## Choice Under Uncertainty

Chapters $1 \& 3$ : Preference relation $\geq$ on X (a set of alternatives) Utility function $\mathrm{u}: \mathrm{X} \rightarrow \mathfrak{R}$ s.t. $\mathrm{u}(\mathrm{x}) \geq \mathrm{u}(\mathrm{y}) \leftrightarrow \mathrm{x} \geq \mathrm{y}$

## Alternatives $\rightarrow$ Outcomes

on which individuals have pref. relations
If one alternative $\rightarrow$ one outcome, then Chap. 1 is OK.
If there is uncertainty, then one alternative may produce two or more outcomes with certain probabilities.
If mixed strategies are used,
then two or more outcomes may be produced w/ certain prob; even if $\exists$ one-to-one correspondence between alternatives and outcomes
$\rightarrow$ von Neumann-Morgenstern expected utility theory

## Outcomes and Lotteries

C : set of outcomes
Assume C is finite; $\mathrm{C}=\{1,2, \ldots, \mathrm{~N}\}$
Lottery (or simple lottery) on C : L = ( $\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}$ ),

$$
\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}} \geq 0, \quad \mathrm{p}_{1}+\ldots+\mathrm{p}_{\mathrm{N}}=1
$$

Compound lottery: ( $\mathrm{L}_{1}, \ldots, \mathrm{~L}_{\mathrm{K}} ; \alpha_{1}, \ldots, \alpha_{\mathrm{K}}$ ) denote: $\alpha_{1} \mathrm{~L}_{1}+\ldots+\alpha_{\mathrm{K}} \mathrm{L}_{\mathrm{K}}$

$$
\mathrm{L}_{\mathrm{k}}=\left(\mathrm{p}_{1}^{\mathrm{k}}, \ldots, \mathrm{p}_{\mathrm{N}}^{\mathrm{k}}\right) \quad \mathrm{k}=1, \ldots, \mathrm{~K}
$$

Reduced lottery: $\mathrm{L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right)$

$$
\mathrm{p}_{\mathrm{n}}=\alpha_{1} \mathrm{P}_{\mathrm{n}}^{1}+\ldots+\alpha_{\mathrm{K}} \mathrm{P}_{\mathrm{n}}^{\mathrm{K}} \quad \mathrm{n}=1, \ldots, \mathrm{~N}
$$

Assume: Each compound lottery is equivalent to its reduced lottery. $\rightarrow$ Figure 6.B. 3

## Preferences on Lotteries

$L$ : set of all simple lotteries over the set of outcomes $\mathrm{C}=\{1, \ldots, \mathrm{~N}\}$
$\geq$ : preference relation on $L$, (rational, i.e., complete, transitive) definition of $\rangle$ and $\sim$
$\geq$ is complete $\Leftrightarrow \forall \mathrm{L}, \mathrm{L}, \in L \quad \mathrm{~L} \geq \mathrm{L}^{\prime}$ or $\mathrm{L}^{\prime} \geq \mathrm{L}$
$\geq$ is transitive $\Leftrightarrow \forall \mathrm{L}, \mathrm{L}, \mathrm{L}$ " $\in L$

$$
\mathrm{L} \geq \mathrm{L}^{\prime} \text { and } \mathrm{L}^{\prime} \geq \mathrm{L} " \rightarrow \mathrm{~L} \geq \mathrm{L} \text { " }
$$

$\mathrm{L}\rangle \mathrm{L}^{\prime} \Leftrightarrow \mathrm{L} \geq \mathrm{L}^{\prime}$ and $\operatorname{not}\left(\mathrm{L}^{\prime} \geq \mathrm{L}\right)$
$\mathrm{L} \sim \mathrm{L}^{\prime} \Leftrightarrow \mathrm{L} \geq \mathrm{L}^{\prime}$ and $\mathrm{L}^{\prime} \geq \mathrm{L}$

## Preferences on Lotteries

$L$ : set of all simple lotteries over the set of outcomes $\mathrm{C}=\{1, \ldots, \mathrm{~N}\}$
$\geq$ : preference relation on $L$, (rational, i.e., complete, transitive) definition of $>$ and $\sim$
$\geq$ is continuous if $\forall \mathrm{L}, \mathrm{L}^{\prime}, \mathrm{L} " \in L$,

$$
\begin{aligned}
& \left\{\alpha \in[0,1] \mid \alpha \mathrm{L}+(1-\alpha) \mathrm{L}^{\prime} \geq \mathrm{L}^{\prime}\right\} \subseteq[0,1] \text { is closed and } \\
& \left\{\alpha \in[0,1] \mid \mathrm{L}^{\prime} \geq \alpha \mathrm{L}+(1-\alpha) \mathrm{L}^{\prime}\right\} \subseteq[0,1] \text { is closed. }
\end{aligned}
$$

$\geq$ satisfies independence axiom (simply called independence)

$$
\begin{aligned}
& \text { if } \forall L^{\prime}, L^{\prime}, L^{\prime \prime} \in L \text { and } \alpha \in(0,1), \\
& L \geq L^{\prime} \leftrightarrow \alpha L^{\prime}+(1-\alpha) L^{\prime \prime} \geq \alpha L^{\prime}+(1-\alpha) L^{\prime \prime}
\end{aligned}
$$

## von Neumann - Morgenstern Expected Utility

$\mathrm{U}: L \rightarrow \mathfrak{R}$ is a $\mathrm{vN}-\mathrm{M}$ expected utility function if
(1) U is a utility function, i.e.,

$$
\mathrm{L} \geq \mathrm{L}^{\prime} \leftrightarrow \mathrm{U}(\mathrm{~L}) \geq \mathrm{U}\left(\mathrm{~L}^{\prime}\right) \quad \forall \mathrm{L}^{\prime}, \mathrm{L}^{\prime} \in L
$$

(2) $\exists$ numbers $\mathrm{u}_{1}, \ldots, \mathrm{u}_{\mathrm{N}}$ s.t.
for every simple lottery $L=\left(p_{1}, \ldots, p_{N}\right)$,

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}}
$$

Prop.6.B.3 (modified):
$\geq$ rational, continuous, independent, (compound lottery)
$\leftrightarrow \quad \exists$ vN-M expected utility fcn U, i.e., $\exists$ utility fcn $\mathrm{U}: \mathrm{L} \rightarrow \mathfrak{R}$

$$
\mathrm{L} \geq \mathrm{L}^{\prime} \quad \Leftrightarrow \quad \mathrm{U}(\mathrm{~L}) \geq \mathrm{U}^{\prime}\left(\mathrm{L}^{\prime}\right) \quad \forall \mathrm{L}, \mathrm{~L}^{\prime} \in L
$$

such that $\exists$ numbers $\mathrm{u}_{1}, \ldots, \mathrm{u}_{\mathrm{N}}$ s.t.

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}} \quad \forall \mathrm{~L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right) \in L
$$

## Proof of Prop. 6.B. 3

Prop.6.B.3 (modified):
$\geq$ rational, continuous, independent, (compound lottery)
$\leftrightarrow \exists$ utility fcn U:L $\rightarrow \mathfrak{R}$

$$
\text { i.e., } \mathrm{L} \geq \mathrm{L}^{\prime} \quad \Leftrightarrow \quad \mathrm{U}(\mathrm{~L}) \geq \mathrm{U}^{\prime}\left(\mathrm{L}^{\prime}\right) \quad \forall \mathrm{L}, \mathrm{~L}^{\prime} \in L
$$

such that $\exists$ numbers $\mathrm{u}_{1}, \ldots, \mathrm{u}_{\mathrm{N}}$ satisfying

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}} \quad \forall \mathrm{~L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right) \in L
$$

Proof: $\leftarrow)$ Clear since $\mathrm{U}(\cdot)$ gives real numbers and U is conts. (Ex)
$\rightarrow$ Step 0: $\exists \mathrm{L}^{+}, \mathrm{L}^{-} \in L$ s.t. $\mathrm{L}^{+} \geq \mathrm{L} \geq \mathrm{L}^{-}$for all $\mathrm{L} \in L$. ( $\rightarrow$ Ex. 6.B.3)
If $\mathrm{L}^{+} \sim \mathrm{L}^{-}$, then clear. Suppose $\left.\mathrm{L}^{+}\right\rangle \mathrm{L}^{-}$.
Show: For any $\mathrm{L} \in L, \exists$ unique $\alpha_{\mathrm{L}}$ s.t. $\mathrm{L} \sim \alpha_{\mathrm{L}} \mathrm{L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-}$

## Proof of Prop. 6.B. 3

Show: For any $\mathrm{L} \in L$,

$$
\exists \text { unique } \alpha_{\mathrm{L}} \text { s.t. } \mathrm{L} \sim \alpha_{\mathrm{L}} \mathrm{~L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-} \quad \text { (Step 3) }
$$

Existence: Let $A=\left\{\alpha_{L} \in[0,1] \mid \alpha_{L} L^{+}+\left(1-\alpha_{L}\right) L^{-} \geq \mathrm{L}\right\}$ and $B=\left\{\alpha_{L} \in[0,1] \mid L \geq \alpha_{L} L^{+}+\left(1-\alpha_{L}\right) L^{-}\right\}$

By continuity of $\geq$, A and B are closed.
By completeness of $\geq, A \cup B=[0,1]$
$A \cup B \subseteq[0,1]:$ clear $[0,1] \subseteq A \cup B:$ completeness of $\geq$

A, B are closed and bounded, $\exists$ min A and max B . Let $\mathrm{a}=\min \mathrm{A}$ and $\mathrm{b}=\max \mathrm{B}$.

## Proof of Prop. 6.B. 3

$\mathrm{A}, \mathrm{B}$ are closed and bounded, $\exists$ min A and max B .
Let $\mathrm{a}=\min \mathrm{A}$ and $\mathrm{b}=\max \mathrm{B}$.
Show: $\mathrm{a}=\mathrm{b}$
Suppose $\mathrm{a}>\mathrm{b}$. Then since a , b are reals, $\exists \mathrm{c}$ s.t. $\mathrm{a}>\mathrm{c}>\mathrm{b}$.
Thus $c \in[0,1]$ and $c \notin A \cup B$, contradicting $A \cup B=[0,1]$.
Suppose $\mathrm{a}<\mathrm{b}$. Then since $\mathrm{a}, \mathrm{b}$ are reals, $\exists \mathrm{c}$ s.t. $\mathrm{a}<\mathrm{c}<\mathrm{b}$
To show a contradiction, we need the following fact.

$$
\begin{aligned}
& \alpha, \beta \in[0,1] . \\
& \left.\quad \beta \mathrm{L}^{+}+(1-\beta) L^{-}\right\rangle \alpha L^{+}+(1-\alpha) L^{-} \leftrightarrow \beta>\alpha \quad \text { (Step 2) }
\end{aligned}
$$

## Proof of Prop. 6.B. 3

$\alpha, \beta \in[0,1]$.

$$
\left.\beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-} \leftrightarrow \beta>\alpha \quad \text { (Step 2) }
$$

Proof: $\leftarrow)$ Suppose $\beta>\alpha$; thus $1 \geq \beta>\alpha \geq 0$.
Then $\beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-} \sim \gamma \mathrm{L}^{+}+(1-\gamma)\left(\alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-}\right)$
(compound lottery)
where $\gamma=(\beta-\alpha) /(1-\alpha)$ and $1-\gamma=(1-\beta) /(1-\alpha)$


## Proof of Prop. 6.B. 3

$$
\begin{aligned}
& \alpha, \beta \in[0,1] . \\
& \left.\quad \beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-} \leftarrow \beta>\alpha \quad \text { (Step 2) }
\end{aligned}
$$

$$
\beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-} \sim \gamma \mathrm{L}^{+}+(1-\gamma)\left(\alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-}\right)
$$

$$
\text { where } \gamma=(\beta-\alpha) /(1-\alpha) \text { and } 1-\gamma=(1-\beta) /(1-\alpha)
$$

$$
\left.\mathrm{L}^{+}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-}
$$

(follows from $\left.\mathrm{L}^{+}\right\rangle \mathrm{L}^{-}$and independence. Ex.6.B.1)
$\left.\gamma \mathrm{L}^{+}+(1-\gamma)\left(\alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-}\right)\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-}$
(follows from $\left.\mathrm{L}^{+}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-}$and independence. Ex.6.B.1)
$\left.\beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-}$

## Proof of Prop. 6.B. 3

$$
\begin{aligned}
& \alpha, \beta \in[0,1] . \\
& \left.\quad \beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-} \rightarrow \beta>\alpha \quad \text { (Step 2) }
\end{aligned}
$$

Suppose not, i.e., $\beta \leq \alpha$.

$$
\text { If } \beta=\alpha \text {, then } \beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-} \sim \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-} . \mathrm{C} \text { ! }
$$

If $\beta<\alpha$, then the same argument as above shows

$$
\left.\alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-}\right\rangle \beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-} . \mathrm{C}!
$$

## Proof of Prop. 6.B. 3

$\mathrm{A}, \mathrm{B}$ are closed and bounded, $\exists \mathrm{min} \mathrm{A}$ and max B .
Let $\mathrm{a}=\min \mathrm{A}$ and $\mathrm{b}=\max \mathrm{B}$.
Show: $\mathrm{a}=\mathrm{b}$
We showed $\mathrm{a}>\mathrm{b} \rightarrow \mathrm{C}$ !
Suppose $\mathrm{a}<\mathrm{b}$. Then since $\mathrm{a}, \mathrm{b}$ are reals, $\exists \mathrm{c}$ s.t. $\mathrm{a}<\mathrm{c}<\mathrm{b}$
To show a contradiction, we have shown the following fact.

$$
\begin{align*}
& \alpha, \beta \in[0,1] . \\
& \left.\quad \beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-} \leftrightarrow \beta>\alpha \quad \text { (Step 2) } \\
& \mathrm{a}<\mathrm{c}<\mathrm{b} \\
& \mathrm{a}<\mathrm{c} \rightarrow \mathrm{cL}^{+}+(1-\mathrm{c}) \mathrm{L}^{-}>\mathrm{aL}^{+}+(1-\mathrm{a}) \mathrm{L}^{-} \geq \mathrm{L} \\
& \mathrm{c}<\mathrm{b} \rightarrow \mathrm{~L} \geq \mathrm{bL}^{+}+(1-\mathrm{b}) \mathrm{L}^{-}>\mathrm{cL}^{+}+(1-\mathrm{c}) \mathrm{L}^{-} \quad \mathrm{C}!
\end{align*}
$$

Thus $\mathrm{a}=\mathrm{b}$

## Proof of Prop. 6.B. 3

$\mathrm{A}, \mathrm{B}$ are closed and bounded, $\exists \mathrm{min} \mathrm{A}$ and max B .
Let $\mathrm{a}=\min \mathrm{A}$ and $\mathrm{b}=\max \mathrm{B}$. Have shown $\mathrm{a}=\mathrm{b}$
Let $\mathrm{a}=\mathrm{b}=\alpha_{\mathrm{L}} \quad$ Note $\alpha_{\mathrm{L}} \in \mathrm{A} \cap \mathrm{B}$
By the definition of A and B ,

$$
\alpha_{\mathrm{L}} \mathrm{~L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-} \geq \mathrm{L} \quad \text { and } \quad \mathrm{L} \geq \alpha_{\mathrm{L}} \mathrm{~L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-}
$$

Thus we obtain $\alpha_{L} L^{+}+\left(1-\alpha_{L}\right) L^{-} \sim L$.

Uniqueness:
Clear from Step 2.

$$
\begin{aligned}
& \hline \alpha, \beta \in[0,1] . \\
& \left.\quad \beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-} \leftrightarrow \beta>\alpha \quad \text { (Step 2) }
\end{aligned}
$$

## Proof of Prop. 6.B. 3

Have shown
(1) For any $L \in L$,
$\exists$ unique $\alpha_{\mathrm{L}}$ s.t. $\mathrm{L} \sim \alpha_{\mathrm{L}} \mathrm{L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-}$
(Step 3)
(2) $\alpha, \beta \in[0,1]$.

$$
\left.\beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-} \leftrightarrow \beta>\alpha \quad \text { (Step 2) }
$$

Show: $\exists \mathrm{vN}$-M expected utility fca U, i.e.,
(3) $\exists$ function $\mathrm{U}: \mathrm{L} \rightarrow \mathfrak{R}$ s.t.

$$
\mathrm{L} \geq \mathrm{L}^{\prime} \leftrightarrow \mathrm{U}(\mathrm{~L}) \geq \mathrm{U}\left(\mathrm{~L}^{\prime}\right) \quad \forall \mathrm{L}, \mathrm{~L}^{\prime} \in L
$$

(4) $\exists$ numbers $\mathrm{u}_{1}, \ldots, \mathrm{u}_{\mathrm{N}}$ s.t.

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}} \quad \forall \mathrm{~L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right) \in L
$$

## Proof of Prop. 6.B. 3

Show: $\exists \mathrm{vN}$-M expected utility fcn U, i.e.,
(3) $\exists$ function $\mathrm{U}: \mathrm{L} \rightarrow \mathfrak{R}$ s.t.

$$
\mathrm{L} \geq \mathrm{L}^{\prime} \leftrightarrow \mathrm{U}(\mathrm{~L}) \geq \mathrm{U}^{\prime}\left(\mathrm{L}^{\prime}\right) \quad \forall \mathrm{L}, \mathrm{~L}^{\prime} \in L
$$

(4) $\exists$ numbers $u_{1}, \ldots, u_{N}$ s.t.

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}} \quad \forall \mathrm{~L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right) \in L
$$

Pf. of (3)
Let U by $\mathrm{U}(\mathrm{L})=\alpha_{\mathrm{L}} \forall \mathrm{L} \in L$. (Step 3) $\left(\mathrm{U}\left(\mathrm{L}^{+}\right)=1, \mathrm{U}\left(\mathrm{L}^{-}\right)=0\right)$ Then $\mathrm{U}(\mathrm{L}) \geq \mathrm{U}\left(\mathrm{L}^{\prime}\right) \leftrightarrow \mathrm{L} \geq \mathrm{L}^{\prime}$

$$
\text { since } \left.\beta \mathrm{L}^{+}+(1-\beta) \mathrm{L}^{-}\right\rangle \alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{-} \leftrightarrow \beta>\alpha \text { (Step 2) }
$$

## Proof of Prop. 6.B. 3

Show: $\exists \mathrm{vN}-\mathrm{M}$ expected utility fcn U, i.e.,
(3) $\exists$ function $\mathrm{U}: \mathrm{L} \rightarrow \mathfrak{R}$ s.t.

$$
\mathrm{L} \geq \mathrm{L}^{\prime} \leftrightarrow \mathrm{U}(\mathrm{~L}) \geq \mathrm{U}\left(\mathrm{~L}^{\prime}\right) \quad \forall \mathrm{L}, \mathrm{~L}^{\prime} \in L
$$

(4) $\exists$ numbers $u_{1}, \ldots, u_{N}$ s.t.

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}} \quad \forall \mathrm{~L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right) \in L
$$

Pf. of (4)
First show that U is linear, i.e.,
(5) $\left.\forall \mathrm{L}, \mathrm{L}^{\prime} \in \mathrm{L}, \forall \alpha \in[0,1], \mathrm{U}\left(\alpha \mathrm{L}^{+}+(1-\alpha) \mathrm{L}^{\prime}\right)\right)=\alpha \mathrm{U}(\mathrm{L}) .+(1-\alpha) \mathrm{U}\left(\mathrm{L}^{\prime}\right)$

From (1), $\exists \alpha_{L}, \alpha_{L}$, such that

$$
\begin{array}{lr}
\mathrm{L} \sim \alpha_{\mathrm{L}} \mathrm{~L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-}, & \alpha_{\mathrm{L}}=\mathrm{U}(\mathrm{~L}) \\
\mathrm{L}^{\prime} \sim \alpha_{\mathrm{L}^{\prime}} \mathrm{L}^{+}+\left(1-\alpha_{L^{\prime}}\right) \mathrm{L}^{-} & \alpha_{\mathrm{L}}=\mathrm{U}\left(\mathrm{~L}^{\prime}\right)
\end{array}
$$

## Proof of Prop. 6.B. 3

First show that U is linear, i.e.,
(5) $\left.\forall \mathrm{L}, \mathrm{L}^{\prime} \in \mathrm{L}, \forall \alpha \in[0,1], \mathrm{U}\left(\alpha \mathrm{L}+(1-\alpha) \mathrm{L}^{\prime}\right)\right)=\alpha \mathrm{U}(\mathrm{L}) .+(1-\alpha) \mathrm{U}\left(\mathrm{L}^{\prime}\right)$

From (1), $\exists \alpha_{\mathrm{L}}, \alpha_{\mathrm{L}}$, such that

$$
\begin{array}{ll}
\mathrm{L} \sim \alpha_{\mathrm{L}} \mathrm{~L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-}, & \mathrm{U}(\mathrm{~L})=\alpha_{\mathrm{L}} \\
\mathrm{~L}^{\prime} \sim \alpha_{\mathrm{L}^{\prime}} \mathrm{L}^{+}+\left(1-\alpha_{\mathrm{L}^{\prime}}\right) \mathrm{L}^{-}, & \mathrm{U}\left(\mathrm{~L}^{\prime}\right)=\alpha_{\mathrm{L}}
\end{array}
$$

Independence:

$$
\begin{aligned}
\alpha \mathrm{L}+(1-\alpha) \mathrm{L}^{\prime} & \sim \alpha\left(\alpha_{\mathrm{L}} \mathrm{~L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-}\right)+(1-\alpha) \mathrm{L}^{\prime} \\
& \sim \alpha\left(\alpha_{\mathrm{L}} \mathrm{~L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-}\right)+(1-\alpha)\left(\alpha_{\mathrm{L}^{\prime}} \mathrm{L}^{+}+\left(1-\alpha_{L^{\prime}}\right) \mathrm{L}^{-}\right)
\end{aligned}
$$

Compound lottery:

$$
\sim\left(\alpha \alpha_{L}+(1-\alpha) \alpha_{L^{\prime}}\right) L^{+}+\frac{\left(\alpha\left(1-\alpha_{\underline{L}}\right)+(1-\alpha)\left(1-\alpha_{L^{\prime}}\right)\right) L^{-}}{\left.\left(=\underline{1-\left(\alpha \alpha_{L}\right.}+(1-\alpha) \alpha_{\underline{L}^{\prime}}\right)\right)}
$$

Definition of U:

$$
\mathrm{U}\left(\alpha \mathrm{~L}+(1-\alpha) \mathrm{L}^{\prime}\right)=\alpha \alpha_{\mathrm{L}}+(1-\alpha) \alpha_{\mathrm{L}^{\prime}}=\alpha \mathrm{U}(\mathrm{~L})+(1-\alpha) \mathrm{U}\left(\mathrm{~L}^{\prime}\right)
$$

## Proof of Prop. 6.B. 3

Have shown that U is linear, i.e.,
(5) $\left.\forall \mathrm{L}, \mathrm{L}^{\prime} \in \mathrm{L}, \forall \alpha \in[0,1], \mathrm{U}\left(\alpha \mathrm{L}+(1-\alpha) \mathrm{L}^{\prime}\right)\right)=\alpha \mathrm{U}(\mathrm{L}) .+(1-\alpha) \mathrm{U}\left(\mathrm{L}^{\prime}\right)$

Show: (4) $\exists$ numbers $\mathrm{u}_{1}, \ldots, \mathrm{u}_{\mathrm{N}}$ s.t.

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}} \quad \forall \mathrm{~L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right) \in L
$$

For each $L_{n}=(0, \ldots, 0,1,0, \ldots, 0), n=1, \ldots, N$, let $u_{n}=U\left(L_{n}\right)=\alpha_{L n}$,
Take any $L=\left(p_{1}, \ldots, p_{N}\right)=p_{1} L_{1}+\ldots+p_{N} L_{N}$.
Since $U$ is linear and

$$
\begin{gathered}
\mathrm{L}=\mathrm{p}_{1} \mathrm{~L}_{1}+\left(1-\mathrm{p}_{1}\right)\left(\left(\mathrm{p}_{2} /\left(1-\mathrm{p}_{1}\right)\right) \mathrm{L}_{2}+\ldots+\left(\mathrm{p}_{\mathrm{N}} /\left(1-\mathrm{p}_{1}\right)\right) \mathrm{L}_{\mathrm{N}}\right), \\
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{U}\left(\mathrm{~L}_{1}\right)+\left(1-\mathrm{p}_{1}\right) \mathrm{U}\left(\left(\mathrm{p}_{2} /\left(1-\mathrm{p}_{1}\right)\right) \mathrm{L}_{2}+\ldots+\left(\mathrm{p}_{\underline{N}} /\left(1-\mathrm{p}_{1}\right)\right) \mathrm{L}_{\mathrm{N}}\right) \\
\mathrm{p}_{2} \underline{2}\left(1-\mathrm{p}_{1}\right) \mathrm{L}_{2}+\left(1-\mathrm{p}_{2} /\left(1-\mathrm{p}_{1}\right)\right)\left(\mathrm{p}_{3} /\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{L}_{3}+\ldots+\mathrm{p}_{\underline{N}} /\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{L}_{\underline{N}}\right) \\
= \\
=\mathrm{p}_{1} \mathrm{U}\left(\mathrm{~L}_{1}\right)+\mathrm{p}_{2} \mathrm{U}\left(\mathrm{~L}_{2}\right) \\
\quad+\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{U}\left(\mathrm{p}_{3} /\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{L}_{3}+\ldots+\mathrm{p}_{\mathrm{N}}\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{L}_{\mathrm{N}}\right.
\end{gathered}
$$

## Proof of Prop. 6.B. 3

Have shown that U is linear, i.e.,
(5) $\left.\forall \mathrm{L}, \mathrm{L}^{\prime} \in \mathrm{L}, \forall \alpha \in[0,1], \mathrm{U}\left(\alpha \mathrm{L}+(1-\alpha) \mathrm{L}^{\prime}\right)\right)=\alpha \mathrm{U}(\mathrm{L}) .+(1-\alpha) \mathrm{U}\left(\mathrm{L}^{\prime}\right)$

Show: (4) $\exists$ numbers $u_{1}, \ldots, u_{N}$ s.t.

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}} \quad \forall \mathrm{~L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right) \in L,
$$

$$
\begin{aligned}
\mathrm{U}(\mathrm{~L})= & \mathrm{p}_{1} \mathrm{U}\left(\mathrm{~L}_{1}\right)+\left(1-\mathrm{p}_{1}\right) \mathrm{U}\left(\left(\mathrm{p}_{2} /\left(1-\mathrm{p}_{1}\right)\right) \mathrm{L}_{2}+\ldots+\left(\mathrm{p}_{\underline{N}} /\left(1-\mathrm{p}_{1}\right)\right) \mathrm{L}_{\underline{N}}\right) \\
& \mathrm{p}_{2} /\left(1-\mathrm{p}_{1}\right) \mathrm{L}_{2}+\left(1-\mathrm{p}_{2} /\left(1-\mathrm{p}_{1}\right)\right)\left(\mathrm{p}_{2} /\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{L}_{3}+\ldots+\mathrm{p}_{\mathrm{N}}\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{L}_{\underline{N}}\right) \\
= & \mathrm{p}_{1} \mathrm{U}\left(\mathrm{~L}_{1}\right)+\mathrm{p}_{2} \mathrm{U}\left(\mathrm{~L}_{2}\right) \\
& \quad+\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{U}\left(\mathrm{p}_{3} /\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{L}_{3}+\ldots+\mathrm{p}_{\mathrm{N}}\left(1-\mathrm{p}_{1}-\mathrm{p}_{2}\right) \mathrm{L}_{\mathrm{N}}\right. \\
= & \mathrm{p}_{1} \mathrm{U}\left(\mathrm{~L}_{1}\right)+\mathrm{p}_{2} \mathrm{U}\left(\mathrm{~L}_{2}\right)+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{U}\left(\mathrm{~L}_{\mathrm{N}}\right) \\
= & \mathrm{p}_{1} \mathrm{u}_{1}+\mathrm{p}_{2} \mathrm{u}_{2}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}}
\end{aligned}
$$

## Uniqueness of vN-M exp'd utility function

Prop.6.B. 3 (modified):
$\geq$ rational, continuous, independent, (compound lottery)
$\leftrightarrow \exists$ utility fcn U:L $\rightarrow \mathfrak{R}$

$$
\text { i.e., } \mathrm{L} \geq \mathrm{L}^{\prime} \quad \Leftrightarrow \quad \mathrm{U}(\mathrm{~L}) \geq \mathrm{U}\left(\mathrm{~L}^{\prime}\right) \quad \forall \mathrm{L}, \mathrm{~L}^{\prime} \in L
$$

such that $\exists$ numbers $\mathrm{u}_{1}, \ldots, \mathrm{u}_{\mathrm{N}}$ satisfying

$$
\mathrm{U}(\mathrm{~L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}} \quad \forall \mathrm{~L}=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{N}}\right) \in L,
$$

Show: Prop. 6.B.2 : Suppose U is a vN-M utility fcn. Then U ' is another vN-M utility fcn

$$
\leftrightarrow \quad \exists \beta>0 \text { and } \gamma \text { s.t. } \mathrm{U}^{\prime}(\mathrm{L})=\beta \mathrm{U}(\mathrm{~L})+\gamma
$$

## Uniqueness of vN-M exp'd utility function

Show: Prop. 6.B.2 : Suppose $U$ is a vN-M utility fcn. Then U ' is another vN-M utility fcn

$$
\leftrightarrow \exists \beta>0 \text { and } \gamma \text { s.t. } \mathrm{U}^{\prime}(\mathrm{L})=\beta \mathrm{U}(\mathrm{~L})+\gamma \quad \forall \mathrm{L} \in L
$$

Pf: $\leftarrow)$ Suppose $L=\left(p_{1}, \ldots, p_{N}\right)$.
Then since $\beta>0, L \geq L^{\prime} \Leftrightarrow U^{\prime}(\mathrm{L}) \geq \mathrm{U}^{\prime}\left(\mathrm{L}^{\prime}\right) \quad \forall \mathrm{L}, \mathrm{L}^{\prime} \in L$.
Furthermore since $\mathrm{U}(\mathrm{L})=\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}}$,

$$
\begin{aligned}
\mathrm{U}^{\prime}(\mathrm{L}) & =\beta\left(\mathrm{p}_{1} \mathrm{u}_{1}+\ldots+\mathrm{p}_{\mathrm{N}} \mathrm{u}_{\mathrm{N}}\right)+\gamma\left(\mathrm{p}_{1}+\ldots+\mathrm{p}_{\mathrm{N}}\right)\left(\text { Note: } \mathrm{p}_{1}+\ldots+\mathrm{p}_{\mathrm{N}}=1\right) . \\
& =\mathrm{p}_{1}\left(\beta \mathrm{u}_{1}+\gamma\right)+\ldots+\mathrm{p}_{\mathrm{N}}\left(\beta \mathrm{u}_{\mathrm{N}}+\gamma\right)
\end{aligned}
$$

## Uniqueness of vN-M exp'd utility function

Show: Prop. 6.B.2 : Suppose $U$ is a vN-M utility fcn. Then U ' is another $\mathrm{vN}-\mathrm{M}$ utility fcn

$$
\leftrightarrow \exists \beta>0 \text { and } \gamma \text { s.t. } \mathrm{U}^{\prime}(\mathrm{L})=\beta \mathrm{U}(\mathrm{~L})+\gamma \quad \forall \mathrm{L} \in L
$$

Pf: $\rightarrow$ ) If $\mathrm{L}^{+} \sim \mathrm{L}^{-}$, then clear. Thus suppose $\left.\mathrm{L}^{+}\right\rangle \mathrm{L}^{-}$.
Take any $\mathrm{L} \in L$ and suppose $\mathrm{L} \sim \alpha_{\mathrm{L}} \mathrm{L}^{+}+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{L}^{-}$.
Then from the linearity of U

$$
\begin{align*}
& \mathrm{U}(\mathrm{~L})=\alpha_{\mathrm{L}} \mathrm{U}\left(\mathrm{~L}^{+}\right)+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{U}\left(\mathrm{~L}^{-}\right)  \tag{1}\\
& \mathrm{U}^{\prime}(\mathrm{L})=\alpha_{\mathrm{L}} \mathrm{U}^{\prime}\left(\mathrm{L}^{+}\right)+\left(1-\alpha_{\mathrm{L}}\right) \mathrm{U}^{\prime}\left(\mathrm{L}^{-}\right) \tag{2}
\end{align*}
$$

By (1), $\alpha_{\mathrm{L}}=\left(\mathrm{U}(\mathrm{L})-\mathrm{U}\left(\mathrm{L}^{-}\right)\right) /\left(\mathrm{U}\left(\mathrm{L}^{+}\right)-\mathrm{U}\left(\mathrm{L}^{-}\right)\right)$
Substituting it into (2), we obtain

$$
\begin{aligned}
\mathrm{U}^{\prime}(\mathrm{L})= & \left(\left(\mathrm{U}^{\prime}\left(\mathrm{L}^{+}\right)-\mathrm{U}^{\prime}\left(\mathrm{L}^{-}\right)\right) /\left(\mathrm{U}\left(\mathrm{~L}^{+}\right)-\mathrm{U}\left(\mathrm{~L}^{-}\right)\right) \mathrm{U}(\mathrm{~L})\right. \\
& +\left(\mathrm{U}\left(\mathrm{~L}^{+}\right) \mathrm{U}^{\prime}\left(\mathrm{L}^{-}\right)-\mathrm{U}^{\prime}\left(\mathrm{L}^{+}\right) \mathrm{U}\left(\mathrm{~L}^{-}\right)\right) /\left(\mathrm{U}^{\left.\left(\mathrm{L}^{+}\right)-\mathrm{U}\left(\mathrm{~L}^{-}\right)\right)}\right.
\end{aligned}
$$

## Assignments

Problem Set 9 (due July 15)
Exercises 6.B.1, 6.B.2, 6.B.3 (pp.208-209)

