

Nash equilibrium

Definition 8.D.1: (Nash equilibrium)

$s = (s_1, \dots, s_I)$ is a Nash equilibrium

in $\Gamma_N = [N=\{0,1,\dots,I\}, \{S_i\}, \{u_i\}]$

if $\forall i=1, \dots, I, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \forall s'_i \in S_i$.

Note: Nash eq. \rightarrow each player's strategy is a best response to the strategies actually played by her rivals

Rationalizable strategies

\rightarrow best response to some justified strategies of the rivals

Nash equilibrium

Example 8.D.1:

 denotes a best response

(M, m) is the unique Nash eq.

	l	m	r
U	<u>5</u> , 3	0, 4	3, <u>5</u>
M	4, 0	<u>5</u> , <u>5</u>	4, 0
D	3, <u>5</u>	0, 4	<u>5</u> , 3

Example 8.D.2:

(a₂, b₂) is the unique Nash eq.

rationalizable strategies →

{a₁, a₂, a₃} for 1,

{b₁, b₂, b₃} for 2

	b ₁	b ₂	b ₃	b ₄
a ₁	0, <u>7</u>	2, 5	<u>7</u> , 0	0, 1
a ₂	5, 2	<u>3</u> , <u>3</u>	5, 2	0, 1
a ₃	<u>7</u> , 0	2, 5	0, <u>7</u>	0, 1
a ₄	0, <u>0</u>	0, -2	0, <u>0</u>	<u>10</u> , -1

Note: Every strategy in Nash eq. → rationalizable

Nash equilibrium

Example 8.D.3:

_ denotes a best response

(E, E), (C, C) are Nash eq.

	E	C
E	<u>100</u> , <u>100</u>	0, 0
C	0, 0	<u>100</u> , <u>100</u>

Nash eq. theory says nothing which eq. we should expect.

Best-response correspondence

$$b_i: S_{-i} \rightarrow S_i$$

$$b_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \forall s'_i \in S_i\}$$

$s = (s_1, \dots, s_I)$ is a Nash equilibrium

$$\text{in } \Gamma_N = [N=\{0,1,\dots,I\}, \{S_i\}, \{u_i\}]$$

$$\text{iff } s_i \in b_i(s_{-i}) \quad \forall i=1, \dots, I$$

Nash equilibrium — Discussion

Why should we concern ourselves with the concept of Nash eq. ?

How do players reach a Nash eq. ?

1. Nash eq. as a consequence of rational inference
2. Nash eq. as a necessary condition
if there is a unique predicted outcome
3. Focal points
4. Nash eq. as a self-enforcing agreement
5. Nash eq. as a stable social convention

Mixed Strategy Nash equilibrium

Definition 8.D.1:

$\sigma = (\sigma_1, \dots, \sigma_I)$ is a Nash equilibrium

in $\Gamma_N = [N=\{0,1,\dots,I\}, \{\Delta(S_i)\}, \{u_i\}]$

if $\forall i=1, \dots, I, u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \forall \sigma'_i \in \Delta(S_i)$.

Example 8.D.4 :

$((1/2, 1/2), (1/2, 1/2))$ is a unique Nash eq.

1's payoff: H $-1 \times 1/2 + 1 \times 1/2 = 0$

T $1 \times 1/2 + (-1) \times 1/2 = 0$

same for 2

	H	T
H	-1, +1	+1, -1
T	+1, -1	-1, +1

Mixed Strategy Nash equilibrium

Proposition 8.D.1: $S_i^+ \subseteq S_i$ set of pure str. played with positive prob.
in $\sigma = (\sigma_1, \dots, \sigma_I)$. σ is a Nash eq. in

$\Gamma_N = [N = \{0, 1, \dots, I\}, \{\Delta(S_i)\}, \{u_i\}]$ iff $\forall i = 1, \dots, I$,

$$(i) \quad u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \quad \forall s_i, s'_i \in S_i^+$$

$$(ii) \quad u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \quad \forall s_i \in S_i^+, \forall s'_i \notin S_i^+$$

Pf : \rightarrow) First show that $\forall i = 1, \dots, I$

$$u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \quad \forall s_i \in S_i^+, \forall s'_i \in S_i$$

Suppose not, i.e., $\exists i, s_i \in S_i^+, s'_i \in S_i$ s.t. $u_i(s'_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i})$.

Let σ'_i be s.t.

$$\begin{aligned} \sigma'_i(s''_i) &= \sigma_i(s''_i) && \text{for } s''_i \neq s_i, s'_i \\ &= \sigma_i(s'_i) + \sigma_i(s_i) && \text{for } s''_i = s'_i \\ &= 0 && \text{for } s''_i = s_i \end{aligned}$$

Then $u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$, contradicting that σ is a Nash eq.

Mixed Strategy Nash equilibrium

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$$(ii) \ u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \quad \forall s_i \in S_i^+, \forall s'_i \notin S_i^+$$

Pf: \rightarrow) Next show that $\forall i=1, \dots, I$

$$u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \quad \forall s_i, s'_i \in S_i^+,$$

Suppose not, i.e., $\exists i, s_i, s'_i \in S_i^+$ s.t. $u_i(s'_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i})$.

(Remember: have shown $u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \quad \forall s_i \in S_i^+, \forall s'_i \in S_i$)

Let σ'_i be s.t.

$$\begin{aligned} \sigma'_i(s''_i) &= \sigma_i(s''_i) && \text{for } s''_i \neq s_i, s'_i \\ &= \sigma_i(s'_i) + \sigma_i(s_i) && \text{for } s''_i = s'_i \\ &= 0 && \text{for } s''_i = s_i \end{aligned}$$

Then $u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$, contradicting that σ is a Nash eq.

Mixed Strategy Nash equilibrium

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in $\sigma = (\sigma_1, \dots, \sigma_I)$. σ is a Nash eq. in

$\Gamma_N = [N=\{0,1,\dots,I\}, \{\Delta(S_i)\}, \{u_i\}]$ iff $\forall i=1, \dots, I,$

$$(i) \quad u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \quad \forall s_i, s'_i \in S_i^+$$

$$(ii) \quad u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \quad \forall s_i \in S_i^+, \forall s'_i \notin S_i^+$$

Pf : \leftarrow) Suppose that σ is not a Nash eq. .

Then $\exists i, \sigma'_i \in \Delta(S_i)$ s.t. $u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$.

Then $\exists s'_i \in S_i$ s.t. $u_i(s'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$ with $\sigma'_i(s'_i) > 0$.

From (i), $u_i(s_i, \sigma_{-i}) = u_i(\sigma_i, \sigma_{-i})$ for all $s_i \in S_i^+$.

Thus $s'_i \notin S_i^+$, contradicting (ii).

Note: To see a Nash eq. or not,

it suffices to check deviations to pure strategies.

Mixed Strategy Nash equilibrium

Corollary 8.D.1:

$s = (s_1, \dots, s_I)$ is a Nash eq. of $\Gamma_N = [N=\{0,1,\dots,I\}, \{S_i\}, \{u_i\}]$
iff it is a Nash eq. of $\Gamma'_N = [N=\{0,1,\dots,I\}, \{\Delta(S_i)\}, \{u_i\}]$

Pf: \leftarrow) clear.

\rightarrow) Since s is a Nash eq. of $\Gamma_N = [N=\{0,1,\dots,I\}, \{S_i\}, \{u_i\}]$,

$$\forall i=1, \dots, I \quad u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \forall s'_i \notin S_i^+$$

Thus (i), (ii) in Proposition 8.D.1 trivially hold since $S_i^+ = \{s_i\}$

Thus by Prop.8.D.1,

s is a Nash eq. of $\Gamma'_N = [N=\{0,1,\dots,I\}, \{\Delta(S_i)\}, \{u_i\}]$.

Mixed Strategy Nash equilibrium

Example 8.D.5:

S's mixed strategy: $(\sigma_s, 1-\sigma_s)$

T: play E $\rightarrow 1000\sigma_s$
 play C $\rightarrow 100(1-\sigma_s)$

		S	
		E	C
T	E	<u>1000</u> , <u>1000</u>	0, 0
	C	0, 0	<u>100</u> , <u>100</u>

Suppose T's mixed strategy $(\sigma_T, 1-\sigma_T)$ satisfies $0 < \sigma_T < 1$.

Then $S_T^+ = \{E, C\}$.

Prop. 8.D.1 $\rightarrow 1000\sigma_s = 100(1-\sigma_s)$

$\rightarrow \sigma_s = 1/11 \rightarrow S$'s mixed strategy $(1/11, 10/11)$

Similarly, T's strategy $(1/11, 10/11)$

Nash eq. $((1/11, 10/11), (1/11, 10/11))$

Mixed Strategy ???

What is a mixed strategy ?

It just makes the rival indifferent over his strategies
(The player has no preference over the probabilities.)

Is a mixed strategy useful ?

1 Players have a pure strategy that gives the same payoff.

→ why randomize them ?

→ Players may not actually randomize; but they make definite choices that are affected by signals.

2 Stability of mixed strategy Nash eq.

players do not have an incentive to use the exact probability

→ may not arise as a social convention,
but as a self-enforcing agreement

Correlated Strategies

Example 8.D.5:

Public signal $\theta \in [0,1]$

$\theta \geq 1/2 \rightarrow$ both play E

$\theta < 1/2 \rightarrow$ both play C

		S	
		E	C
T	E	<u>1000, 1000</u>	0, 0
	C	0, 0	<u>100, 100</u>

This is equilibrium.

If T (S) follows, then S (T) has no incentive to deviate.



Correlated equilibrium

Existence of Nash equilibrium

Proposition 8.D.2:

$\Gamma_N = [N=\{0,1,\dots,I\}, \{\Delta(S_i)\}, \{u_i\}]$ in which S_1, \dots, S_I have a finite number of elements has a mixed strategy Nash eq.

Proposition 8.D.3:

A Nash eq. exists in $\Gamma_N = [N=\{0,1,\dots,I\}, \{S_i\}, \{u_i\}]$ if $\forall i=1, \dots, I$

- (i) S_i is a nonempty, convex, and compact subset of \mathfrak{R}^m , and
- (ii) $u_i(s_1, \dots, s_I)$ is continuous in (s_1, \dots, s_I) and quasi-concave in s_i .

$u_i(s_1, \dots, s_I)$ is quasi-concave in s_i

if $\forall s'_i, s''_i, \alpha \in [0,1]$

$$u_i(\alpha s'_i + (1-\alpha)s''_i, s_{-i}) \geq \min(u_i(s'_i, s_{-i}), u_i(s''_i, s_{-i}))$$

Assignments

Problem Set 4 (due May 13):

Exercises (pp.262-266):

1. 8.D.3, 8.D.4, 8.D.5, 8.D.9
2. Read (i) – (v) on the concept of Nash equilibrium (pp.248-249) and summarize them.

Reading Assignments:

Text Chapter 8, pp.253-257