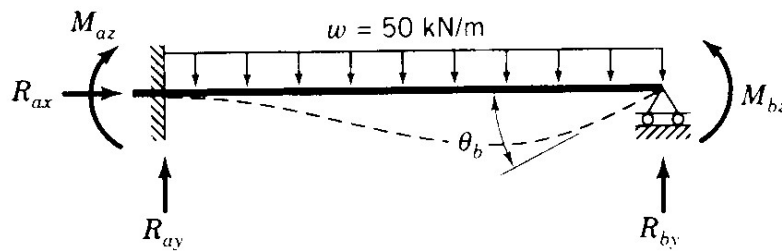$$\theta_{b1} + \theta_{b2} + \theta_{b3} = \theta_b$$

(9.6)

$\Delta_{b1} = wl^4/8EI(-)$

$\theta_{b1} = wl^3/6EI(-)$

$\theta_{b2} = R_{by}l^2/2EI$

$\Delta_{b2} = R_{by}l^3/3EI$

R_{by}

$\theta_{b3} = M_{bz}l/EI$
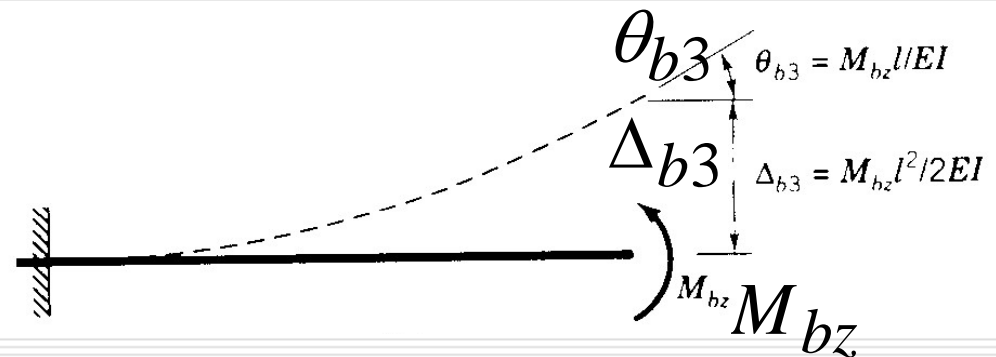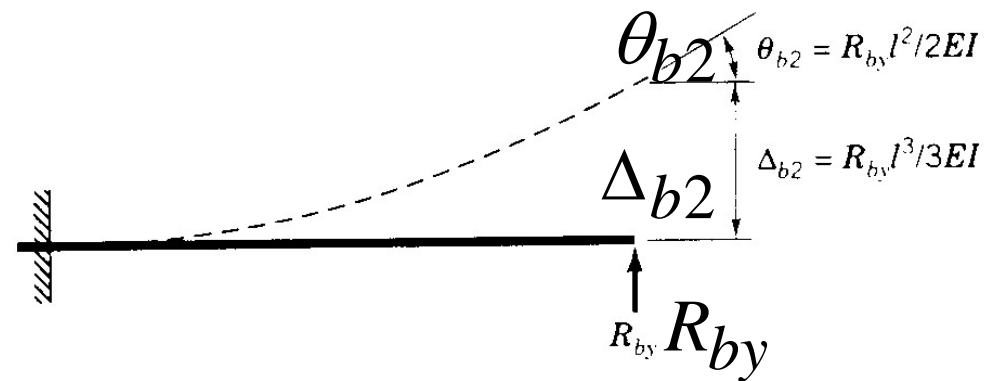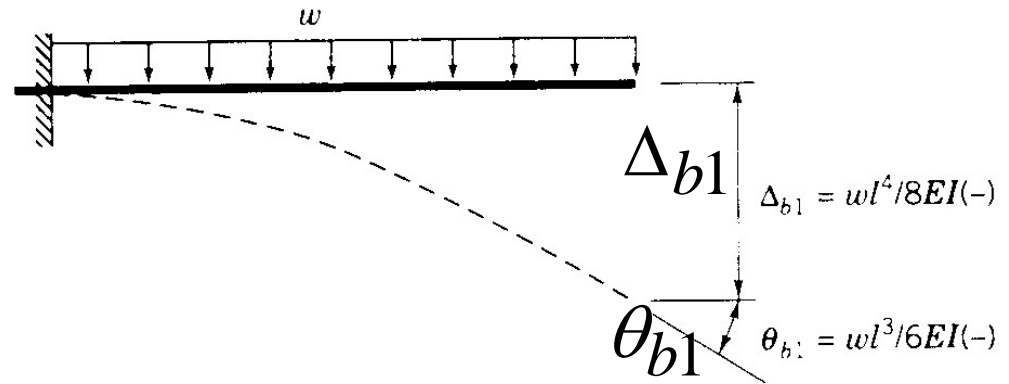
$\Delta_{b3} = M_{bz}l^2/2EI$

M_{bz}

Fig. 9.6 (b)

Fig. 9.6(a)

●Substitution of the displacement quantities into Eq. (9.6) yields

$$\frac{R_{by}l^3}{3EI} + \frac{M_{bz}l^2}{2EI} = \frac{wl^4}{8EI}$$

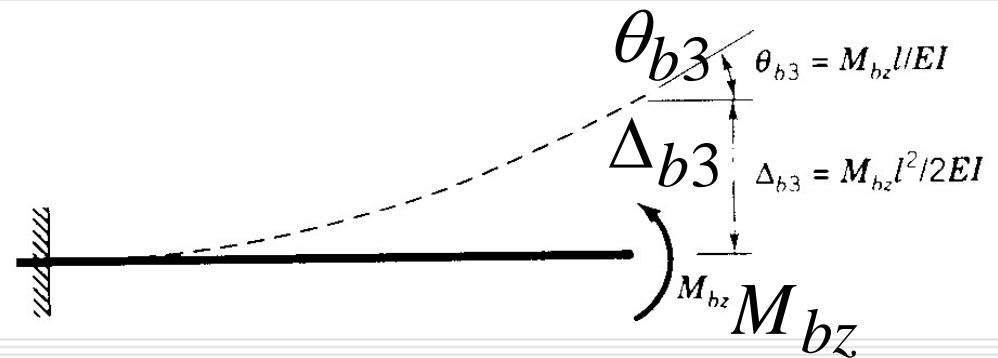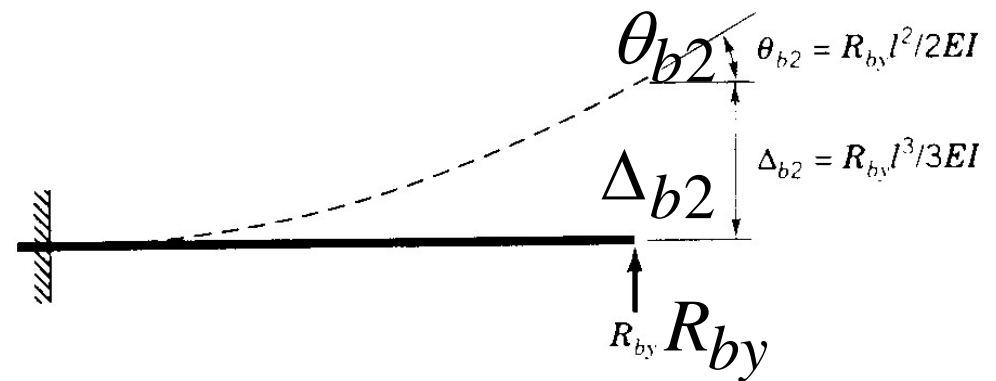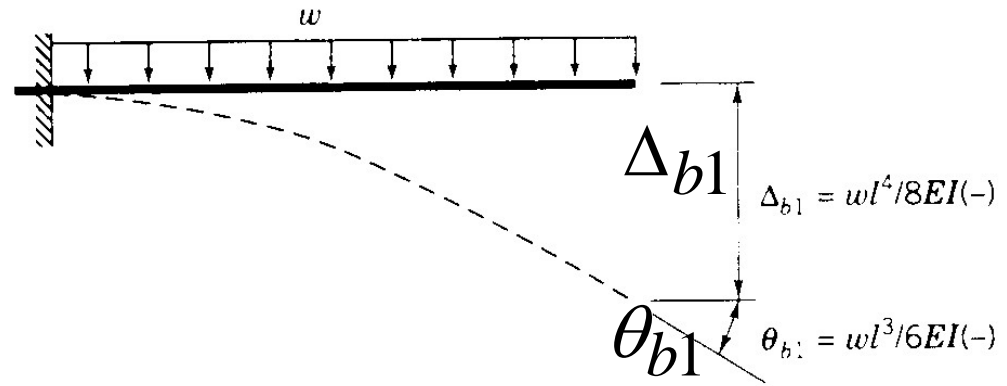$$\frac{R_{by}l^2}{2EI} + \frac{M_{bz}l}{EI} = \frac{wl^3}{6EI} + \theta_b$$

(9.7)

$\Delta_{b1} = wl^4/8EI(-)$

$\theta_{b1} = wl^3/6EI(-)$

$\theta_{b2} = R_{by}l^2/2EI$

$\Delta_{b2} = R_{by}l^3/3EI$

R_{by}

$\theta_{b3} = M_{bz}l/EI$

$\Delta_{b3} = M_{bz}l^2/2EI$

M_{bz}

Fig. 9.6 (b)

●Solving Eqs.(9.7) for M_{bz} and $R_{by,}$ and then applying statics to structure of Fig. 9.6(a), we obtain
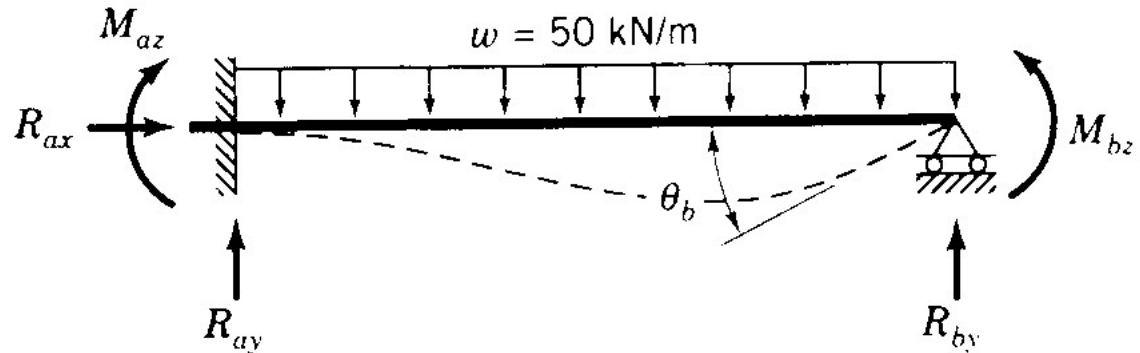

Fig. 9.6(a) Loading & deformation of a statically indeterminate beam

$$R_{ay} = \frac{wl}{2} + \frac{3k_b}{2l}\theta_b \qquad\qquad R_{by} = \frac{wl}{2} - \frac{3k_b}{2l}\theta_b$$

$$M_{az} = -\frac{wl^2}{12} - \frac{k_b}{2}\theta_b \qquad\qquad M_{az} = -\frac{wl^2}{12} + k_b\theta_b$$
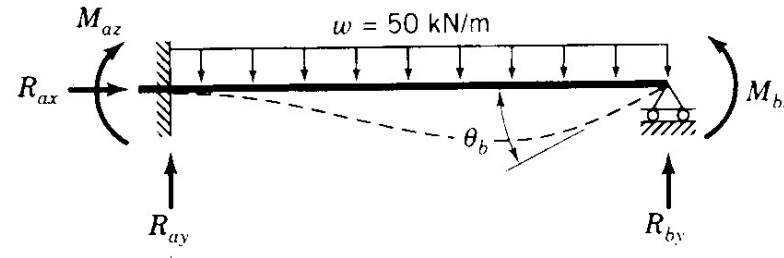
(9.8)

where, $$k_b = \frac{4EI}{l}$$

●Eq. (9.8) expresses all of the response quantities in terms of the single kinematic degree of freedom θ_b

Static Equilibrium

$$R_{ay} + R_{by} = wl$$

$$M_{az} + R_{ay}l - wl\frac{l}{2} - M_{bz} = 0$$

Using the above two requirements as well as Eq. (9.9), one obtains

$$
\begin{bmatrix}
1 & 0 & 1 & 0 \\
l & 1 & 0 & -1 \\
0 & 0 & l^3/3EI & l^2/2EI \\
0 & 0 & l^2/2EI & l/EI
\end{bmatrix}
\begin{Bmatrix}
R_{ay} \\
M_{az} \\
R_{by} \\
M_{bz}
\end{Bmatrix}
=
\begin{Bmatrix}
wl \\
wl^2/2 \\
wl^4/8EI \\
wl^3/6EI + \theta_b
\end{Bmatrix}
$$

●Assuming

$$EI\theta_b = 400kNm^2,$$

R_{ay}=+274kN,
M_{az}=-496.7kNm,
R_{by}=+226kN, and
M_{bz}=-256.7kNm.

$R_{ay} = +275kN$

$R_{by} = +226kN$

$EI\theta_b = 400kNm^2$
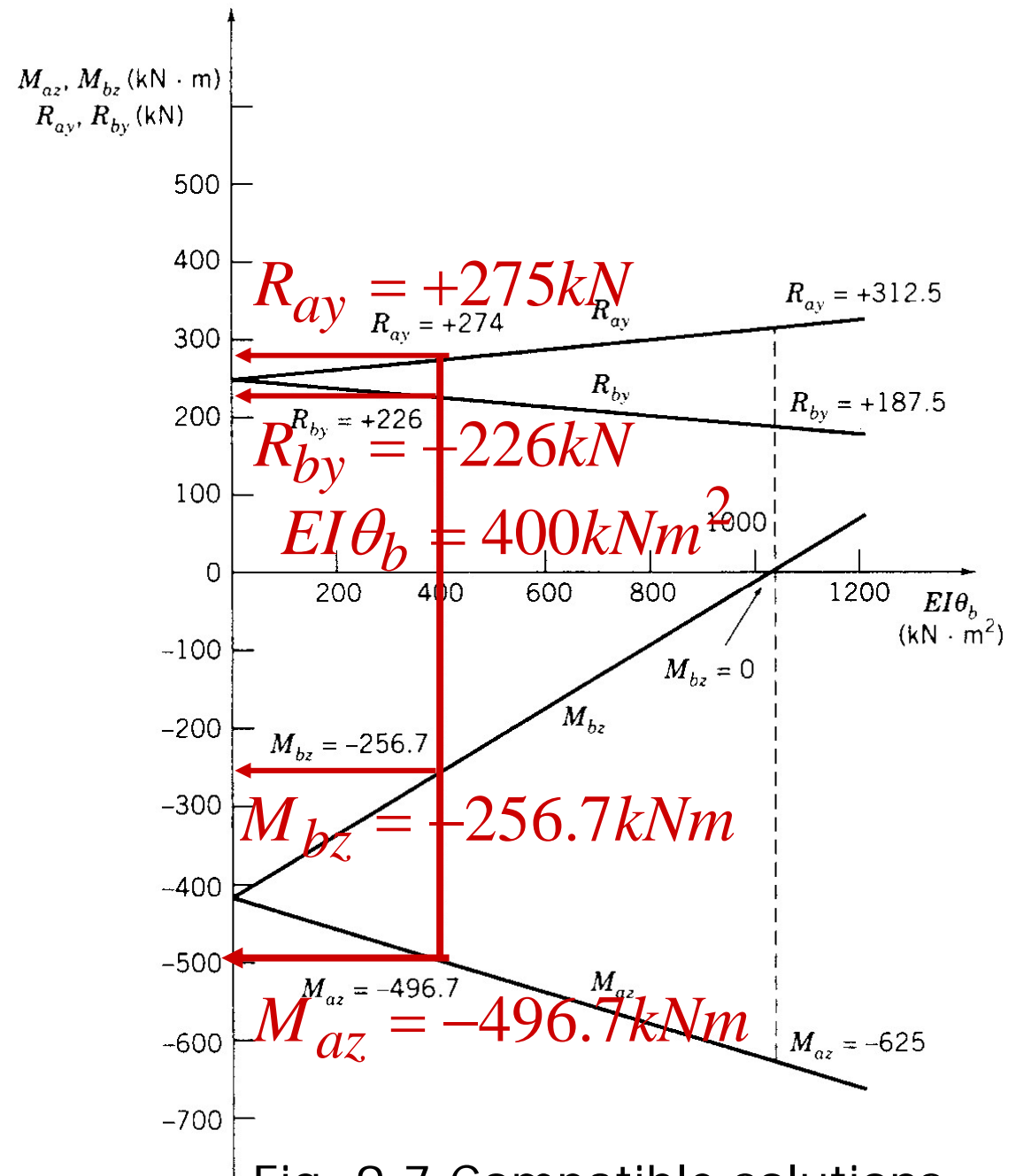
$M_{bz} = -256.7kNm$

$M_{az} = -496.7kNm$

Fig. 9.7 Compatible solutions
for structure of Fig. 9.6 (a)

●Eq. (9.8) can provide solutions of the kinemetrically determinate primary structure, by substituting $\theta_b = 0$

$$R_{ay} = \frac{wl}{2} + \frac{3k_b}{2l}\theta_b \qquad\qquad R_{by} = \frac{wl}{2} - \frac{3k_b}{2l}\theta_b$$

(9.8)

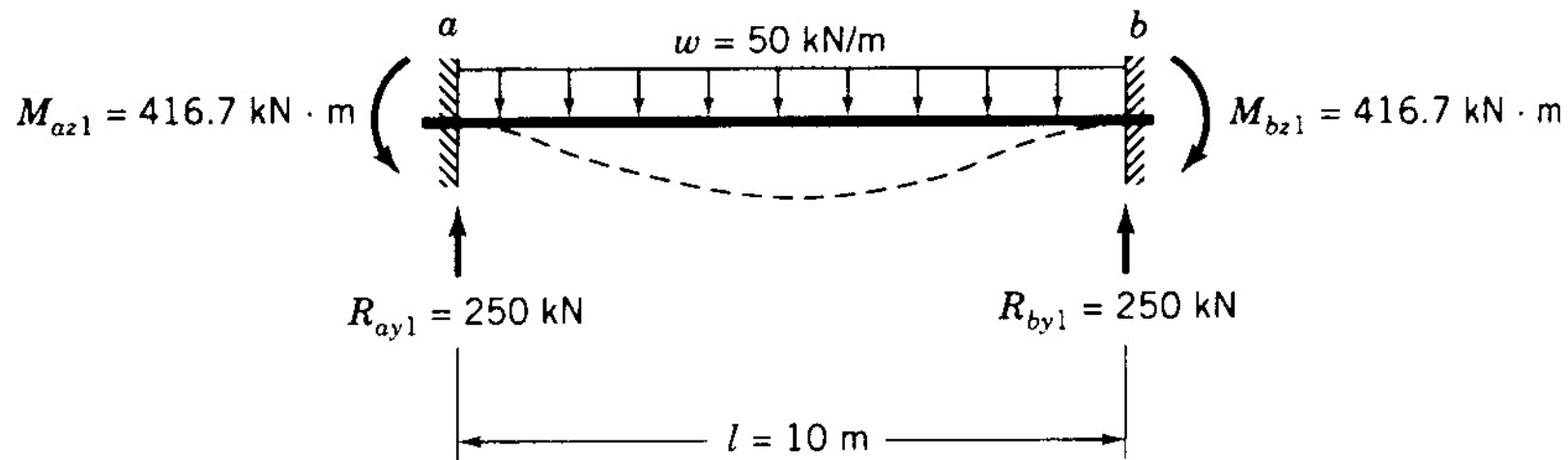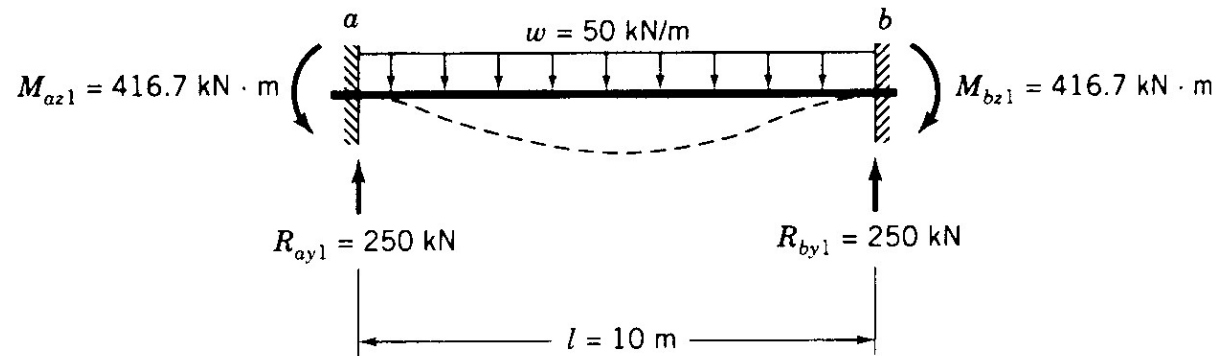$$M_{az} = -\frac{wl^2}{12} - \frac{k_b}{2}\theta_b \qquad\qquad M_{bz} = -\frac{wl^2}{12} + k_b\theta_b$$

Fig. 9.8 (a) Primary structure subjected to given loading

●Although this solution is a valid equilibrium solution, it violates the required force boundary condition on the moment at point b.

(a) Primary structure subjected to given load

(b) Primary structure subjected to redundant rotation

●To remedy this problem, the primary structure is subjected to the redundant displacement θ_b
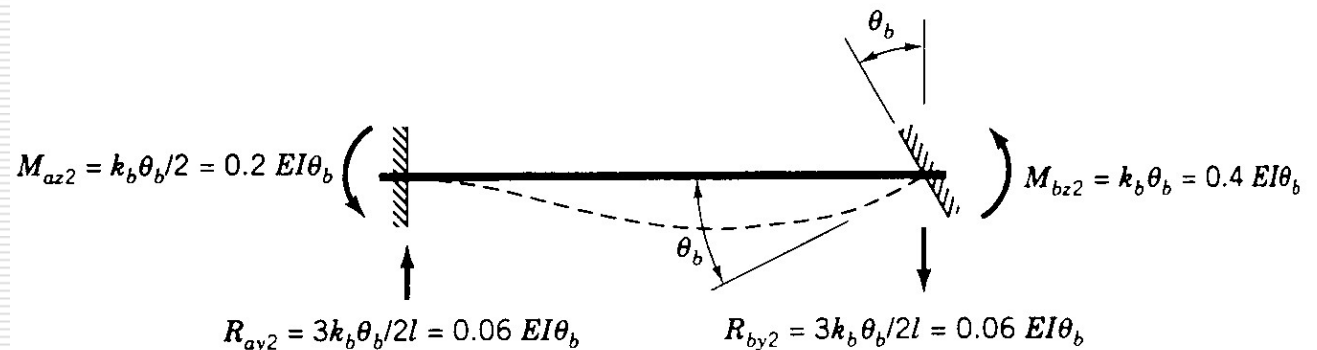
Fig. 9.8 Kinematic primary structure

(b) Primary structure subjected to redundant rotation

●Assuming that w=0 in Eq. (9.8), and based on the associated end moments and reactions take on the values as

$$\frac{wl}{2} - \frac{3k_b}{2l}\theta_b$$

$$:-\frac{wl^2}{12} + k_b\theta_b$$

(9.8)
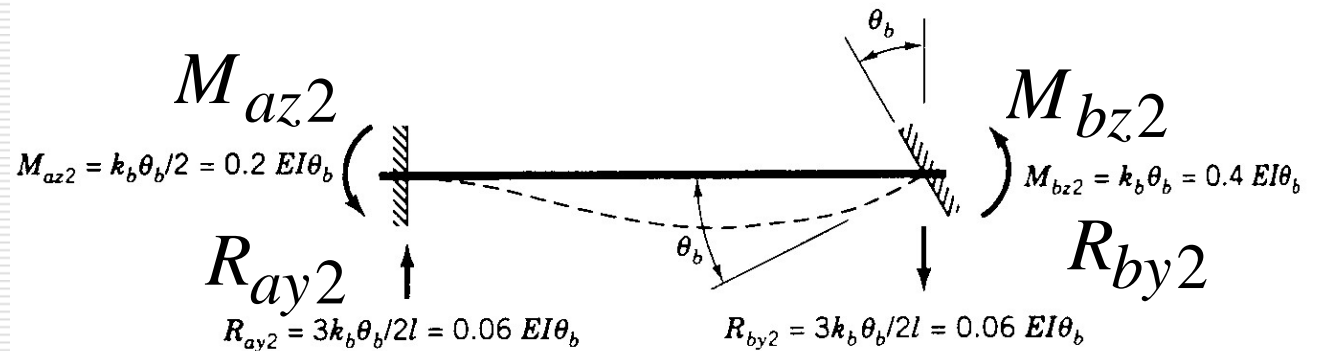
$$R_{ay2} = \frac{3k_b}{2l}\theta_b = 0.06EI\theta_b$$
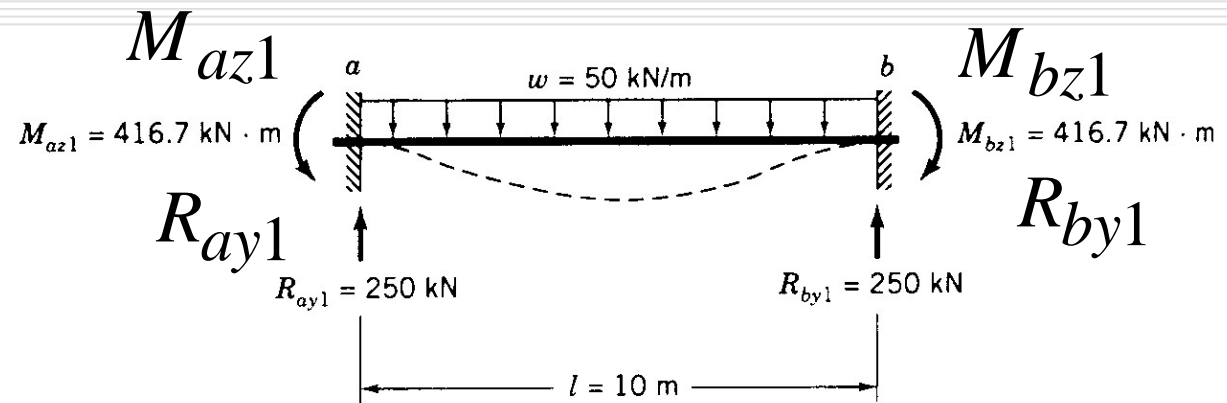
$$R_{by2} = \frac{3k_b}{2l}\theta_b = 0.06EI\theta_b$$

$$M_{az2} = -\frac{k_b}{2}\theta_b = -0.2EI\theta_b$$
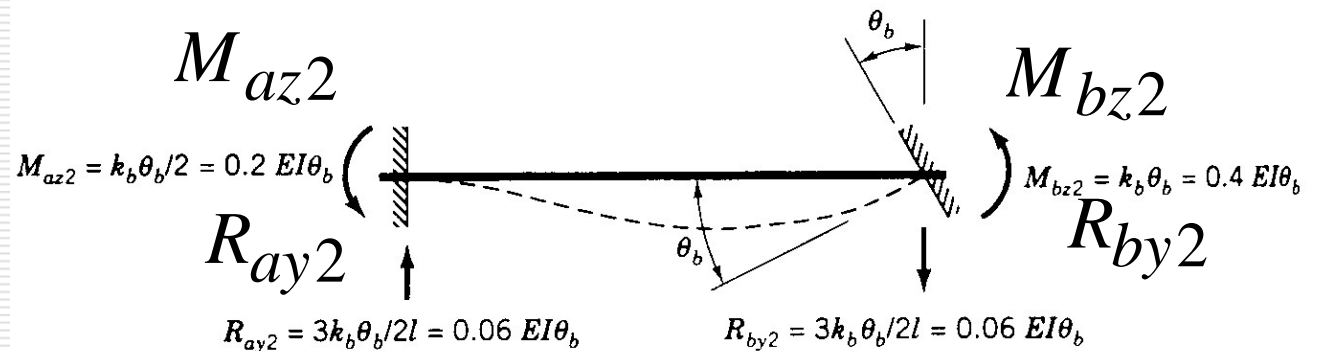
$$M_{bz2} = k_b\theta_b = 0.4EI\theta_b$$

●For a solution that reflects the proper loading and satisfies the designated force boundary conditions at point b, the solution shown in Figs. 9.8(a) and 9.8(b) must be superimposed so that the final moment at point b is zero.

●Thus,

$$M_{bz1} + M_{bz2} = 0$$

M_{az1}

M_{bz1}

R_{ay1}

R_{by1}

$M_{az1} = 416.7 \text{ kN} \cdot \text{m}$

$w = 50 \text{ kN/m}$

$M_{bz1} = 416.7 \text{ kN} \cdot \text{m}$

$R_{ay1} = 250 \text{ kN}$

$R_{by1} = 250 \text{ kN}$

$l = 10 \text{ m}$

(a) Primary structure subjected to given load

M_{az2}

M_{bz2}

R_{ay2}

R_{by2}

θ_b

$M_{az2} = k_b \theta_b / 2 = 0.2 \, EI\theta_b$

$M_{bz2} = k_b \theta_b = 0.4 \, EI\theta_b$

θ_b

$R_{ay2} = 3k_b \theta_b / 2l = 0.06 \, EI\theta_b$

$R_{by2} = 3k_b \theta_b / 2l = 0.06 \, EI\theta_b$

(b) Primary structure subjected to redundant rotation

Fig. 9.8 Kinematic primary structure

- Upon substitution,

$$-416.7 + 0.4EI\theta_b = 0$$

- from which

$$EI\theta_b = 1047.7 kNm^2$$

- Retuning to Fig. 9.8,

$$R_{ay} = +312.5kN$$

$$R_{by} = 187.5kN$$

$$M_{az} = -625kNm$$

$$M_{bz} = 0$$

$$R_{ay} = +312.5kN$$
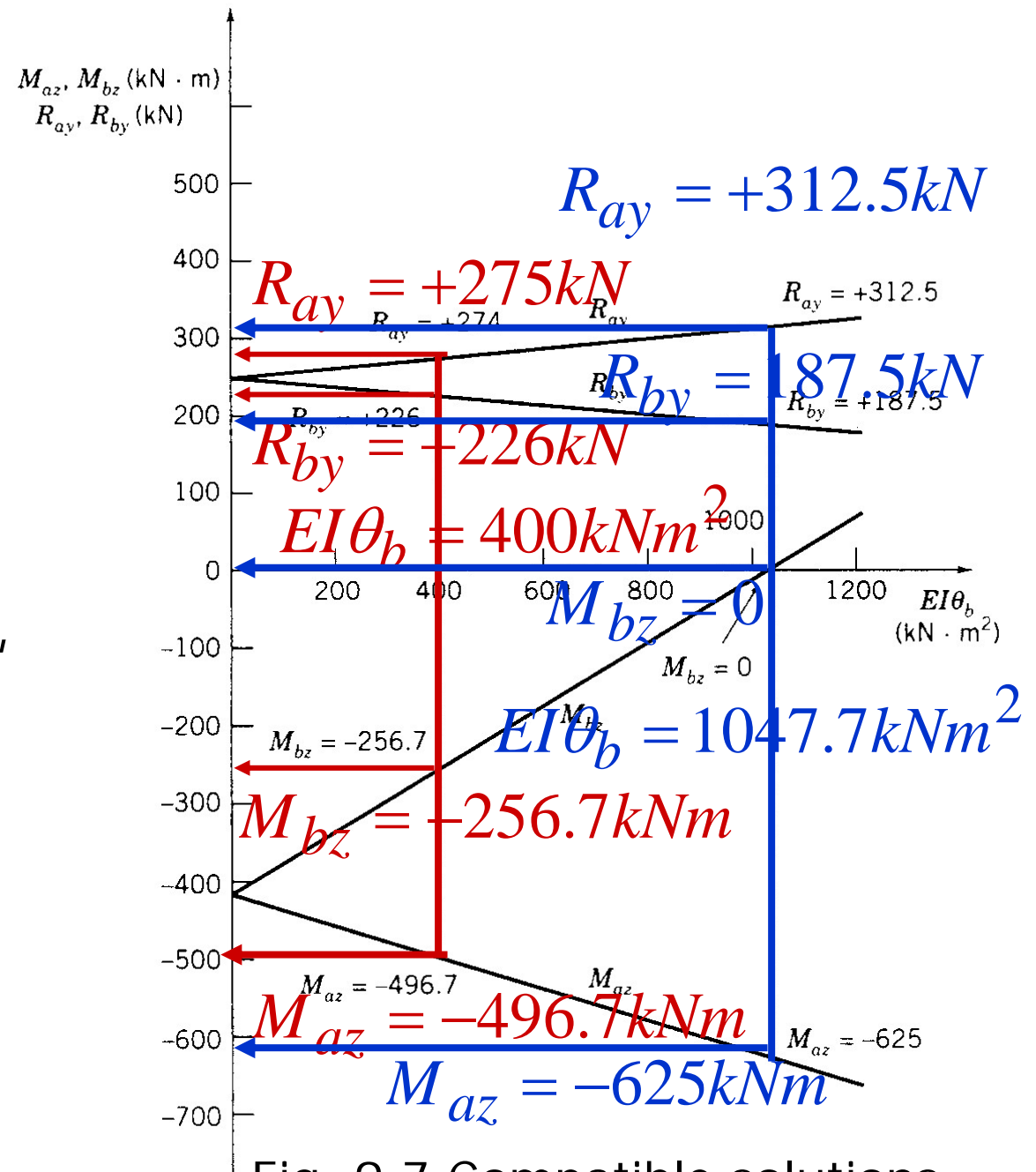$$R_{ay} = +275kN$$
$$R_{by} = 187.5kN$$
$$R_{by} = +226kN$$
$$EI\theta_b = 400kNm^2$$
$$M_{bz} = 0$$
$$EI\theta_b = 1047.7kNm^2$$
$$M_{bz} = -256.7kNm$$
$$M_{az} = -496.7kNm$$
$$M_{az} = -625kNm$$

Fig. 9.7 Compatible solutions for structure of Fig. 9.6 (a)

9.6 STATIC VERSUS KINEMATIC INDETERMINACY

1) Which is better between static and kinematic determinacy?

●The above example problem had only one static redundancy, R_{by}, and only one kinematic redundancy, θ_b. So whether the static indeterminacy or kinematic determinacy is better does not matter.

●However, as the structure becomes more complex, static vs. kinematic indeterminacy becomes important

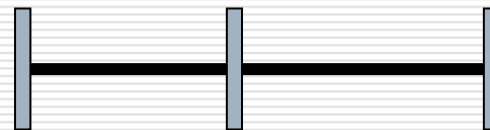## 2) Example of static and kinematic indeterminacy

| Structure 1 | Primary structure | Degree of Indeterminacy |
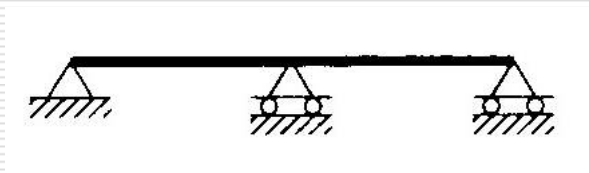
Static classification

1
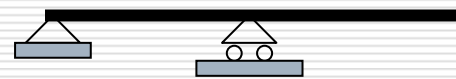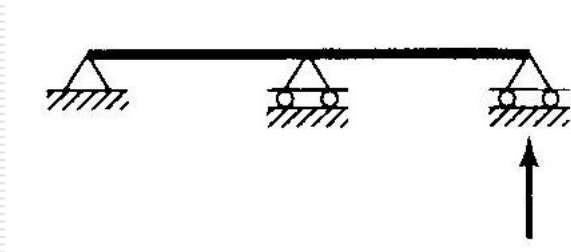
Kinematic classification
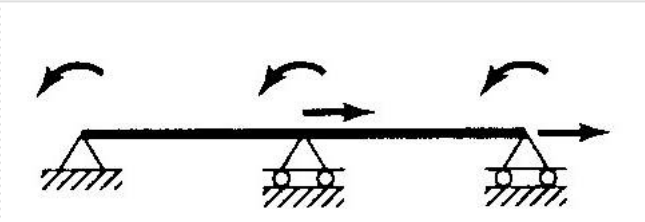
5

Structure 2
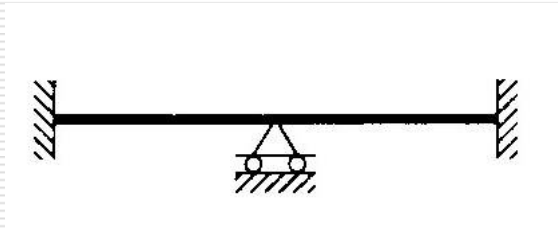
Primary structure

Degree of Indeterminacy

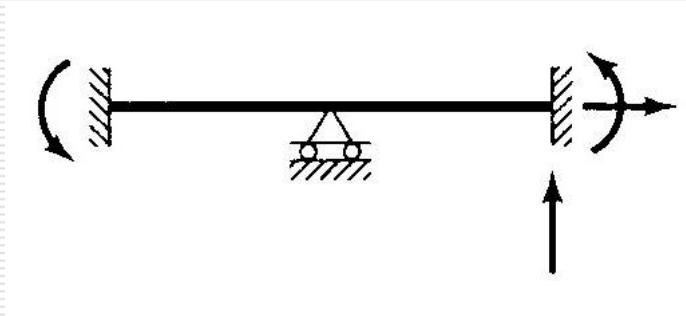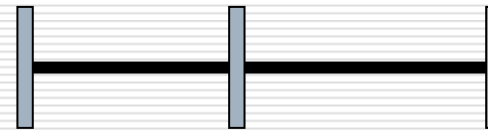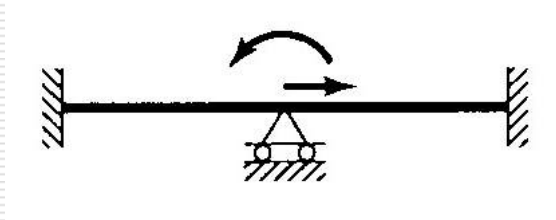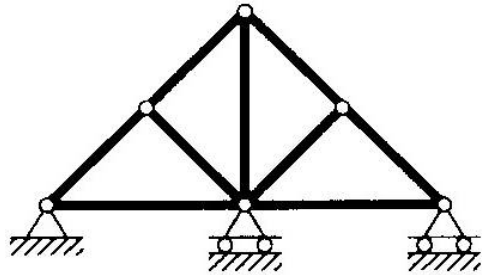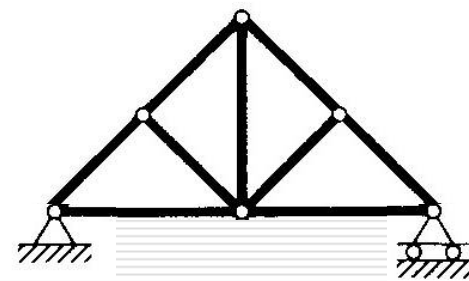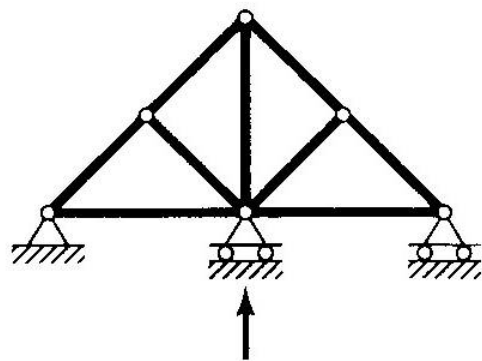## Static classification
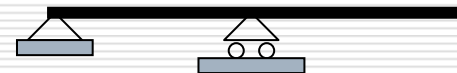
4

Kinematic classification
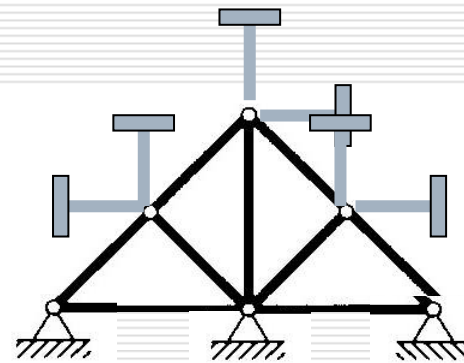

2

Structure 3

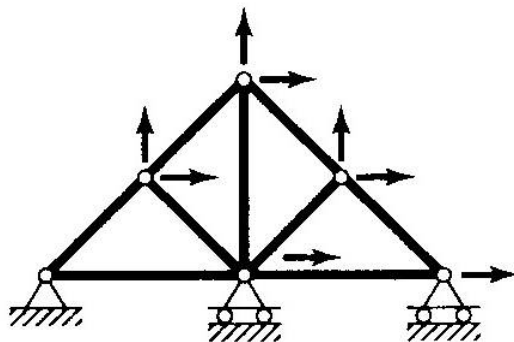## Primary structure
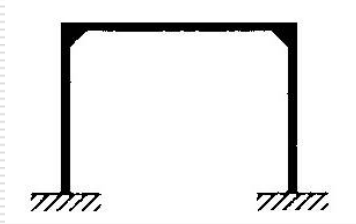
Degree of Indeterminacy
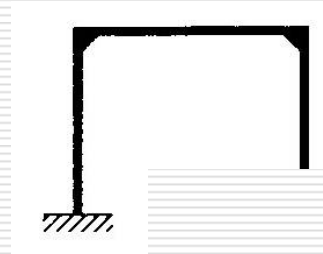
Static classification
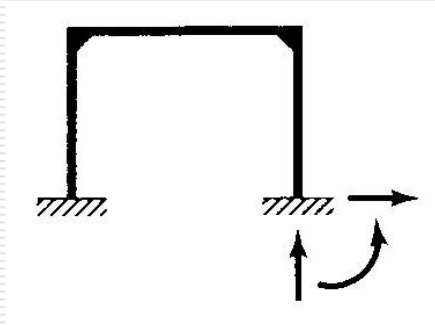

1

Kinematic classification

8

Structure 4

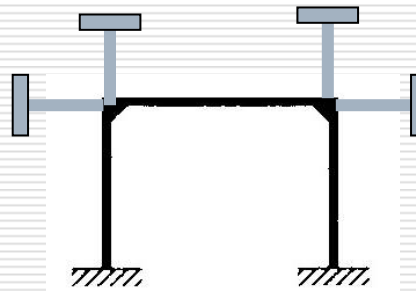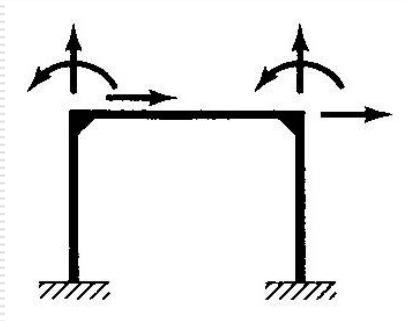
	Primary structure	Degree of Indeterminacy

Static classification

3

Kinematic classification

6

3) Important observation for static versus kinematic indeterminacy

- In establishing the statically determinate primary structure, there are a number of alternative selection patters for redundant force quantities.

- However, in developing the kinematically determinate primary structure, no selection process is necessary. Instead, all of the displacement quantities that are necessary to describe the structure's response automatically become redundant quantities.

9.7 COMPATIBILITY METHOD OF ANALYSIS

1) What is the compatibility method?

● The compatibility method is based on the solution of a set of equations that express compatibility relationship throughout the structure.

● The compatibility method is also referred as the force method   or the flexibility method (   ), since the unknowns in the governing equations are forces and the coefficients of those unknowns are flexibility (displacement quantities)(

.

2) Flexibility and Stiffness

$$F=ku \qquad (A1)$$

where,

F: force (   )
u: displacement (   )
K: stiffness (   )

Equilibrium method, or displacement method, or stiffness method

$$u=fF \qquad (A2)$$

where,

F: force (   )
u: displacement (   )
f: flexibility (   )

Compatibility method, or Force method, or Flexibility method

) Example of the compatibility method

●Thus, we have the compatibility condition

Condition of compatibility

$$\Delta_b = \Delta_{b1} + \Delta_{b2} \qquad (9.3)$$

2 500 kN · m (−)

500 kN

$\Delta_{b1} = \dfrac{62\ 500\ kN \cdot m^3\ (−)}{EI}$

10 m

(a) Primary structure subjected to given loading

●Upon substitution and rearrangement,

$$\frac{333R_{by}}{EI} = \frac{62500}{EI} \qquad (9.4)$$

$\Delta_{b2} = \dfrac{333R_{by}\ m^3}{EI}$

$10R_{by}$

R_{by}

●From which
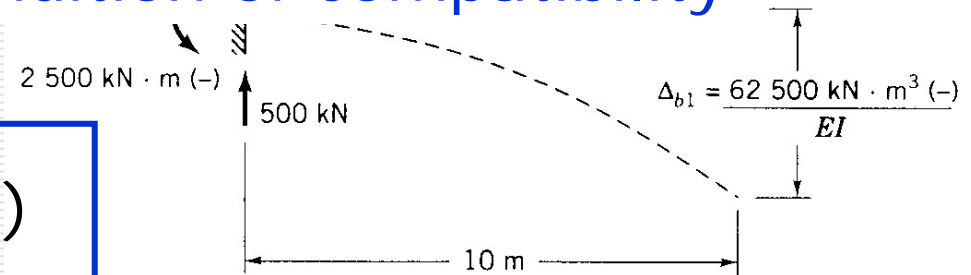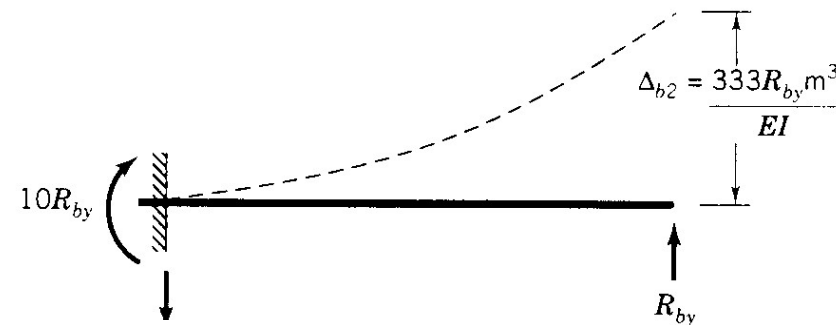
The coefficient of R_by is a flexibility quantity

...ucture subjected to redundant reaction

$$R_{by} = 187.5\ \text{...}$$

Fig. 9.4 Primary structure of Fig. 9.3

4) Are requirements of equilibrium approved in the compatibility method?

● The final solution must satisfy both the conditions of compatibility and the requirements of equilibrium.

● When the method is properly formulated, the compatibility equations represent a superposition of a set of partial solutions, each of which satisfies the requirements of equilibrium.

9.8 EQUILIBRIUM METHOD OF ANALYSIS（ ）

1) What is the equilibrium method?

● The equilibrium method is based on the solution of a set of equations that express the equilibrium requirements for the structure.

● The equilibrium method is also called the displacement method (   ) or the stiffness method (   ).

2) Example of the equilibrium method (Refer to Eq. (9.9))

$$M_{bz1} + M_{bz2} = 0$$

Equilibrium requirement at point b

● Upon substitution,

$$-416.7 + 0.4EI\theta_b = 0$$
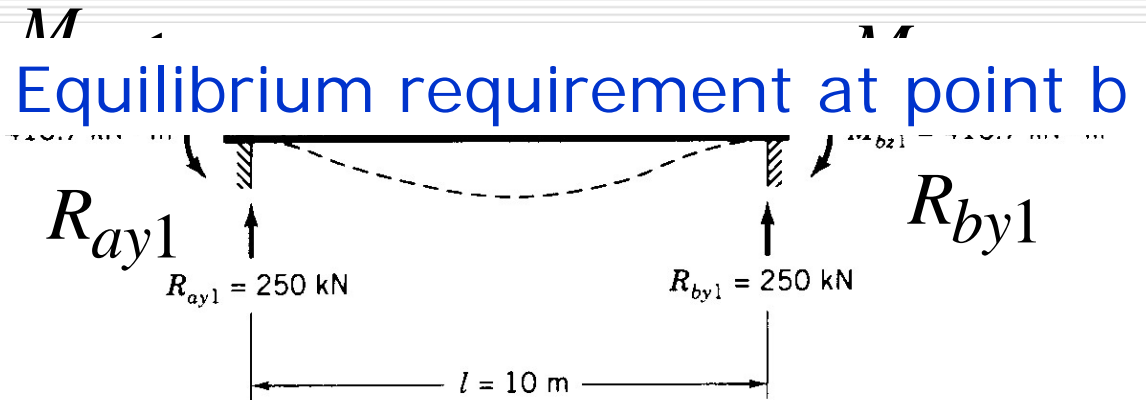
Coefficient of unknown (displacement) is stiffness (force) quantity

● from which

$$EI\theta_b = 1047.7 kNm^2$$

● Retuning to Fig. 9.8

$$R_{ay} = +312.5 kN$$

$$R_{by} = 187.5 kN$$

$$M_{az} = -625 kNm$$

$$M_{bz} = 0$$

R_{ay1}
R_{by1}

$R_{ay1} = 250$ kN
$R_{by1} = 250$ kN

$l = 10$ m

$M_{az2} = k_b\theta_b/2 = 0.2\ EI\theta_b$
$M_{bz2} = k_b\theta_b = 0.4\ E$

R_{ay2}
R_{by2}

θ_b

$R_{ay2} = 3k_b\theta_b/2l = 0.06\ EI\theta_b$
$R_{by2} = 3k_b\theta_b/2l = 0.06\ EI\theta_b$
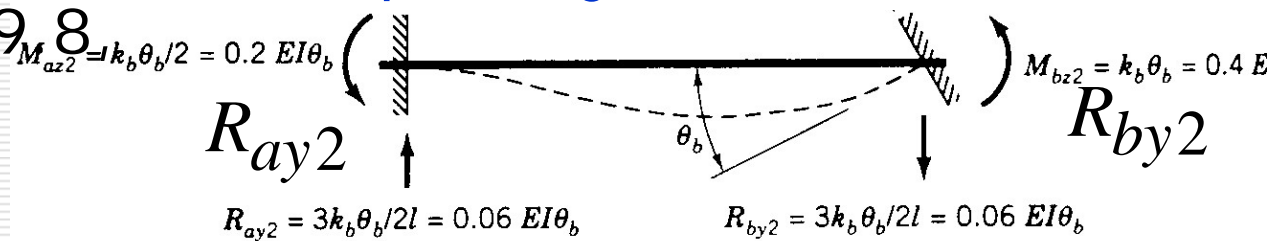
(b) Primary structure subjected to redundant rotation

Fig. 9.8 Kinematic primary structure

3) Advantage of equilibrium method

●For larger structures, there are many redundant displacement quantities. There is one redundant displacement quantity corresponding to each kinematic degree of freedom.

●Therefore, a number of equilibrium equations must be solved simultaneously for the displacement quantities.

●However unlike the compatibility method, there is no selectivity in choosing the redundants. All the kinematic degree of freedom are taken as redundant displacements.

●This is a major advantage for the automation that is needed in computer programming.

9.9 BEHAVIORAL CHARACTERISTICS OF STATICALLY INDETERMINATE STRUCTURES

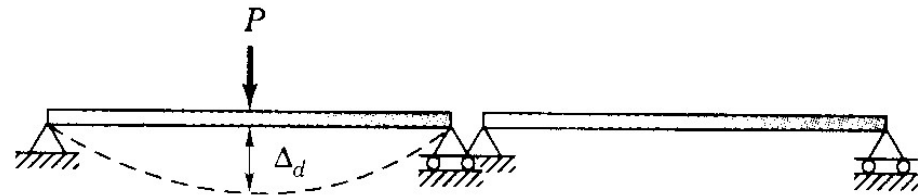## 1) Why are statically indeterminate structures used?

(1) A statically indeterminate structure displays greater stiffness in resisting load than does a comparable statically determinate structure.
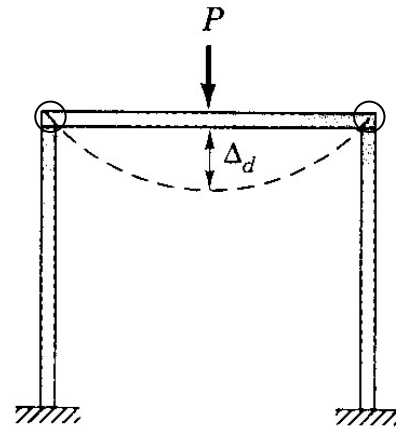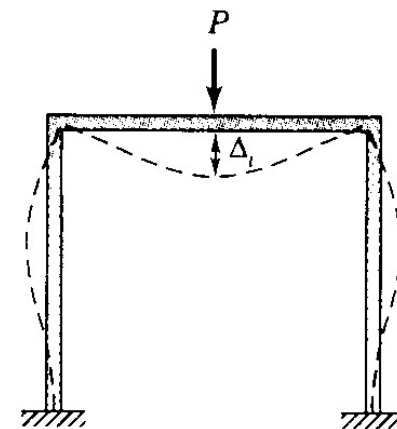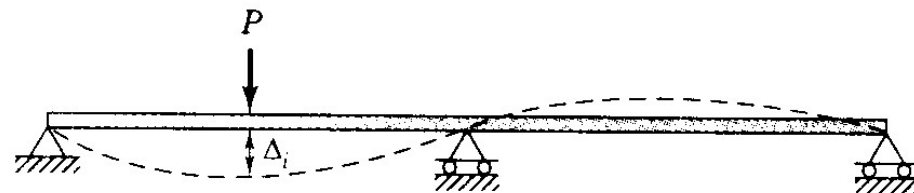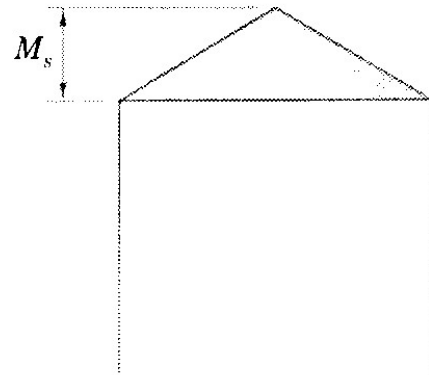
(a) Post and lintel   (b) Portal frame

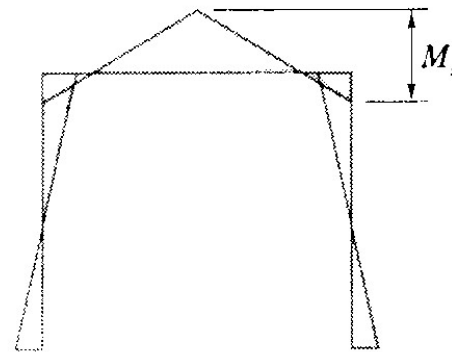(c) Two simply supported beams

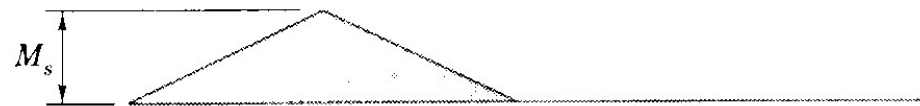(d) Two-span continuous beam

Fig. 9.9

(2) A statically indeterminate structure has lower stress intensities than would a comparative statically determinate structure.
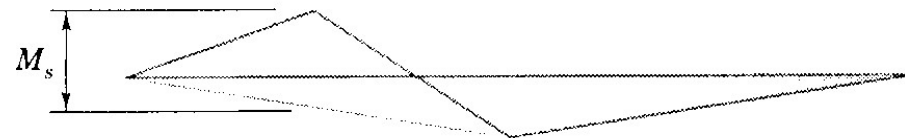

(a) Post and lintel

(b) Portal frame

(c) Two simply supported beams

(d) Two-span continuous beam

Fig. 9.10 Comparative moment diagrams

(3) A statically indeterminate structure is safer than would a comparative statically determinate structure.

●There are either internal force or external reactions that are not needed for stability in a statically indeterminate structure.

●No matter how these force components are removed, the structure will not become unstable.
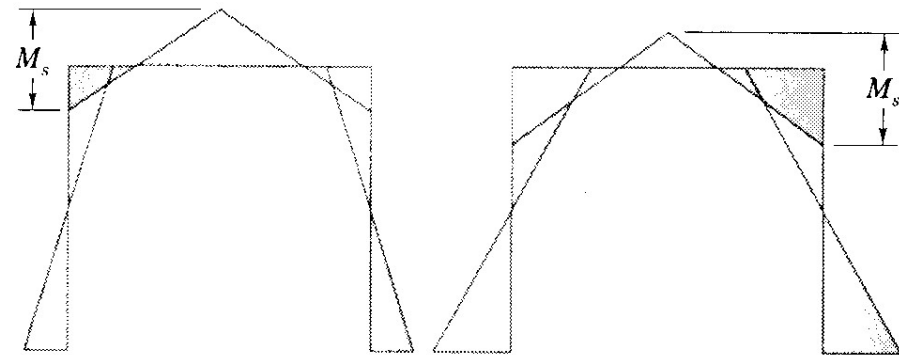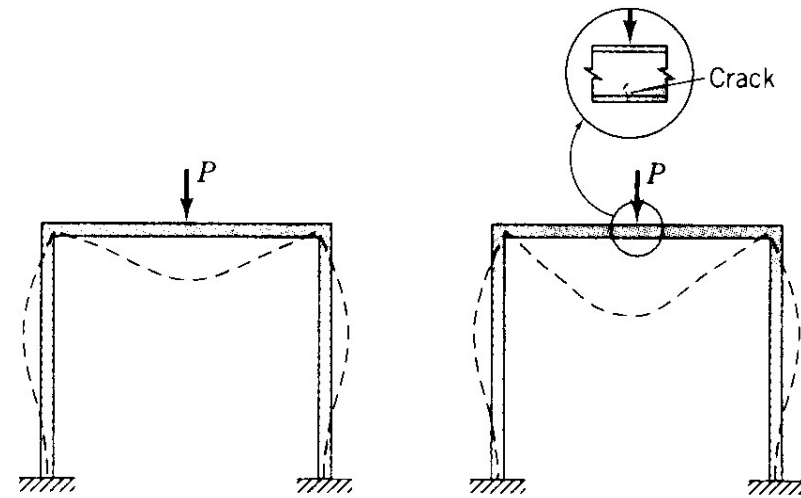
(a) Statically indeterminate frame

(b) Partially collapsed frame

Fig. 9.11 Redistribution of moment

## 3) What is disadvantage of a statically indeterminate structure?

●Continuity which is the advantage of a statically indeterminate structure can be the primary disadvantage.

●If point b of a three-span continuous beam settled as shown in Fig. 9.12 (b), bending moment as shown in Fig. 9.12 (c) would be developed. Thus, without any loading on the structure, sizable moments would be introduced at the interior support points.

●In situations where such settlements can occur, a statically determinate structure should be employed as shown in Fig. 9.13.
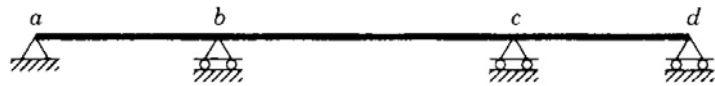
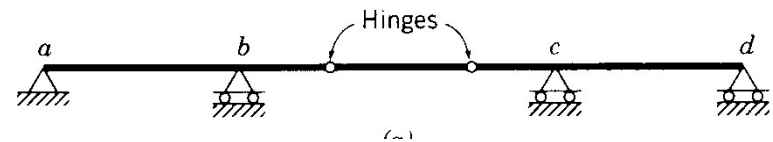(a) Three-span continuous beam
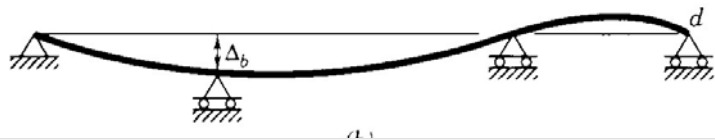
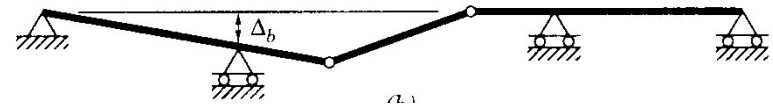(b) Deformations caused by relative settlement
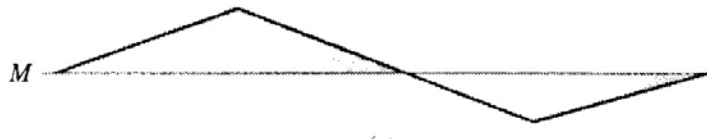
(c) Moment induced by settlement

Fig. 9.12

(a) Cantilever structure

(b) Relative settlement

Fig. 9.13