Nucleolus

Excess (Dissatisfaction) of coalition S at imputation x

$$e(S,x) = v(S) - \sum_{i\in S} x_i$$

Core → Every coalition has non-positive excess

$$C = \{x=(x_1,\ldots,x_n) \mid \sum_{i\in N} x_i = v(N), \quad x_i \geq v(\{i\}) \;\; \forall i\in N$$
$$e(S, x) \leq 0 \quad \forall S\subseteq N \}$$

Core can be empty or very large

Split the excess equally → Nucleolus

Comparing imputations with respect to excess

For any two imputations x,y ,

construct for each x a vector of excess of each coalition S, $e(S, x)$ $e(S,y)$ in descending order

For these two vectors, compare the values in descending order, starting with the largest.

Consider the first component in which the values are not equal. The imputation with the excess vector that has the lower value at that point is said to be more acceptable（permissible） than the other imputation.

The set of imputations such that there is no imputation that is more acceptable is called the nucleolus.

Comparing imputations in ex. 6-3

Characteristic Function $v(\{A,B,C\}) = 20,$

$v(\{A,B\}) = 6,\ v(\{A,C\}) = 0,\ v(\{B,C\}) = 8,$

$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$

For any imputation $x = (x_A, x_B, x_C)$

$$v(\{A,B,C\}) - (x_A + x_B + x_C) = 0, \quad v(\varnothing) - \sum_{i \in \varnothing} x_i = 0$$

$\rightarrow$ can disregard $\{A, B, C\}, \varnothing$

sufficient to compare $\{A, B\}, \{A,C\}, \{B, C\}, \{A\}, \{B\}, \{C\}$

Example

Compare (6, 0, 14) and (12, 4, 4)

	(6, 0, 14)	(12, 4, 4)
{A,B}	$6 - (6 + 0) = 0$	$6 - (12 + 4) = -10$
{A,C}	$0 - (6 + 14) = -20$	$0 - (12 + 4) = -16$
{B,C}	$8 - (0 + 14) = -6$	$8 - (4 + 4) = 0$
{A}	$0 - 6 = -6$	$0 - 12 = -12$
{B}	$0 - 0 = 0$	$0 - 4 = -4$
{C}	$0 - 14 = -14$	$0 - 4 = -4$

$$(6, 0, 14) \rightarrow (0, \ 0, -6, -6, -14, -20)$$

$$(12, 4, 4) \rightarrow (0, -4, -4, -10, -12, -16)$$

$-4 < 0$; thus, (12, 4, 4) is more acceptable than (6, 0, 14)

Finding the Nucleolus

Nucleolus ： There does not exist an imputation more acceptable than any imputation in the nucleolus

→ must minimize the maximum excess

Linear Programming

$M \rightarrow$ minimize

such that $e(S, x) \leq M \quad \forall S \subseteq N$

$x \in A$

→ 「minimum core」

→ Of these solutions, the second-highest excess must be minimized

…………

Nucleolus is obtained as a solution set of successive linear programming problems.

Nucleolus of ex. 6-3 (1)

Excess of imputation $x = (x_A, x_B, x_C)$

$\{A,B\}$	$v(\{A,B\}) - (x_A+x_B) = 6 - (x_A+x_B)$
$\{A,C\}$	$v(\{A,C\}) - (x_A+x_C) = 0 - (x_A+x_C)$
$\{B,C\}$	$v(\{B,C\}) - (x_B+x_C) = 8 - (x_B+x_C)$
$\{A\}$	$v(\{A\}) - x_A = - x_A$
$\{B\}$	$v(\{B\}) - x_B = - x_B$
$\{C\}$	$v(\{C\}) - x_C = - x_C$

Minimization problem: $\min \ M$

$$6 - (x_A+x_B) \le M, \ -(x_A+x_C) \le M, \ 8 - (x_B+x_C) \le M$$

$$-x_A \le M, \ -x_B \le M, \ -x_C \le M$$

$$x_A + x_B + x_C = 20, \quad x_A, x_B, x_C \ge 0$$

Nucleolus of ex. 6-3 (2)

Minimization problem: $\min\ M$

$$6-(x_A+x_B)\le M,\ -(x_A+x_C)\le M,\ 8-(x_B+x_C)\le M$$

$$-x_A\le M,\ -x_B\le M,\ -x_C\le M$$

$$x_A+x_B+x_C=20,\quad x_A,x_B,x_C\ge 0$$

$$x_A+x_B+x_C=v(\{A,B,C\})=20$$

$$\downarrow$$

$$6-(20-x_C)=-14+x_C\le M,\quad -(20-x_B)=-20+x_B\le M,$$

$$8-(20-x_A)=-12+x_A\le M,\quad -x_A\le M,\ -x_B\le M,\ -x_C\le M$$

$$\downarrow$$

$$-M\le x_A\le 12+M,\ -M\le x_B\le 20+M,\ -M\le x_C\le 14+M$$

$$-3M\le x_A+x_B+x_C=20\le 3M$$

Nucleolus of ex. 6-3 (3)

$-M \leq x_A \leq 12 + M, \quad -M \leq x_B \leq 20 + M, \quad -M \leq x_C \leq 14 + M$

$-3M \leq x_A + x_B + x_C = 20 \leq 46 + 3M$

$\downarrow$

Conditions for (x_A, x_B, x_C) to satisfy the above inequalities

$M \geq -6, \quad M \geq -10, \quad M \geq -7$

$M \geq -20/3, \quad M \geq -26/3$

$\downarrow$

The minimum value of M that satisfies the above inequalities $M = -6$

$\rightarrow \quad x_A = 6 \ (x_B + x_C = 14), \quad 6 \leq x_B \leq 14, \quad 6 \leq x_C \leq 8$

Nucleolus of ex. 6-3 (4)

Minimize the second-highest excess of x that satisfy

$x_A = 6\ (x_B + x_C = 14),\ \ 6 \le x_B \le 14,\ \ 6 \le x_C \le 8$

Excess of imputation $x = (6, x_B, 14 - x_B)\ \ (6 \le x_B \le 8)$

$\{A,B\}$	$v(\{A,B\}) - (x_A + x_B) = 6 - (6 + x_B) = -x_B$
$\{A,C\}$	$v(\{A,C\}) - (x_A + x_C) = 0 - (6 + 14 - x_B) = -20 + x_B$
$\{B,C\}$	$v(\{B,C\}) - (x_B + x_C) = 8 - 14 = -6$
$\{A\}$	$v(\{A\}) - x_A = -6$
$\{B\}$	$v(\{B\}) - x_B = -x_B$
$\{C\}$	$v(\{C\}) - x_C = -14 + x_B$

Nucleolus of ex. 6-3 (5)

Min problem $\min \; M'$

$$-x_B \le M', \; -20 + x_B \le M',$$

$$-x_B \le M', \; -14 + x_B \le M'$$

$\downarrow$

$$-M' \le x_B \le 14 + M'$$

$\downarrow$

Solution $M' = -7$

$$\rightarrow \quad x_B = 7, \; x_C = 14 - x_B = 7$$

Nucleolus $\rightarrow$ $(6, 7, 7)$

Splitting the Costs by nucleolus in ex 6-3

Characteristic Function（Amount of Cost Reduction）

$$v(\{A,B,C\}) = 20,$$

$$v(\{A,B\}) = 6,\ \ v(\{A,C\}) = 0,\ \ v(\{B,C\}) = 8,$$

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

Nucleolus → Split 20 into 6, 7, 7

The amount of cost reduction 70 + 55+ 65 − 170 = ¥20 million

→ divide into ¥ 6 million, ¥7 million, ¥7 million

Payments:	A：	70 − 6 = ¥64 million
	B：	55 − 7 = ¥48 million
	C：	65 − 7 = ¥58 million

Shapley Value

Contribution of player i towards coalition S $(i \notin S)$

$$v(S \cup \{i\}) - v(S)$$

Consider the contribution of each player when the grand coalition (N) is formed by one player joining at a time.

For example, if the grand coalition is formed in the order,

$1, \ldots, i, \ldots, n$, then i's contribution is

$$v(\{1, \ldots, i-1, i\}) - v(\{1, \ldots, i-1\})$$

Assuming that each formation process of the grand coalition occurs with equal probability, each player's expected contribution

$\longrightarrow$ **Shapley value**

Shapley value of ex. 6-3

Characteristic Function $v(\{A,B,C\}) = 20,$

$v(\{A,B\}) = 6, \; v(\{A,C\}) = 0, \; v(\{B,C\}) = 8,$

$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$

	Contribution		
Order that N forms	A	B	C
$A \leftarrow B \leftarrow C$	0	6	14
$A \leftarrow C \leftarrow B$	0	20	0
$B \leftarrow A \leftarrow C$	6	0	14
$B \leftarrow C \leftarrow A$	12	0	8
$C \leftarrow A \leftarrow B$	0	20	0
$C \leftarrow B \leftarrow A$	12	8	0

Shapley value (5, 9, 6)

Splitting Costs by Shapley Value

Characteristic function（amount of cost reduction）

$v(\{A,B,C\}) = 20,$

$v(\{A,B\}) = 6,\ \ v(\{A,C\}) = 0,\ \ v(\{B,C\}) = 8,$

$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$

Shapley value → split 20 by 5,9,6

The total amount of cost reduction 70 + 55+ 65ー170 = ¥20 million

→ split into ¥5 million, ¥9 million, ¥6 million

Payment for each player

A： 70ー5 = ¥65 million

B： 55ー9 = ¥46 million

C： 65ー6 = ¥59 million

Is the Shapley value an imputation?

Characteristic function of ex. 6-3

$$v(\{A,B,C\}) = 20,$$

$$v(\{A,B\}) = 6,\ \ v(\{A,C\}) = 0,\ \ v(\{B,C\}) = 8,$$

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

Shapley value $(5, 9, 6)$ $\longrightarrow$ imputation

In general, the Shapley value

satisfies group rationality

satisfies individual rationality if (N, v) is <u>superadditive</u>

Shapley value and Group rationality

Order	A	B	C
$A \leftarrow B \leftarrow C$	$v(A)-v(\emptyset)$	$v(AB)-v(A)$	$v(ABC)-v(AB)$
$A \leftarrow C \leftarrow B$	$v(A)-v(\emptyset)$	$v(ABC)-v(AC)$	$v(AC)-v(A)$
$B \leftarrow A \leftarrow C$	$v(AB)-v(B)$	$v(B)-v(\emptyset)$	$v(ABC)-v(AB)$
$B \leftarrow C \leftarrow A$	$v(ABC)-v(BC)$	$v(B)-v(\emptyset)$	$v(BC)-v(B)$
$C \leftarrow A \leftarrow B$	$v(AC)-v(C)$	$v(ABC)-v(AC)$	$v(C)-v(\emptyset)$
$C \leftarrow B \leftarrow A$	$v(ABC)-v(BC)$	$v(BC)-v(C)$	$v(C)-v(\emptyset)$

The Shapley value of each player is calculated by taking the sum along each column and dividing the result by 3!=6

The summation of the Shapley value for each player

$\rightarrow$ sum of all the above entries divided by $3!$

Addition by rows gives $v(ABC)$

$\rightarrow$ the sum of all entries above $3!\,v(ABC)$

The sum of the Shapley values $v(ABC)$

Shapley value and Individual Rationality

Order	A	B	C
$A \leftarrow B \leftarrow C$	$v(A) - v(\varnothing)$	$v(AB) - v(A)$	$v(ABC) - v(AB)$
$A \leftarrow C \leftarrow B$	$v(A) - v(\varnothing)$	$v(ABC) - v(AC)$	$v(AC) - v(A)$
$B \leftarrow A \leftarrow C$	$v(AB) - v(B)$	$v(B) - v(\varnothing)$	$v(ABC) - v(AB)$
$B \leftarrow C \leftarrow A$	$v(ABC) - v(BC)$	$v(B) - v(\varnothing)$	$v(BC) - v(B)$
$C \leftarrow A \leftarrow B$	$v(AC) - v(C)$	$v(ABC) - v(AC)$	$v(C) - v(\varnothing)$
$C \leftarrow B \leftarrow A$	$v(ABC) - v(BC)$	$v(BC) - v(C)$	$v(C) - v(\varnothing)$

By superadditivity,

every entry in column A $\geq v(A)$, every entry in column B $\geq v(B)$,

every entry in column C $\geq v(C)$

$\rightarrow$ A's Shapley value $\geq v(A)$, same for B,C

Axiomatization of the Shapley Value （L.S.Shapley）

Characeristic function form game (N, v)

$\rightarrow$ solution $x^* = (x^*_1, \ldots, x^*_n)$

Properties that x^* should satisfy

1	Group Rationality
2	Null Player Property
3	Symmetry
4	Additivity

There is only one solution x^* that satisfies the above 4 properties: the Shapley value.

4 Properties (1)

1 Group Rationality

$$\sum_{i \in N} x^*_i = v(N)$$

2 Null Player Property

Player i is a null player

$\Leftrightarrow$ For every coalition S $(i \notin S)$, $v(S \cup \{i\}) - v(S) = 0$

If player i is a null player, then $x^*_i = 0$

4 Properties (2)

3 Equal Treatment Property (Symmetry)

Players i, j are symmetric

$\Leftrightarrow$ For every coalition S ($i, j \notin S$), $v(S \cup \{i\}) = v(S \cup \{j\})$

If players i, j are symmetric, $x^*_i = x^*_j$

4 Additivity

Given two games (N, v), (N, u) , define a new game (N, w) such that for every coalition S, $w(S) = v(S) + u(S)$

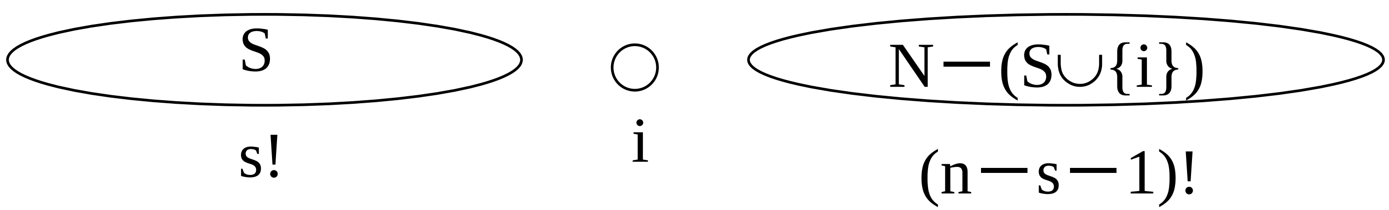
Then, if the solutions to (N, v), (N, u), (N, w) are x^*, y^*, z^* respectively, then for any player $i \in N$, $z^*_i = x^*_i + y^*_i$

The Formula of Shapley Value

Shapley value

the expected contribution of each player over all n ! orderings occurring with equal probability

Contribution of player i towards coalition S (the term $v(S\cup\{i\})-v(S)$) occurs in $s! \times (n-s-1)!$ orderings.



Player i’s Shapley value:

$$x^*_i = (1/n!) \sum_{S\subseteq N,\ i\notin S} s! \times (n-s-1)!(v(S\cup\{i\})-v(S))$$

Assignment due next lecture

<u>Reading assignment</u>

Handout： Multi-person cooperative game

(nucleolus and Shapley value)

<u>Homework</u>

Problem Set 2: #1, 2, 3 (nucleolus and Shapley value)

（Use A4-size paper,

and staple on the upper left-hand side）