Games in Characteristic Function Form

1. Chataceristic Function Form Games

   - $(N = \{1, 2, ..., n\}, v)$
     $N = \{1, 2, ..., n\}$: set of players
     $v : 2^N \to \Re$: characteristic function
     2^N: collection of subsets of N, $S \subseteq N$: coalition
   - $v(S)$: the maximum payoff that a coalition S can guarantee
   - (N, v) is a **superadditive** game $\Leftrightarrow$
     for every $S, T \subseteq N, S \cap T = \emptyset,, v(S) + v(T) \leq v(S \cup T)$

2. imputation

   - $x = (x_1, x_2, ..., x_n)$: payoff vector
     payoff vector $x = (x_1, x_2, ..., x_n)$ is an **imputation** $\Leftrightarrow$
     $\sum_{i=1}^{n} x_i = v(N)$ (efficiency, group rationality)
     $x_i \geq v(\{i\})\ \forall i = 1, ..., n$ (individual rationality)
   - Set of imputations, A, can be expressed as
     $A = \{x = (x_1, x_2, ..., x_n) \in \Re^n | \sum_{i=1}^{n} x_i = v(N), x_i \geq v(\{i\})\ \forall i = 1, ..., n\}$

3. Core

   - A set of imputations C is the **core** $\Leftrightarrow$
     $C = \{x \in A | \sum_{i \in S} x_i \geq v(S)\ \forall S \subseteq N\}$
   - $\sum_{i \in S} x_i \geq v(S)$: coalitional rationality
   - $e(S, x) = v(S) - \sum_{i \in S} x_i$: excess of coalition S at imputation x
   - core $\Leftrightarrow$ a set of imputations in which no coalition S has a positive excess value
   - Dominance Core
     - Dominance:
       For two imputations $x, y \in A$, if there is a coalition $S \subseteq N$ such that the two conditions below are satisfied, then x is said to dominate y via coalition S, (noted as $x\ dom_S\ y$)
       * $x_i > y_i\ \forall i \in S$
       * $\sum_{i \in S} x_i \leq v(S)$

       If there exists some S such that $x\ dom_S\ y$, then x is said to dominate y, written as $x\ dom\ y$.
     - The set of imputations that are not dominated DC is called the **dominance core**. That is,

       $$DC = \{x \in A | \text{there does not exist } y \in A \text{ such that } y\ dom\ x\}$$

     - $C \subseteq DC$ always holds.

– If (N, v) is superadditive, $DC \subseteq C$ also holds, and $C = DC$.

4. Nucleolous

- For every $x \in A$, denote by $\theta(x)$ an ordered vector that orders the components of $e(S,x)$ $(S \subseteq N, S \neq N, \emptyset)$ in descending order.
  $\theta(x) = (e(S_1,x), e(S_2,x), ..., e(S_{2^n-2}))$
  $e(S_1,x) \geq e(S_2,x) \geq ... \geq e(S_{2^n-2}))$
- For any two imputations $x, y \in A$, x is more **acceptable** than $y \Leftrightarrow$
  $\theta(y)$ is *lexicographically greater* than $\theta(x)$ (denoted $\theta(y) >_L \theta(x)$) $\Leftrightarrow$
  there exists $k \in \{1, ..., 2^n - 2\}$ such that
  $\theta_i(x) = \theta_i(y)\ \forall i = 1, ..., k-1$
  $\theta_k(x) < \theta_k(y)$
- A set of imputations L is the **nucleolus**
  $L = \{x \in A |$there is no y such that y is more acceptable than x $\}$
- The nucleolus always exists and contains exactly one element.
- If the core is nonempty, then the nucleolus is contained in the core.

5. Shapley value

- Marginal contribution of player $i \in N$ towards coalition $S, i \notin S$
  $v(S \cup \{i\}) - v(S)$
- given a permutation (or reordering) of players $\pi = (\pi(1), \pi(2), ..., \pi(n))$ contribution of player $\pi(k)$
  $v(\{\pi(1), ..., \pi(k-1), \pi(k)\}) - v(\{\pi(1), ..., \pi(k-1)\})$
  $\pi(1), ..., \pi(k-1)$: players that precede $\pi(k)$ according to permutation π
- contribution of i with respect to permutation π
  $v(P^{\pi,i} \cup \{i\}) - v(P^{\pi,i})$
  $P^{\pi,i}$: the set of players that precede i with respect to permutation π
- Shapley value of player i
  $\psi_i = \frac{1}{n!} \sum_{\pi \in \Pi} (v(P^{\pi,i} \cup \{i\}) - v(P^{\pi,i}))$
  Π : set of all permutations

  Shapley value
  $\psi = (\psi_1, ..., \psi_n)$
  assuming that a permutation of a set of n players ($n!$ of them) occurs with equal probability, Shapley value is each player's expected contribution
- Shapley value satisfies efficiency.
  If (N, v) is supseradditive, then the Shapley value is individually rational; thus, it is an imputation.
- An alternative expression of the Shapley value
  $\psi_i = \sum_{S:S \subseteq N, i \notin S} \frac{s!(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S))$
  $s = |S|$: number of players in coalition S

6. Axiomatization of the Shapley value

- Fix a set of players N, Denote by V the set of all superadditive characteristic functions $v : 2^N \to \Re$.
  For every game $(N, v), v \in V$, let ϕ be a function $\phi : V \to \Re^n$ and $\phi(v) = (\phi_1(v), ..., \phi_n(v))$.
- Axioms
  (a) Efficiency
  For every $v \in V$, $\sum_{i \in N} \phi_i(v) = v(N)$
  (b) Null Player Property
  A player $i \in N$ is a **null player** $\Leftrightarrow v(S \cup \{i\}) - v(S) = 0 \ \forall S \subseteq N, i \notin S$
  If player i is a null player, $\phi_i(v) = 0$
  (c) Symmetry (Equal Treatment)
  Players $i, j \in N$ are **symmetric** $\Leftrightarrow v(S \cup \{i\}) = v(S \cup \{j\}) \ \forall S \subseteq N, i, j \notin S$
  If players i, j are symmetric, then $\phi_i(v) = \phi_j(v)$
  (d) Additivity
  For any two characteristic functions $v, u \in V$, define $w \in V$ by
  $w(S) = v(S) + u(S) \ \forall S \subseteq N$.
  Then, $\phi(w) = \phi(v) + \phi(u)$
- Theorem
  There is only function ϕ that satisfies efficiency, no award for null players, symmetry, and additivity and for each game (N, v), ϕ is given by
  $\phi_i(v) = \sum_{S: S \subseteq N, i \notin S} \frac{s!(n-s-1)!}{n!}(v(S \cup \{i\}) - v(S)) \ \forall i \in N$