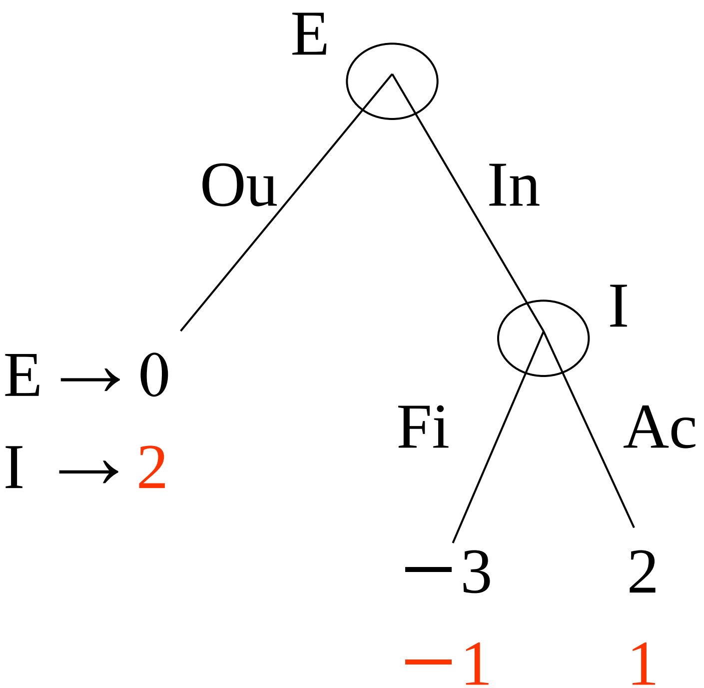
Example 9.B.1



I

E	Fi	Ac
Ou	0, 2	0, 2
In	−3, −1	2, 1

Nash eq (in pure str.)

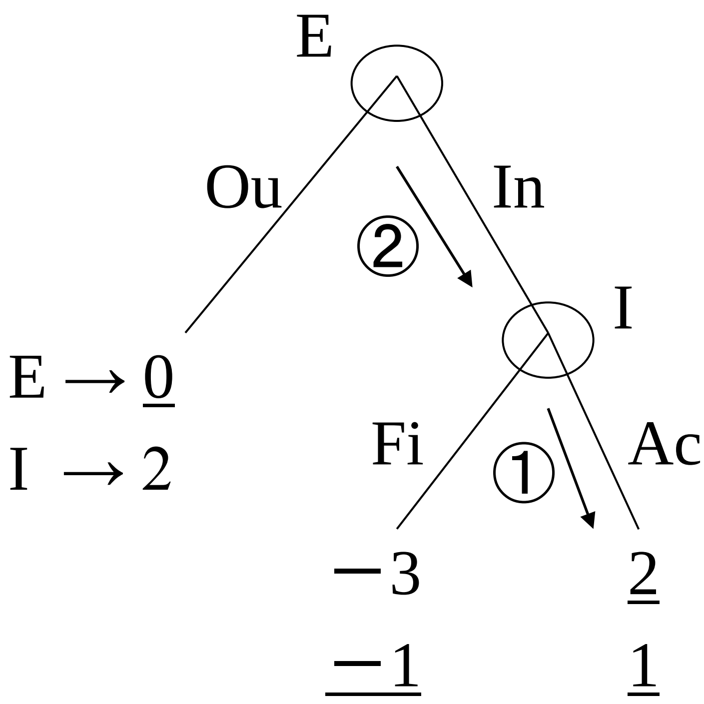
→ (Ou, Fi), (In, Ac)

(Ou, Fi) → rational ? ? ?

Fi : I's incredible threat

If E really plays "In", I will play "Ac". ($1 > -1$)

Backward Induction



Backward induction

① 1 > −1 → I plays Ac

② 2 > 0 → E plays In

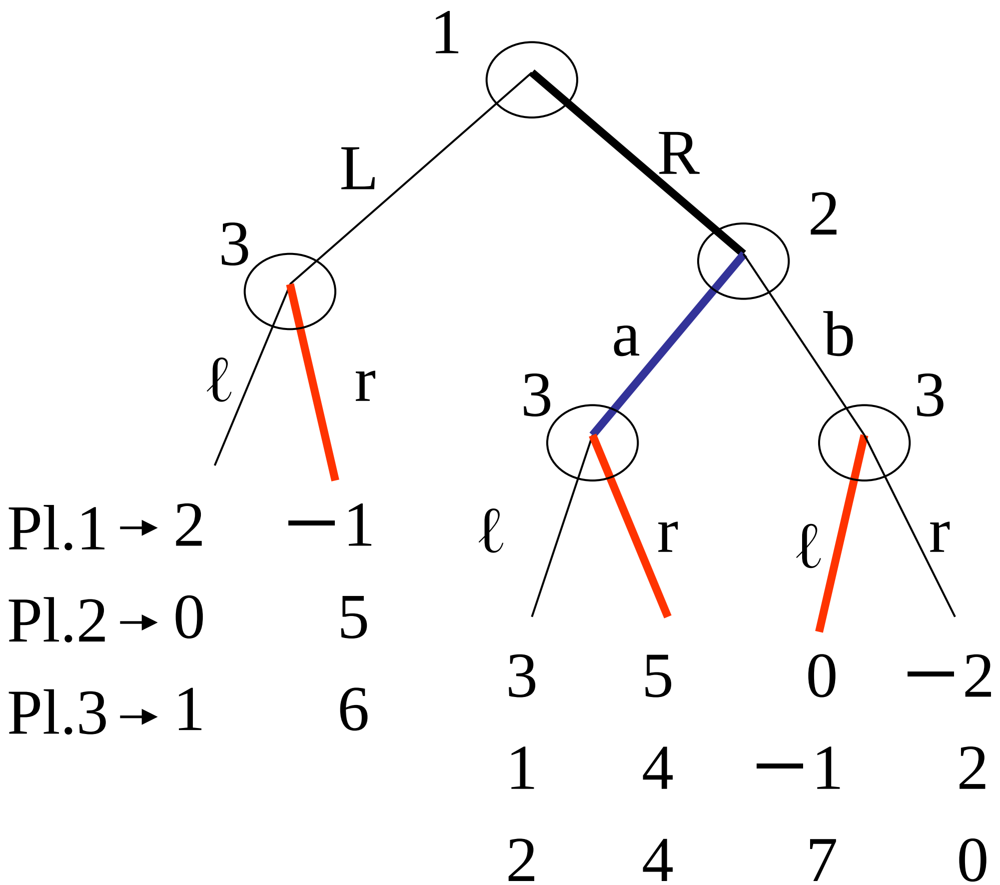
↓

(In, Ac)

Games with perfect information

→ every information set has one decision point.

Backward Induction (Example 9.B.2)



3's decision

$1 < 6 \rightarrow r$

$2 < 4 \rightarrow r$

$7 > 0 \rightarrow \ell$

2's decision

$4 > -1 \rightarrow a$

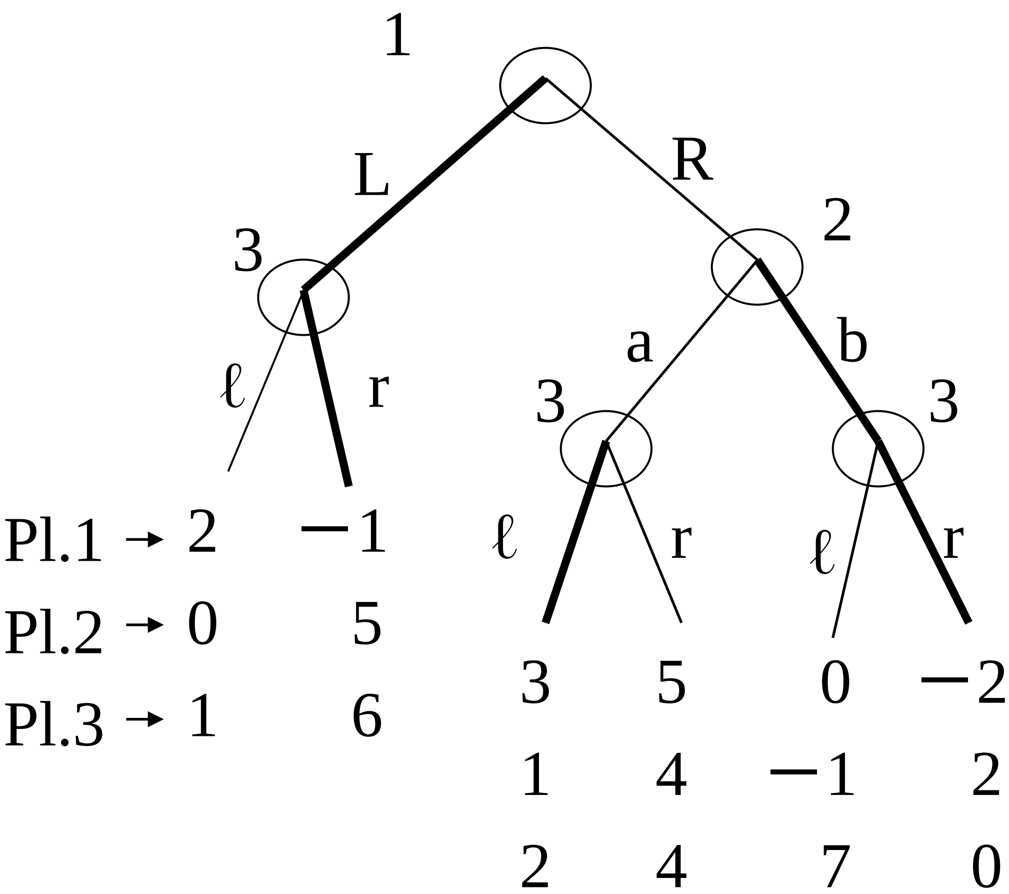
1's decision

$-1 < 5 \rightarrow R$

Backward induction $\rightarrow$ (R, a, (r, r, ℓ)) $\rightarrow$ Nash eq.

Other Nash eq. $\rightarrow$ (L, b, (r, ℓ, r))

Other Nash Equilibria (Example 9.B.2)



Backward induction

→ (R, a, (r, r, ℓ))

→ Nash eq.

Other Nash eq.

→ (L, b, (r, ℓ, r))

Nash Equilibria in Games with Perfect Information

Prop. 9.B.1 (Zermelo's Theorem) : Every finite game w/ perfect information has a pure strategy Nash equilibrium produced by backward induction. If no player has the same payoffs, then ∃ unique Nash eq. derived in this manner.

Pf: a finite game w/ perfect information

→ backward induction is well-defined

no player has the same payoffs

→ a unique strategy combination

Let $(\sigma_1, \ldots, \sigma_I)$ be the strategy combination derived thru backward induction

Show $(\sigma_1, \ldots, \sigma_I)$ is a Nash eq.

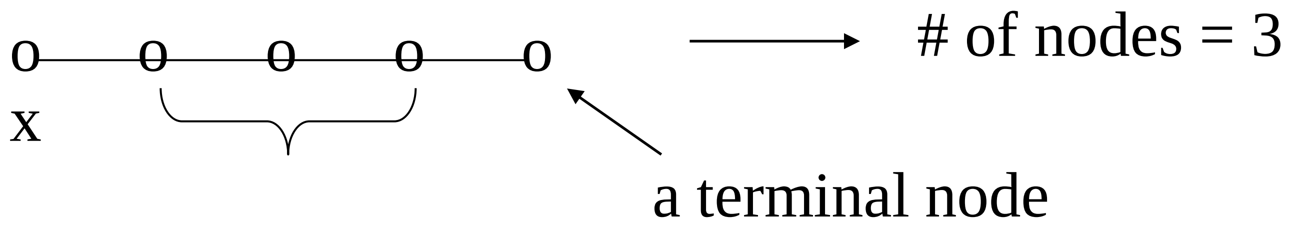
Proof

Show $\forall i \ \ \forall \hat{\sigma}_i \quad u_i(\sigma_i, \ \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \ \sigma_{-i})$

Take any $\hat{\sigma}_i$ and define i's strategy $\hat{\sigma}_i(n)$ as follows.

For each node x,

let d(x) = <u>max</u> # of nodes between x and terminal nodes



$$\text{Let } \hat{\sigma}_i(n)(x) = \begin{cases} \sigma_i(x) & \text{if } d(x) \leq n \\ \hat{\sigma}_i(x) & \text{if } d(x) > n \end{cases}$$

<u>Note</u>:

$$\begin{cases} \hat{\sigma}_i(0)(x) = \begin{cases} \sigma_i(x) & \text{if } d(x) = 0 \\ \hat{\sigma}_i(x) & \text{if } d(x) > 0 \end{cases} \\ \hat{\sigma}_i(N)(x) = \sigma_i(x) \quad \forall x \quad \leftarrow \quad N = \max_x d(x) \end{cases}$$

Proof

Show $u_i(\sigma_i\wedge(N) = \sigma_i, \ \sigma_{-i}) \geq u_i(\sigma\wedge_i, \ \sigma_{-i})$: induction on n

(1) $n = 0$: $\sigma\wedge_i(0)(x) =$ $\sigma_i(x)$ if $d(x) = 0$

$\sigma\wedge_i(x)$ if $d(x) > 0$

$\sigma_i(x)$ chooses an alternative at x that max i's payoff

$\rightarrow \ u_i(\sigma\wedge_i(0), \ \sigma_{-i}) \geq u_i(\sigma\wedge_i, \ \sigma_{-i})$

(2) Suppose for $n = k-1$

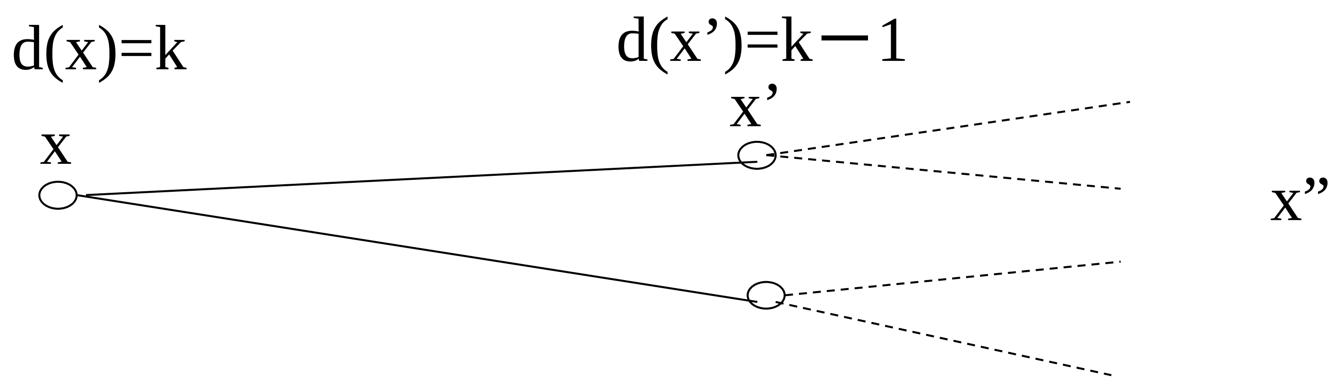
$u_i(\sigma\wedge_i(k-1), \ \sigma_{-i}) \geq u_i(\sigma\wedge_i, \ \sigma_{-i})$ holds.

(3) For $n = k$, show $u_i(\sigma\wedge_i(k), \ \sigma_{-i}) \geq u_i(\sigma\wedge_i, \ \sigma_{-i})$

Proof

(2) Suppose for $n = k-1$, $u_i(\sigma\wedge_i(k-1), \sigma_{-i}) \geq u_i(\sigma\wedge_i, \sigma_{-i})$ ①

(3) For $n = k$, <u>show</u> $u_i(\sigma\wedge_i(k), \sigma_{-i}) \geq u_i(\sigma\wedge_i, \sigma_{-i})$ ②



$\sigma\wedge_i(k)(x) = \sigma_i(x)$ $\quad$ $\sigma\wedge(k)(x') = \sigma_i(x')$ $\quad$ $\sigma_i(x'')$ ・・・

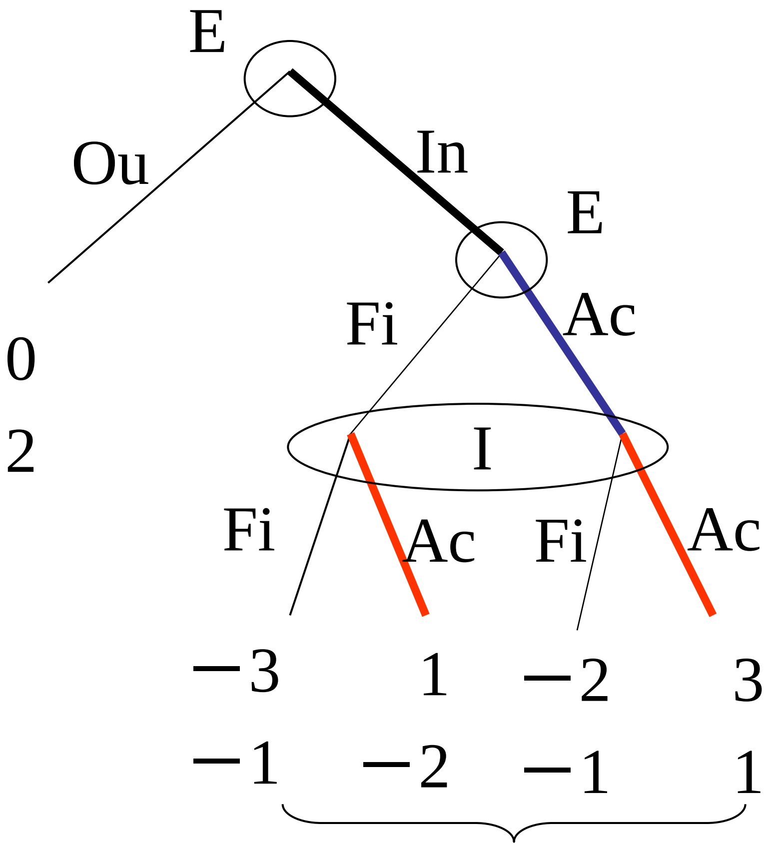
$\sigma\wedge_i(k-1)(x) = \sigma\wedge_i(x)$ $\quad$ $\sigma\wedge_i(k-1)(x') = \sigma_i(x')$ $\quad$ $\sigma_i(x'')$

By the definition of σ_i, $u_i(\sigma\wedge_i(k), \sigma_{-i}) \geq u_i(\sigma\wedge_i(k-1), \sigma_{-i})$ ③

① and ③ → ② holds.

Eventually $u_i(\sigma_i, \sigma_{-i}) = u_i(\sigma\wedge_i(N), \sigma_{-i}) \geq u_i(\sigma\wedge_i, \sigma_{-i})$ Q.E.D.

A Game with Imperfect Information (Example 9.B.3)



		I	
		Ac	Fi
E	Ac	3, 1	−2, −1
	Fi	1, −2	−3, −1

Nash eq. (Ac, Ac)

		I	
		Ac	Fi
	Ou Ac	0, 2	0, 2
E	Ou Fi	0, 2	0, 2
	In Ac	3, 1	−2, −1
	In Fi	1, −2	−3, −1

Nash eq. ((Ou Ac), Fi),
((Ou, Fi), Fi),
((In, Ac), Ac)

Subgames

Defn. 9.B.1: A subgame of an extensive form game is a subset of the game having the following properties:

(1) It begins with an information set containing only one node.

(2) It contains all successors of the node and no other node.

(3) For each successor, any node, in the information set that contains the successor, is in the subset.

Note: (1) whole game → a subgame

(2) Fig.9.B.1 → two subgames

(3) Fig.9.B.3 → five subgames

(games with perfect information

→ each node initiates a subgame)

(4) Fig.9.B.4 → two subgames

(5) Fig.9.B.5 → parts of the game that are not subgames

Subgame Perfect Equilibrium (definition)

Defn. 9.B.2: A strategy profile $\sigma = (\sigma_1, \dots, \sigma_I)$ of an extensive form game is SPNE if it induces a Nash equilibrium in every subgame of the game.

Note: (1) SPNE $\rightarrow$ Nash equilibrium (whole game is a subgame.)

(2) SPNE $\rightarrow$ SPNE of each subgame

(3) Fig.9.B.1 $\rightarrow$ (In, Ac)

(4) Fig.9.B.2 $\rightarrow$ (R, a, (r, r, ℓ))

(5) Fig.9.B.3 $\rightarrow$ ((In, Ac), Ac)

SPNE in Games with Perfect Information

Prop. 9.B.2 : Every finite game w/ perfect information has a pure strategy SPNE. If no player has the same payoffs, then ∃ unique SPNE

Pf: clear from Prop. 9.B.1 and the definition of SPNE

Properties of SPNE (Prop. 9.B.3)

Prop. 9.B.3 : Γ_E : an extensive form game, S : a subgame

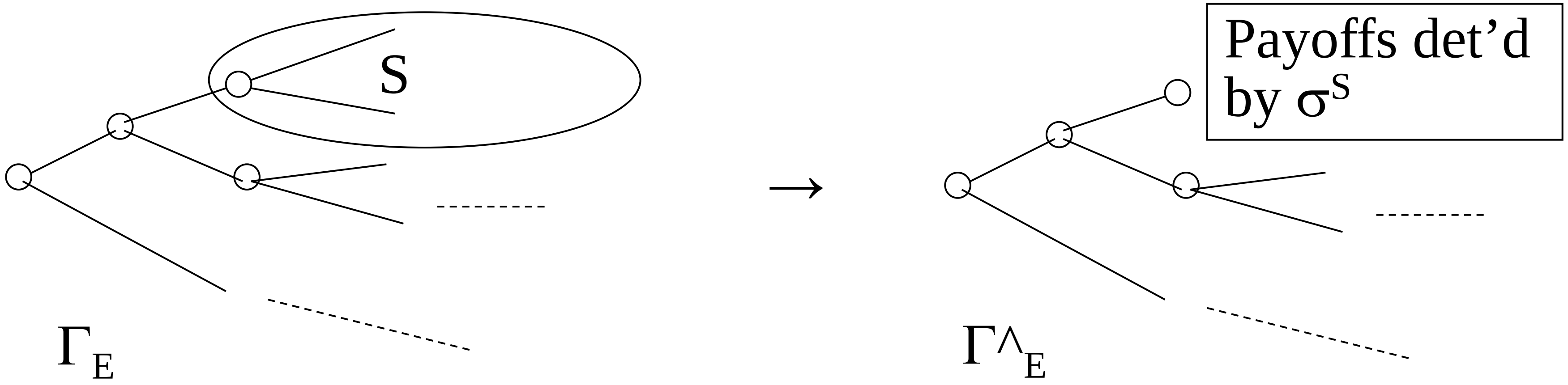
σ^S : an SPNE of subgame S

$\Gamma^\wedge_E$: the reduced game replacing the subgame S by a terminal node with payoff determined by σ^S

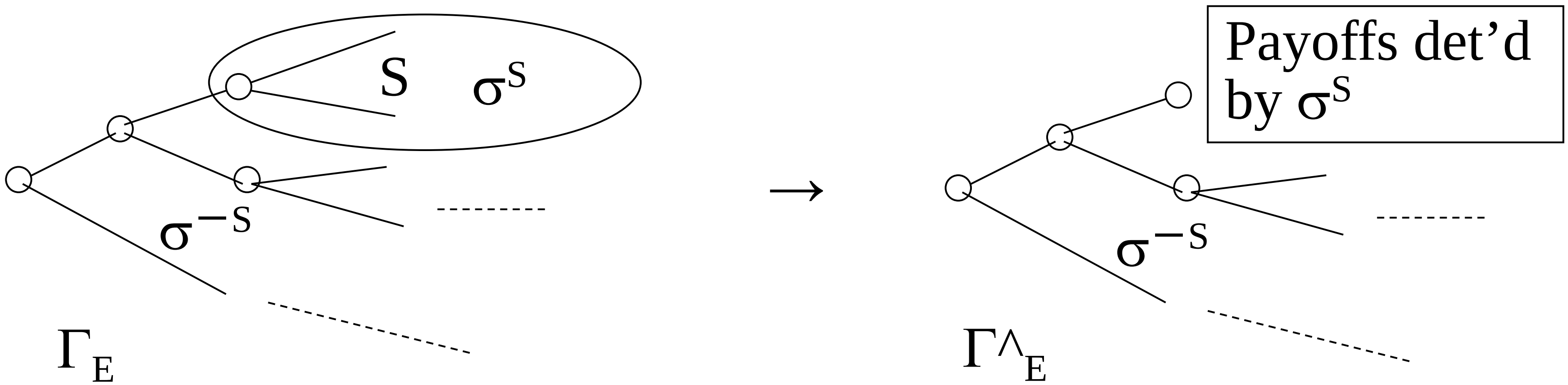
(1) σ : an SPNE of Γ_E s.t. restriction of σ to S is σ^S.

σ^{-S}, the restriction of σ to outside S $\rightarrow$ σ^{-S} is an SPNE of $\Gamma^\wedge_E$

(2) $\sigma^\wedge$: an SPNE of $\Gamma^\wedge_E$ $\rightarrow$ $(\sigma^\wedge, \sigma^S)$ is an SPNE of Γ_E



Proof of Prop. 9.B.3



(1) σ : an SPNE of Γ_E $\quad$ σ^S : restriction of σ to S

σ^{-S} : restriction of σ to outside S

$\rightarrow$ σ^{-S} is an SPNE of $\Gamma\hat{}_E$

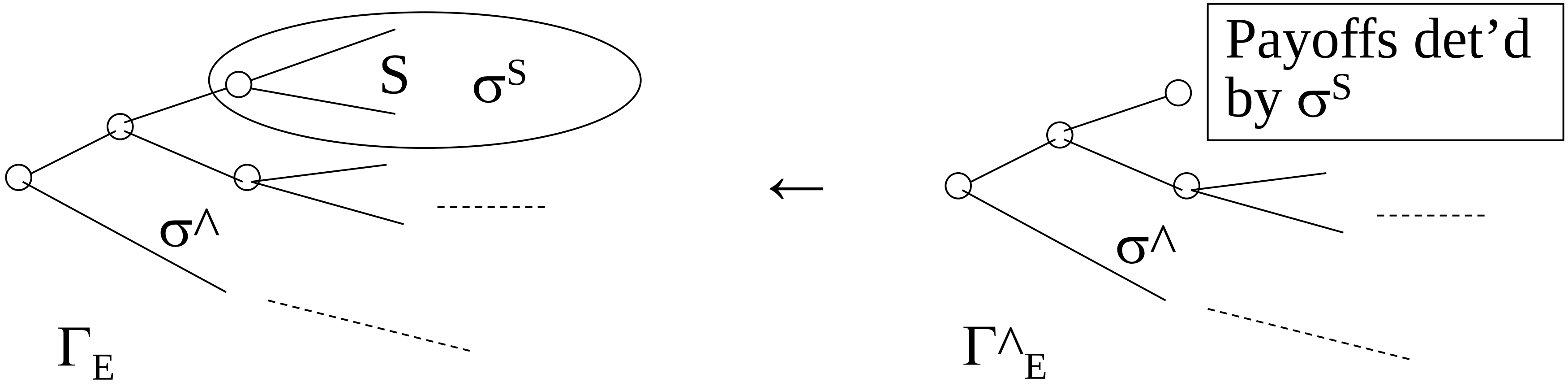
Pf: Suppose σ^{-S} is not an SPNE of $\Gamma\hat{}_E$.

Then $\exists$ a subgame T of $\Gamma\hat{}_E$ s.t. σ^T is not a Nash eq. in $\Gamma\hat{}_E$.

$\exists$i who can increase his payoff by deviating from σ^T in $\Gamma\hat{}_E$.

i can increase his payoff in Γ_E by the same deviation.

Proof of Prop. 9.B.3



(2) $\sigma^\wedge$: an SPNE of $\Gamma^\wedge_E$ $\rightarrow$ $(\sigma^\wedge, \sigma^S)$ is and SPNE of Γ_E

<u>Pf</u>: Let $\sigma' = (\sigma^\wedge, \sigma^S)$. Take any subgame T.

If $T \subseteq S$ or $T \subseteq {}^{-}S$, then σ'^T is a Nash eq. of T.

If not, T contains S.

Suppose $\exists$ i who can gain more by deviating from σ'_i .

Since σ^S is an SPNE of S, i changes his choice outside S.

Then i can gain more also in $\Gamma^\wedge_E$. C! Q.E.D.

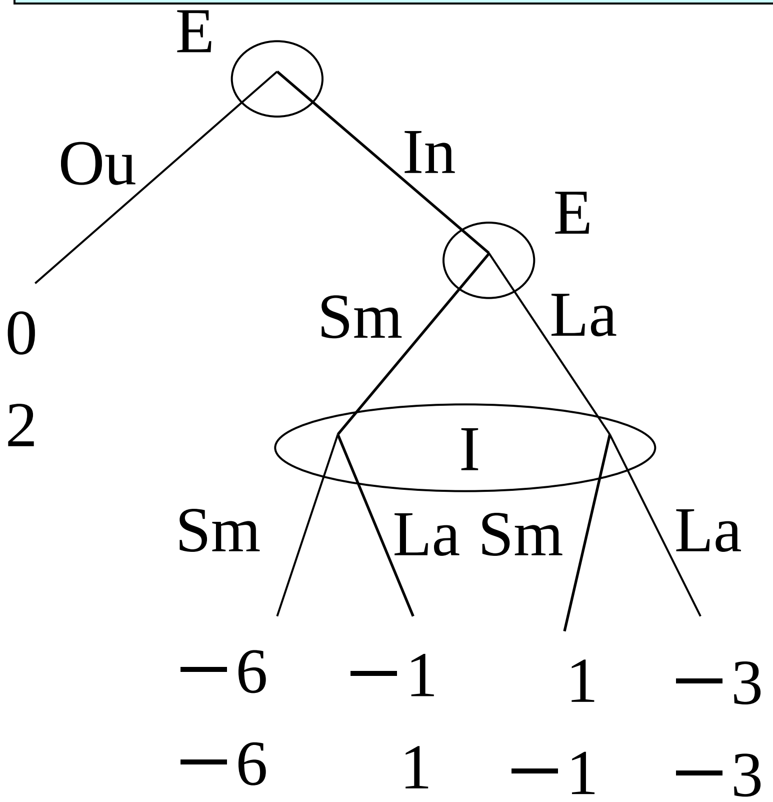
Generalized Backward Induction

1 Start at the end of the game tree. Identify Nash eq. in each of the final subgames.

2 Select one Nash eq. in each of the final subgames, and derive the reduced extensive form game by replacing each subgame by a terminal node with payoffs of the selected Nash eq.

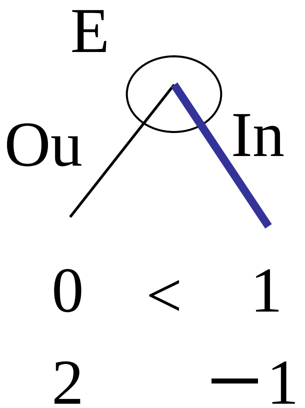
3 Repeat this procedure until every move in the original extensive form game is determined.

Example 9.B.4



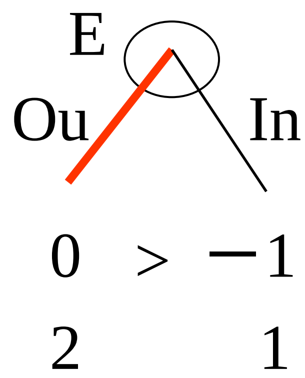
		I	
		Sm	La
E	Sm	−6, −6	−1, 1
	La	1, −1	−3, −3

Nash eq. (La, Sm), (Sm, La)



SPNE

((In, La), Sm)



SPNE

((Ou, Sm), La)

Prop. 9.B.4

<u>Prop. 9.B.4</u> : Γ^t_E : simultaneous move game, $t = 1, 2, \dots, T$.

Γ_E : successive play of Γ^t_E

Each player's payoff = sum of his payoffs in T periods

Each player knows others' choices just after each game is played.

If $\exists$ a unique Nash equilibrium σ^t in Γ^t_E,

then there is a unique SPNE in Γ_E

in which each player i plays σ^t_i in $t = 1, 2, \dots, T$.

<u>Pf:</u> Induction on T. If $T = 1$, clear.

Suppose the claim is true for all $T \leq n - 1$.

Show the claim holds when $T = n$.

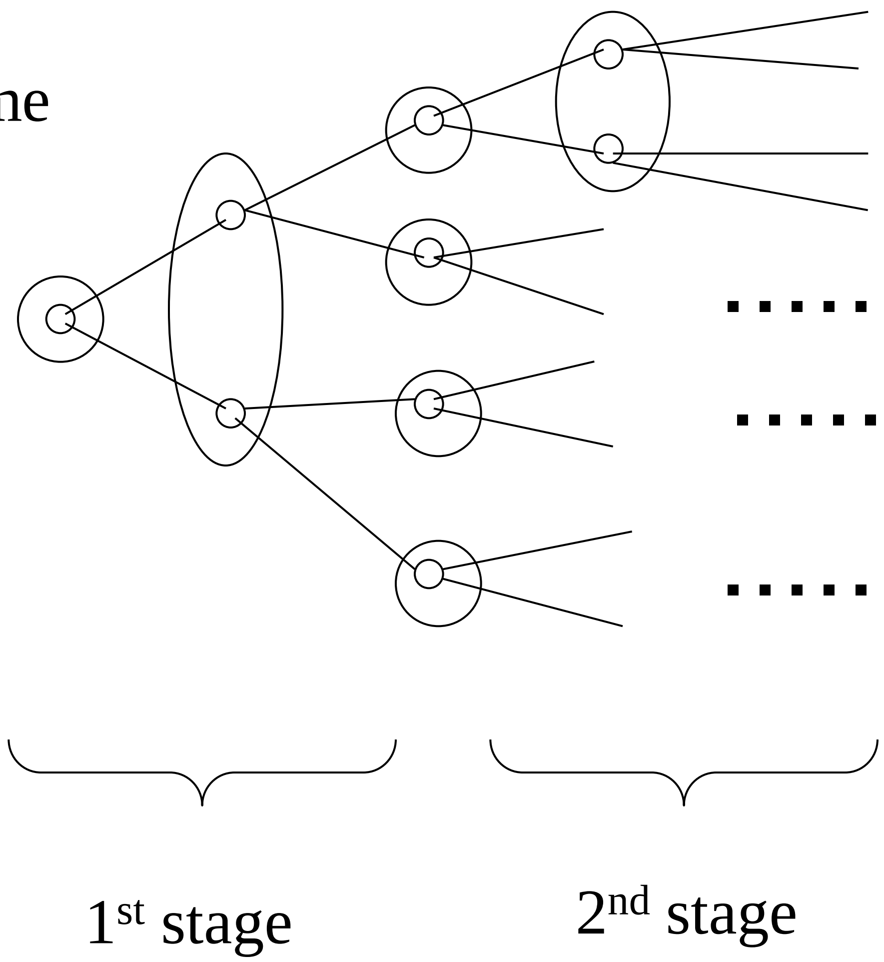
After the first period is over, we have $n - 1$ period game.

Thus from the induction hypothesis, the conclusion easily follows.

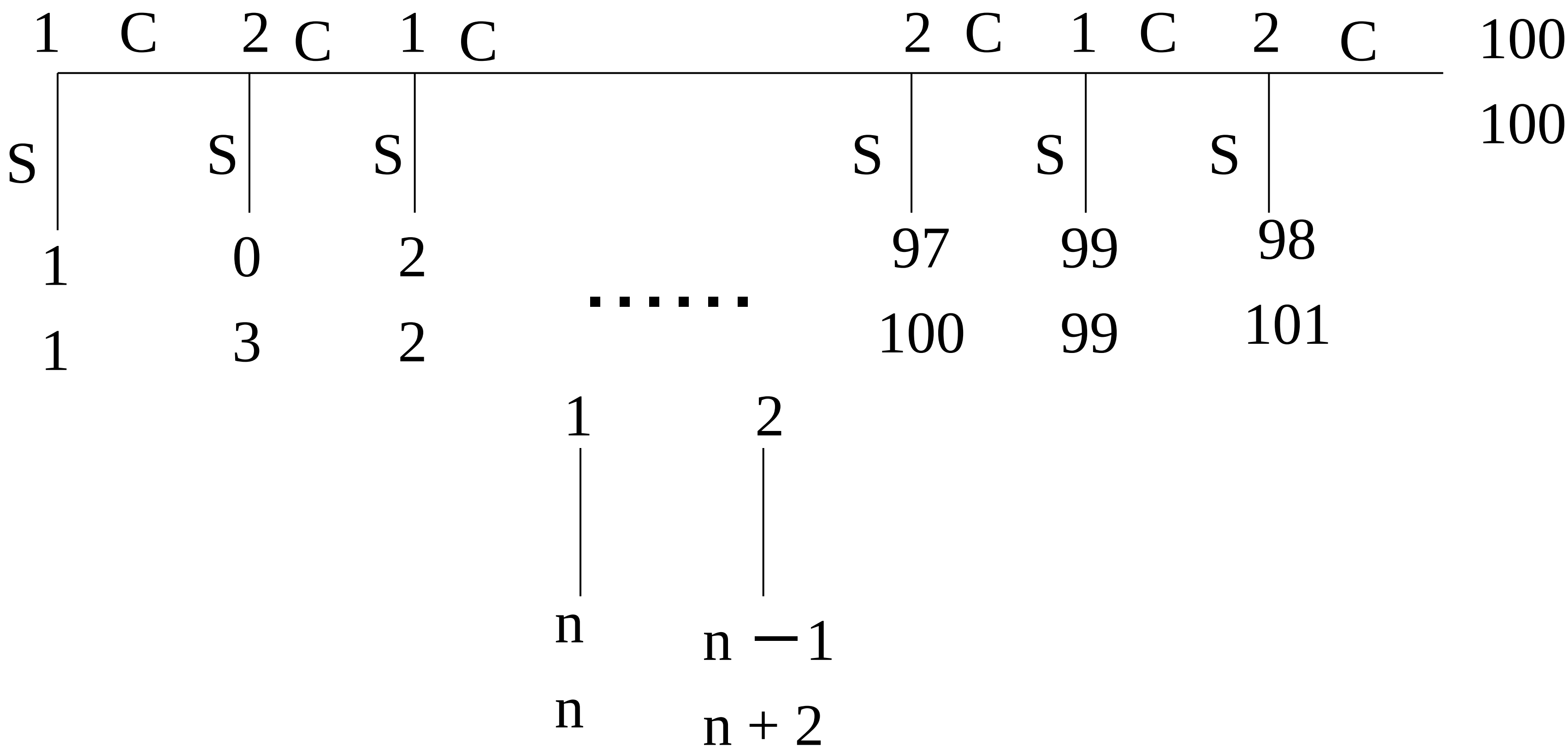
Repeated Game

$N = \{1, 2\}$, $S_1 = \{a, b\}$, $S_2 = \{c, d\}$

Two-stage game



Centipede Game



SPNE (S, S, … , S), (S, S, … , S))

Assignments

Problem Set 7 (due June 24)

Exercises (pp.301-305)

9.B.3, 9.B.6, 9.B.9, 9.B.10

Reading Assignment:

Text, Chapter 9, pp.282-291