統計的推測

母数推定

多数のデータ$\{X_i\}$から未知の母数(パラメタ)を推定したい.

$$X_i = \theta + \varepsilon_i$$

ε_i:確率変数 → PDF が分かっている

1)最小分散不偏推定量:標本平均の修正

$$\overline{X} = \frac{X_1 + \cdots + X_n}{n}$$

$$\hat{\theta}^* = \overline{X} - E_0\left(\overline{X} \middle| Y_1, \cdots, Y_{n-1}\right)$$

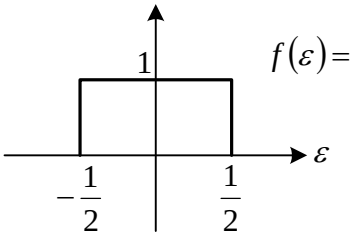
但し, E_0:母数が $\theta = 0$ の時の $\overline{X}$ の平均値

2) $\hat{\theta}^* = \dfrac{\int \theta f(X_1 - \theta) \cdots f(X_n - \theta)\, d\theta}{\int f(X_1 - \theta) \cdots f(X_n - \theta)\, d\theta}$:Pitman の推定量(アルゴリズム)

$f(\quad)$:誤差ε_iの PDF

3) $f(\quad)$が一様分布の場合

$\hat{\theta}^* = \dfrac{\mathrm{Max}\{X_i\} + \mathrm{Min}\{X_i\}}{2}$ となる



$$f(\varepsilon) = \begin{cases} 1, & |\varepsilon| \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

推定量

位置共変性 $X_i = \theta + \varepsilon_i$

$\forall C$ に対して

$$\hat{\theta}(X_1 + C, X_2 + C, \cdots, X_n + C) = \hat{\theta}(X_1, \cdots, X_n) + C$$

が成立する推定量 → 位置共変推定量

標本平均で推定した場合に<u>失う情報</u>

$\underbrace{X_1 \cdots X_n}$:n ヶのデータがあった

$\overline{X} = \dfrac{X_1 + \cdots + X_n}{n}$:1ヶが生成された 残り$(n-1)$ヶ分のデータは?

$$\left.\begin{array}{c} Y_1 = X_2 - Y_1 \\ \vdots \\ Y_{n-1} = X_n - X_1 \end{array}\right\} \rightarrow \text{残りの } n-1 \text{ヶのデータ}$$

$\hat{\theta}^* = \overline{X} - E_0\left(\overline{X} \middle| Y_1, \cdots, Y_{n-1}\right)$

$\overline{X}$:標本平均という確率変数

定理:任意の位置共変不偏推定量$\hat{\theta}$に対して$E\left[\left(\hat{\theta}^* - \theta\right)^2\right] \le E\left[\left(\hat{\theta} - \theta\right)^2\right]$が成立する.

→ $\hat{\theta}^*$が最小分散推定量である. かつ$\hat{\theta}^*$も位置共変性を有する.

1) $\hat{\theta}^*$の位置共変性

$$\begin{aligned}\hat{\theta}^*\left(X_1 + C, \cdots, X_n + C\right) &= \frac{X_1 + C + \cdots + X_n + C}{n} - E_0\left(\overline{X} \middle| Y_1, \cdots, Y_{n-1}\right) \\ &= \frac{X_1 + \cdots + X_n}{n} + C - E_0\left(\overline{X} \middle| Y_1, \cdots, Y_{n-1}\right) \\ &= \hat{\theta}^*\left(X_1, \cdots, X_n\right) + C\end{aligned}$$

$$\boxed{\begin{aligned} Y_1 &= \left(X_2 + C\right) - \left(X_1 + C\right) = X_2 - X_1 \\ &\vdots \\ Y_{n-1} &= X_n - X_1 \end{aligned}}$$

であるので $Y_1, \cdots, Y_{n-1}$ は変わらない

2) $\hat{\theta}^*$の不偏性($E\left(\overline{X}\right) = \theta = 0$の時)

$$\begin{aligned}E\left(\hat{\theta}^*\right) &= E^{X_1, \cdots, X_n}\left(\hat{\theta}^*\left(X_1, \cdots, X_n\right)\right) = E^{X_1, \cdots Y_{n-1}}\left[\overline{X} - E_0\left(\overline{X} \middle| Y_1, \cdots, Y_{n-1}\right)\right] \\ &= \theta - \underbrace{E^{\left(\overline{X}, Y_1, \cdots, Y_n\right)}\left(E_0\left(\overline{X} \middle| Y_1, \cdots, Y_n\right)\right)}_{\theta} \\ &= \theta - 0 = \theta\end{aligned}$$

3) $\hat{\theta}^*$の最小分散性

(証明のスケッチ) $T \overset{\Delta}{=} \hat{\theta} - \hat{\theta}^*$ とおく

$\hat{\theta}$:任意の位置共変推定量

$$\begin{aligned}T\left(X_1 + C, \cdots X_n + C\right) &= \hat{\theta}\left(X_1 + C, \cdots X_n + C\right) - \hat{\theta}^*\left(X_1 + C, \cdots X_n + C\right) \\ &= \hat{\theta}\left(X_1, \cdots, X_n\right) + C - \hat{\theta}^*\left(X_1, \cdots, X_n\right) - C \\ &= T\left(X_1, \cdots, X_n\right) = T\left(Y_1, \cdots, Y_{n-1}\right)\end{aligned}$$

$X_1, \cdots, X_n \to \overline{X}, \boxed{Y_1, \cdots, Y_{n-1}}$

$\boxed{Y_1, \cdots, Y_{n-1}}$: $+C$ のオフセットの影響を受けない

$$
\begin{aligned}
E_0\left(\hat{\theta}^2\right) &= E_0\left(\hat{\theta}^{*2} + 2\hat{\theta}^* T + T^2\right) \\
&= E_0\left(\hat{\theta}^{*2}\right) + E_0\left(T^2\right) \geq E_0\left(\hat{\theta}^{*2}\right) \\
&\therefore E_0\left(\hat{\theta}^2\right) \geq E_0\left(\hat{\theta}^{*2}\right)
\end{aligned}
$$

$$
\because E_0\left(\hat{\theta}^* T\right) = T \cdot E_0\left(\hat{\theta}^*\right) = T \cdot 0 = 0
$$

Pitman の推定量（p.122　定理 10.3）

誤差 ε_i の確率密度関数を $f\left(\varepsilon_i\right)$ とする.

この時, 最小分散位置共役不偏推定量は

$$
\hat{\theta}^*\left(X_1, \cdots, X_n\right) = \frac{\int \theta \prod_{i=1}^{n} f\left(X_i - \theta\right) d\theta}{\int \prod_{i=1}^{n} f\left(X_i - \theta\right) d\theta}
$$

証明）

$$
\begin{aligned}
n\left(\overline{X} - X_1\right) &= n\overline{X} - \left(X_1 + \cdots + X_1\right) \\
&= \left(X_1 + \cdots + X_n\right) - \left(X_1 + \cdots X_1\right) \\
&= \left(X_2 - X_1\right) + \cdots + \left(X_n - X_1\right) = Y_1 + \cdots + Y_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
E_0\left(X_1 \middle| Y_1, \cdots, Y_{n-1}\right) &= E_0\left(X_1 - \overline{X} + \overline{X} \middle| Y_1, \cdots, Y_{n-1}\right) \\
&= E_0\left(X_1 - \overline{X} \middle| Y_1, \cdots, Y_{n-1}\right) + E_0\left(\overline{X} \middle| Y_1, \cdots, Y_{n-1}\right) \\
&= X_1 \underbrace{- \overline{X} + E_0\left(\overline{X} \middle| Y_1, \cdots, Y_{n-1}\right)}_{-\hat{\theta}^*}
\end{aligned}
$$

$$
\therefore \hat{\theta}^*\left(X_1, \cdots, X_n\right) = \overline{X} - E_0\left(\overline{X} \middle| Y_1, \cdots, Y_{n-1}\right) = X_1 - E_0\left(X_1 \middle| Y_1, \cdots, Y_{n-1}\right)
$$

$X_1, \cdots, X_n$ の同時確率密度関数: $f\left(x_1, \cdots, x_n\right) = \prod_{i=1}^{n} f\left(x_i\right)$

［$X_1, \cdots, X_n$: i.i.d.（同一独立な分布）］

変数変換

$$
\begin{array}{lcl}
x_1 & \rightarrow & x_1 \\
x_2 & \rightarrow & y_1 = x_2 - x_1 \\
\vdots & & \vdots \\
x_n & \rightarrow & y_{n-1} = x_n - x_{n-1}
\end{array}
$$

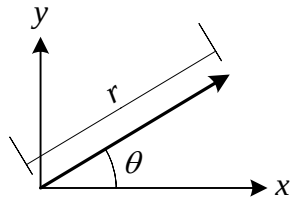
$J = 1$　変数変換のヤコビアン

$$
\begin{aligned}
f\left(x_1, y_1, \cdots, y_{n-1}\right) &= f\left(x_1, \cdots, x_n\right) \\
&= f\left(x_1\right) f\left(x_2\right) \cdots f\left(x_n\right) \\
&= \underbrace{f\left(x_1\right) f\left(y_1 + x_1\right) \cdots f\left(y_{n-1} + x_{n-1}\right)}_{\text{誤差}\varepsilon_i\text{の}\mathrm{PDF}}
\end{aligned}
$$

同時確率密度 → 条件付き確率密度

$$f\left(x_1 \middle| y_1,\cdots,y_{n-1}\right)=\frac{f\left(x_1,y_1,\cdots,y_{n-1}\right)}{f\left(y_1,\cdots,y_{n-1}\right)}$$
$$=\frac{f\left(x_1,y_1,\cdots,y_{n-1}\right)}{\int f\left(x_1,y_1,\cdots,y_{n-1}\right)dx_1}$$

例）Jacobian



$$dxd\theta = Jdrd\theta$$

$$x=r\cos\theta,\ y=r\sin\theta$$

$$J=\begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$
$$=\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$=r\cos^2\theta+r\sin^2\theta=r$$

$$E_0\left(X_1 \middle| Y_1,\cdots,Y_{n-1}\right)=\int x_1\ f\left(x_1 \middle| y_1,\cdots,y_{n-1}\right)dx_1$$
$$=\frac{\int x_1\ f\left(x_1\right)f\left(y_1+x_1\right)\cdots f\left(y_{n-1}+x_{n-1}\right)dx_1}{\int f\left(x_1\right)\cdot f\left(y_1+x_1\right)\cdots f\left(y_{n-1}+x_{n-1}\right)dx_1}$$

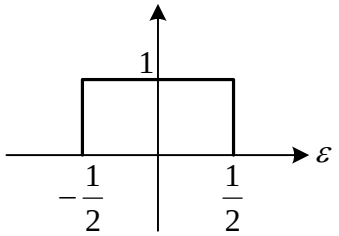
$$\hat{\theta}^*=\overline{X}-E_0\left(\overline{X} \middle| Y_1,\cdots,Y_{n-1}\right)$$
$$=X_1-E_0\left(X_1 \middle| Y_1,\cdots,Y_{n-1}\right)=X_1-\frac{\int x_1\ f\left(x_1\right)f\left(Y_1+x_1\right)\cdots f\left(Y_{n-1}+x_1\right)dx_1}{\int f\left(x_1\right)f\left(Y_1+x_1\right)\cdots f\left(Y_{n-1}+x_1\right)dx_1}$$

$\theta=X_1-x_1$ を代入 $\left[x_1=X_1-\theta\right]$

$$X_1-E_0\left(X_1 \middle| Y_1,\cdots,Y_{n-1}\right)=X_1-\frac{\int\left(X_1-\theta\right)f\left(X_1-\theta\right)f\left(X_2-\theta\right)\cdots f\left(X_n-\theta\right)d\theta}{\int f\left(X_1-\theta\right)f\left(X_2-\theta\right)\cdots f\left(X_n-\theta\right)d\theta}$$
$$=\frac{\int\theta f\left(X_1-\theta\right)f\left(X_2-\theta\right)\cdots f\left(X_n-\theta\right)d\theta}{\int f\left(X_1-\theta\right)f\left(X_2-\theta\right)\cdots f\left(X_n-\theta\right)d\theta}$$

例）一様分布

$$f(\varepsilon)=\begin{cases}1, & |\varepsilon|\le\frac{1}{2}\\ 0, & \text{other wise}\end{cases}$$



$$\frac{\int\theta\prod_{i=1}^{n}f(X_i-\theta)\,d\theta}{\int\prod_{i=1}^{n}f(X_i-\theta)\,d\theta}$$

分母の$\prod_{i=1}^{n}f(X_i-\theta)$が 1 となるのは全ての $f(X_i-\theta)$が 1 の時に限られる.

これを満足するθの範囲

$$\left[\underbrace{\mathrm{Max}(X_i)-0.5}_{\alpha},\ \underbrace{\mathrm{Min}(X_i)+0.5}_{\beta}\right]\therefore\ \frac{\int_\alpha^\beta\theta\cdot 1\,d\theta}{\int_\alpha^\beta 1\,d\theta}=\frac{\frac{1}{2}\left(\beta^2-\alpha^2\right)}{\beta-\alpha}$$

$$=\frac{\alpha+\beta}{2}=\frac{\mathrm{Max}(X_i)+\mathrm{Min}(X_i)}{2}$$

例）$f(\varepsilon)$:ガウス分布　　$f(\varepsilon)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\varepsilon^2}{2\sigma^2}}$

$$\prod_{i=1}^{n}f(X_i-\theta)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(X_i-\theta)^2}{2\sigma^2}}$$

$$=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{\sum_{i=1}^{n}(X_i-\theta)^2}{2\sigma^2}}=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{\left[\sum_{i=1}^{n}X_i^2-2\theta\sum_{i=1}^{n}X_i+n\theta^2\right]}{2\sigma^2}}$$

$$=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{\left(\theta^2-2\theta\frac{\sum X_i}{n}+\frac{\sum X_i^2}{n}\right)}{2\sigma^2/n}}=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{\left[\left(\theta-\frac{\sum X_i}{n}\right)^2+\frac{\sum X_i^2}{n}-\left(\frac{\sum X_i}{n}\right)^2\right]}{2\sigma^2/n}}$$

$$\Rightarrow\frac{\int\theta\prod_{i=1}^{n}f(X_i-\theta)\,d\theta}{\int\prod_{i=1}^{n}f(X_i-\theta)\,d\theta}=\overline{X}\ \text{（標本平均）}$$