

Questions in Cooperative Game Theory

Through communication among players,

- 1 What coalitions will form ?
- 2 How will payoffs be divided among players in the coalitions that are formed ?

Form : Games in Characteristic Function Form
(Coalitional Form)

Coalitional Form Games

$N = \{1, 2, \dots, n\}$: set of players

$S \subseteq N$: Coalition

$v : 2^N \rightarrow \mathbb{R}$: Characteristic Function (TU-games)

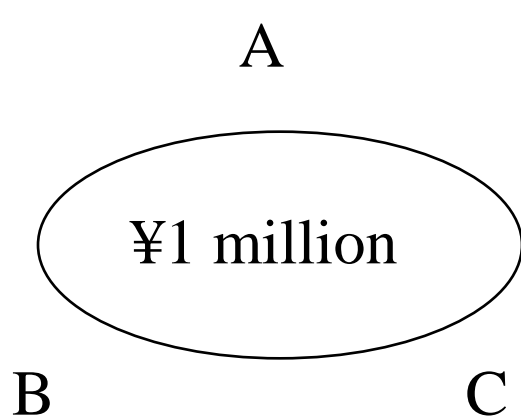
(2^N : set of subsets of N , \mathbb{R} : set of real numbers)

$v(S)$: the maximum payoff that coalition S can obtain for sure regardless of what $N - S$ does

$v(\emptyset) = 0$ (\emptyset : empty set)

(N, v) : Characteristic function form games
(Coalitional form games)

Example 6-1



4 Proposals

- 1 Divide between A and B
- 2 Divide between A and C
- 3 Divide between B and C
- 4 Divide among all three
(renegotiate)

A, B, C each has one vote
Choose one by majority rule

Which coalition will form ?

How do they divide ¥1 million among themselves ?

Ex 6-1 as a coalitional form game

Set of players: $N = \{A, B, C\}$

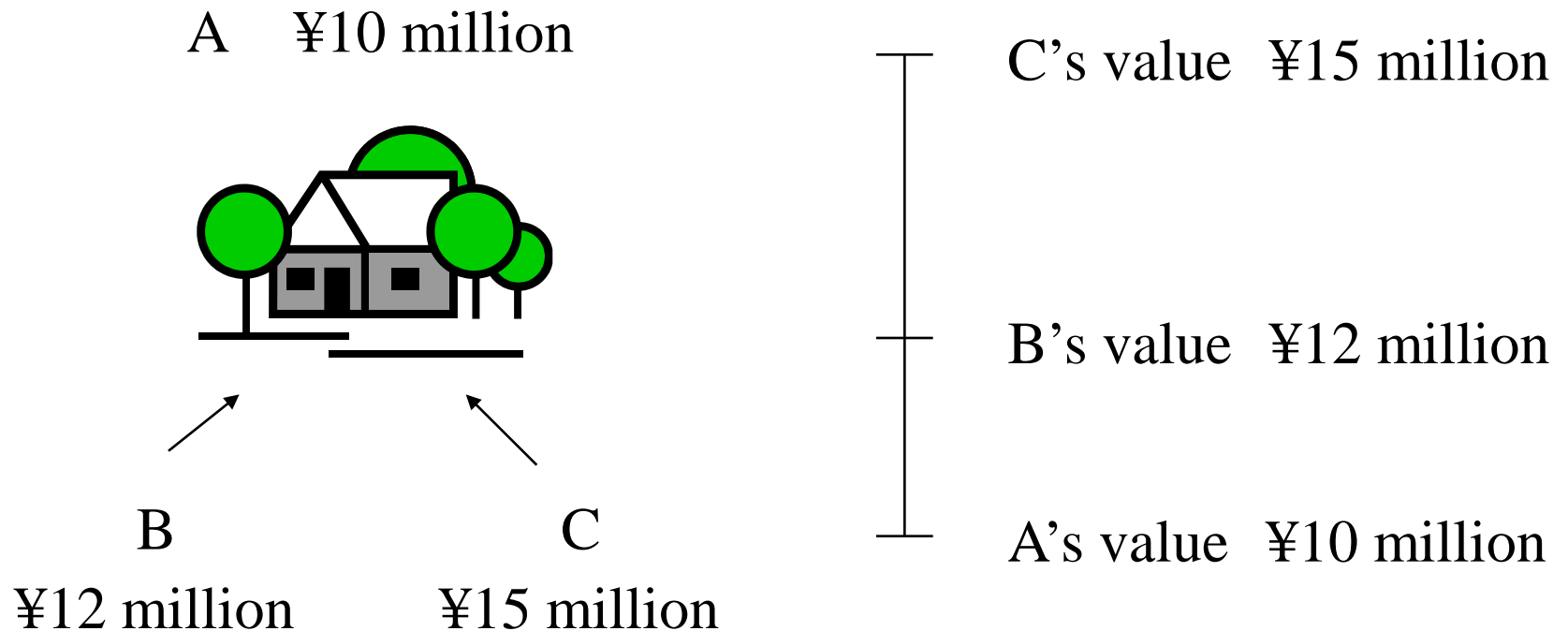
Characteristic function:

$$v(\{A, B, C\}) = 1,$$

$$v(\{A, B\}) = v(\{A, C\}) = v(\{B, C\}) = 1,$$

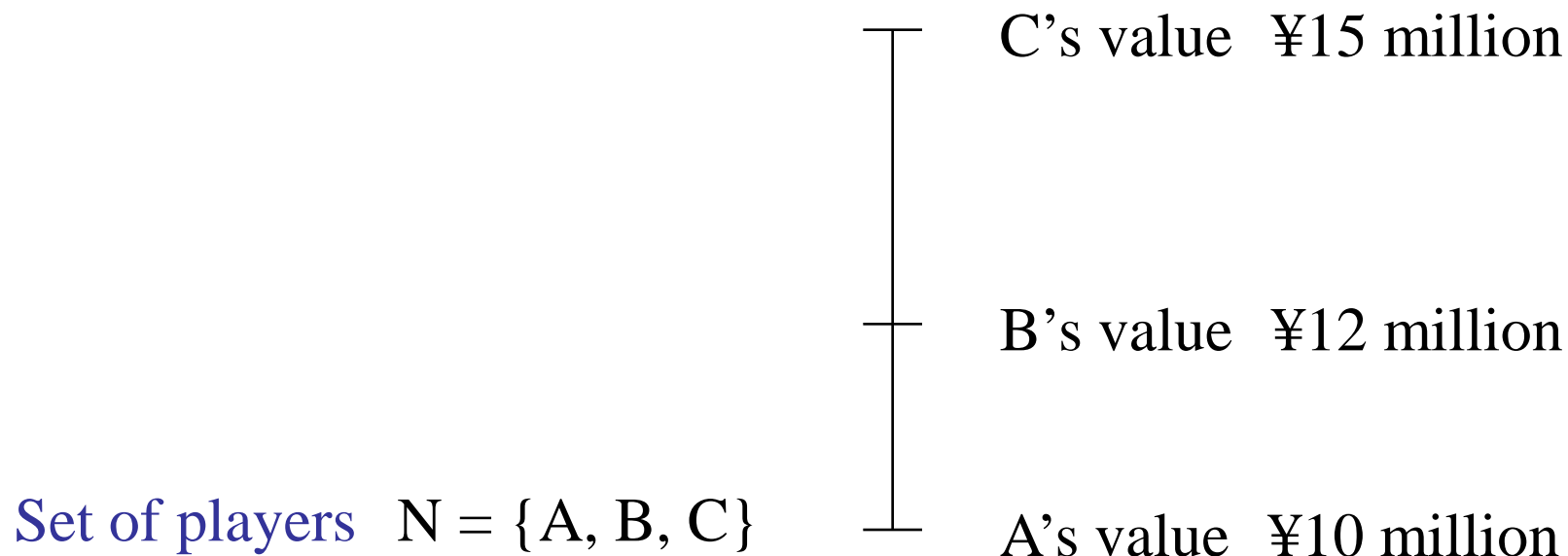
$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

Ex. 6-2



To whom (B or C) and what price will A's house be sold ?

Ex. 6-2 as a coalitional form game



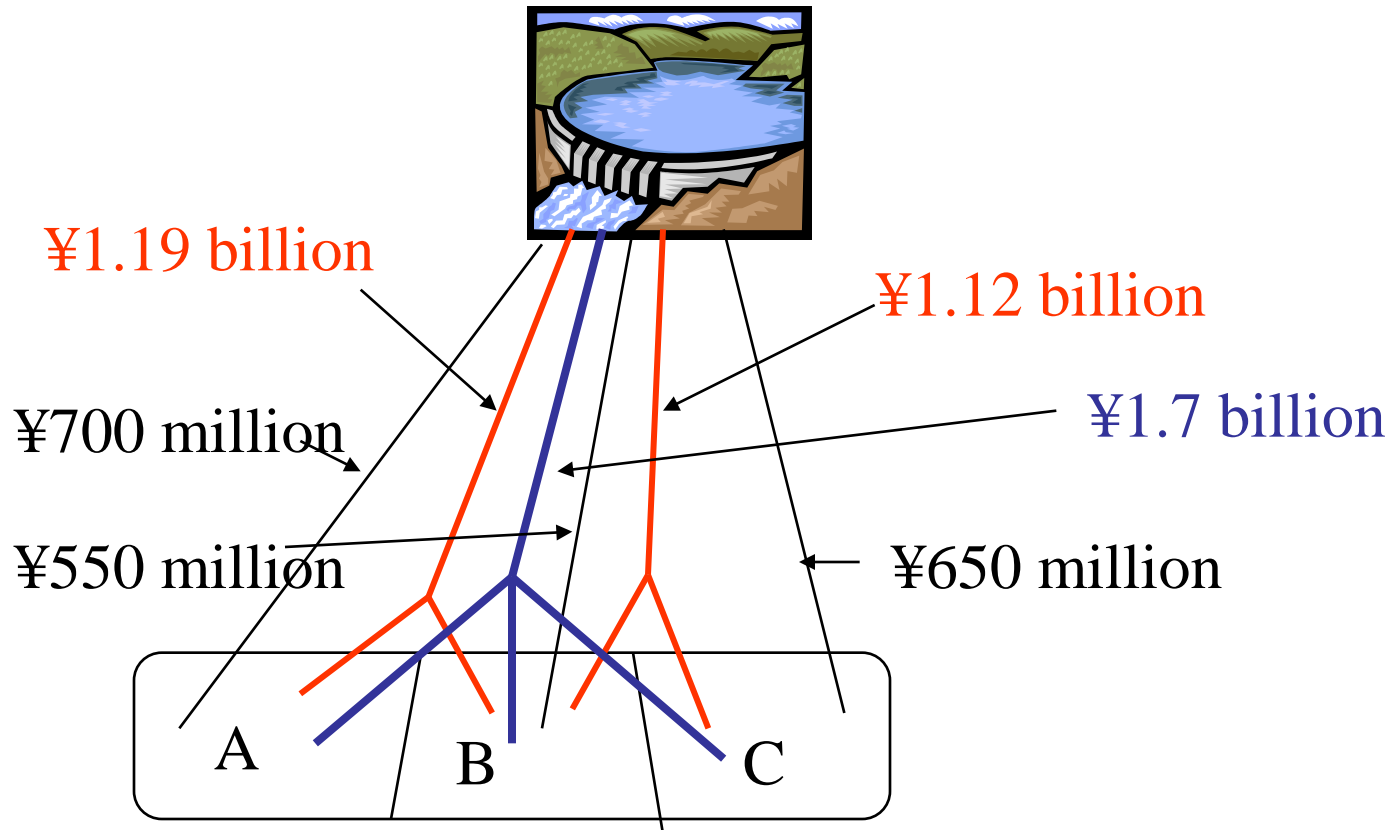
Characteristic function:

$$v(\{A, B, C\}) = 5,$$

$$v(\{A, B\}) = 2, \quad v(\{A, C\}) = 5, \quad v(\{B, C\}) = 0,$$

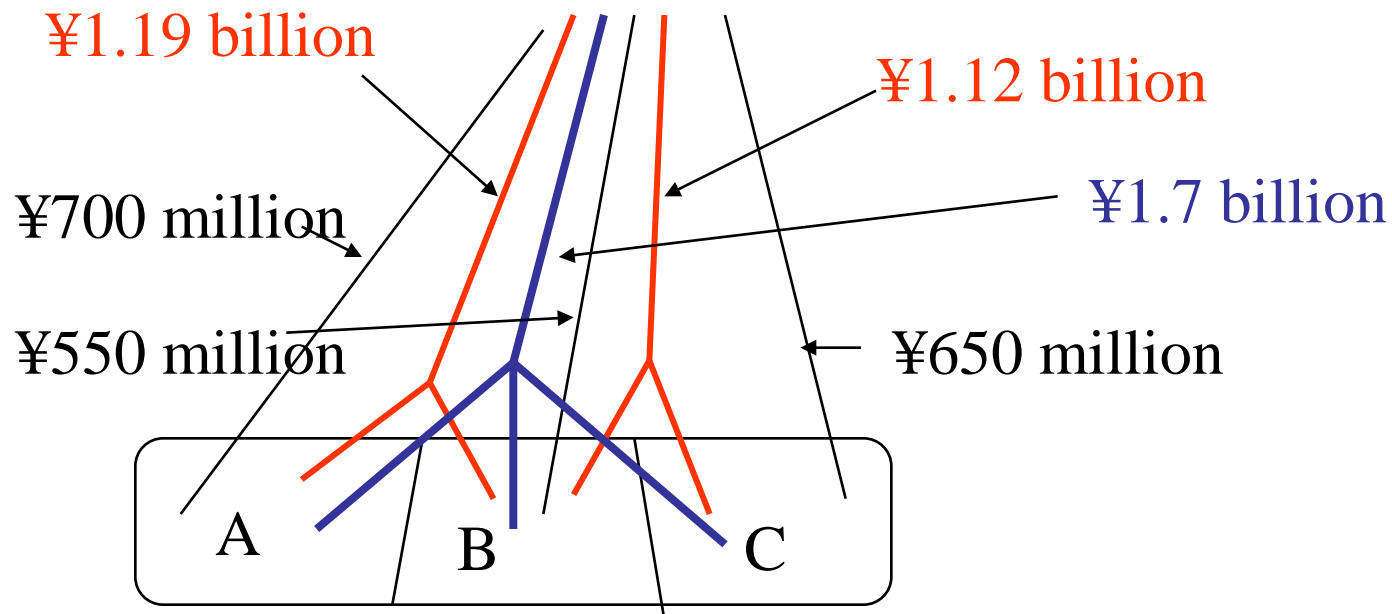
$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

Ex. 6-3



How will A,B, and C cooperate, and divide the joint costs ?

Ex. 6-3 as a coalition form game



Set of players: $N = \{A, B, C\}$

Characteristic function: (cost reduction)

$$v(\{A, B, C\}) = 20,$$

$$v(\{A, B\}) = 6, \quad v(\{A, C\}) = 0, \quad v(\{B, C\}) = 8,$$

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

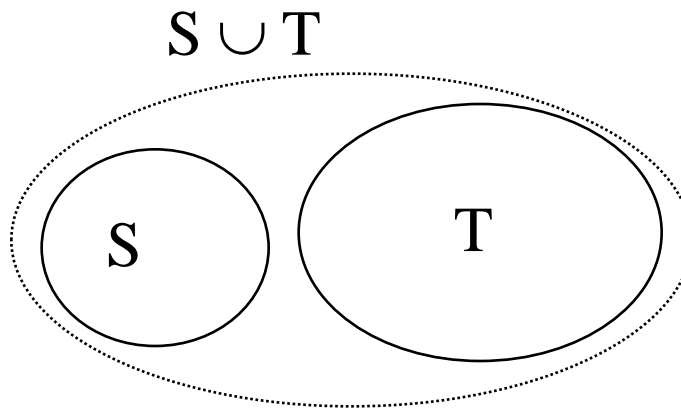
Superadditivity

Superadditivity

Characteristic function form game (N, v) is **superadditive**

\Leftrightarrow for every S, T with $S \cap T = \emptyset$

$$v(S) + v(T) \leq v(S \cup T)$$



Ex. 6-3

Characteristic function of ex. 6-3

$$v(\{A,B,C\}) = 20,$$

$$v(\{A,B\}) = 6, \quad v(\{A,C\}) = 0, \quad v(\{B,C\}) = 8,$$

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

$$v(\{A,B\}) = 6 > 0 = v(\{A\}) + v(\{B\})$$

$$v(\{A,B,C\}) = 20 > 6 = v(\{A,B\}) + v(\{C\}) \quad \text{etc.}$$

Superadditivity and Grand Coalition Formation

Superadditivity

→ grand coalition N will form

Questions in Cooperative Game Theory

1 What kind of coalitions will form ?

→ Assumed that grand coalition will form

2 How will payoffs be divided among players ?

→ cooperative game theory up until now



Recently, more focus on coalition formation

Imputation

How will payoffs be divided among players ?

Payoff vector : $x = (x_1, x_2, \dots, x_n)$

x_i : i 's payoff

Payoff vector : $x = (x_1, x_2, \dots, x_n)$ is an **imputation**

$$1 \quad x_1 + x_2 + \dots + x_n = v(N)$$

$$2 \quad x_i \geq v(\{i\}) \quad \forall i = 1, 2, \dots, n$$

1 Pareto optimality (efficiency) or Group rationality

2 Individual rationality

Ex. 6-3 Imputation Set

$$N = \{A, B, C\}$$

$$v(\{A, B, C\}) = 20,$$

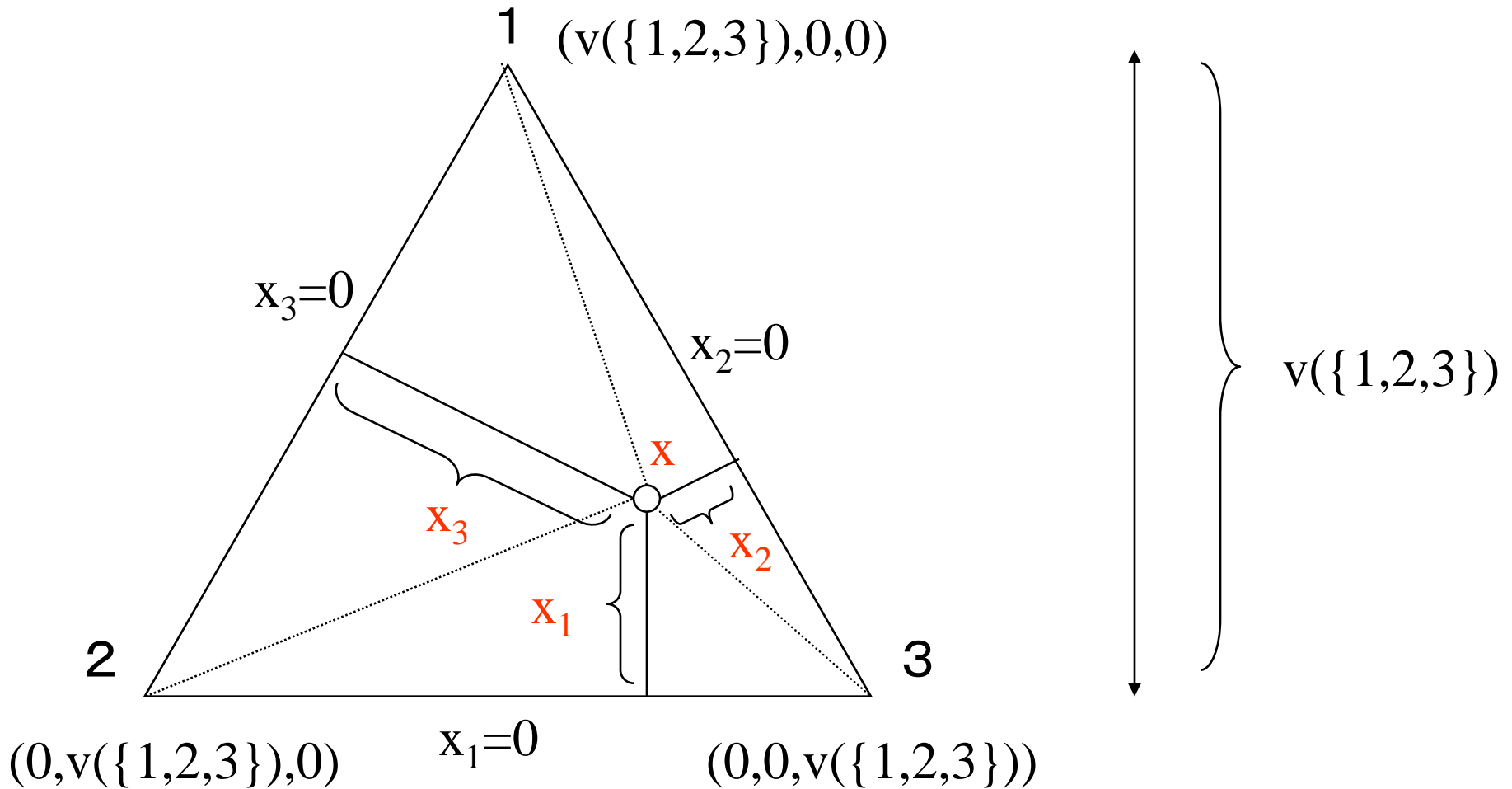
$$v(\{A, B\}) = 6, \quad v(\{A, C\}) = 0, \quad v(\{B, C\}) = 8,$$

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

Imputation Set

$$A = \{x = (x_A, x_B, x_C) \mid x_A + x_B + x_C = 20, \quad x_A, x_B, x_C \geq 0\}$$

Diagram representing Imputation Set



Core

Imputation Set

$$A = \{x = (x_1, \dots, x_n) \mid \sum_{i \in N} x_i = v(N), \quad x_i \geq v(\{i\}) \quad \forall i \in N\}$$

Core $C = \{x = (x_1, \dots, x_n) \mid \sum_{i \in N} x_i = v(N), \quad x_i \geq v(\{i\}) \quad \forall i \in N$

$$\sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N \}$$



Coalitional Rationality

excess of imputation x for coalition S (dissatisfaction of S for x)

$$e(S, x) = v(S) - \sum_{i \in S} x_i$$

Core $C = \{x = (x_1, \dots, x_n) \mid \sum_{i \in N} x_i = v(N), \quad x_i \geq v(\{i\}) \quad \forall i \in N$

$$e(S, x) \leq 0 \quad \forall S \subseteq N \}$$

Core of Ex 6-1

Characteristic function $v(\{A,B,C\}) = 1,$

$$v(\{A,B\}) = v(\{A,C\}) = v(\{B,C\}) = 1,$$

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

Core $C = \{x=(x_A, x_B, x_C) \mid x_A + x_B + x_C = 1, x_A \geq 0, x_B \geq 0, x_C \geq 0,$

$$x_A + x_B \geq 1, x_A + x_C \geq 1, x_B + x_C \geq 1 \}$$

$$x_A + x_B \geq 1, x_A + x_C \geq 1, x_B + x_C \geq 1 \rightarrow x_A + x_B + x_C \geq 3/2$$

$$\rightarrow x_A + x_B + x_C = 1 \text{ (contradiction)}$$



Core $C = \emptyset$

Core of Ex. 6-2

Characteristic function

$$v(\{A,B,C\}) = 5,$$

$$v(\{A,B\}) = 2, \quad v(\{A,C\}) = 5, \quad v(\{B,C\}) = 0,$$

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

$$\text{Core} \quad C = \{x=(x_A, x_B, x_C) \mid x_A + x_B + x_C = 5, x_A \geq 0, x_B \geq 0, x_C \geq 0,$$

$$x_A + x_B \geq 2, x_A + x_C \geq 5, x_B + x_C \geq 0 \}$$

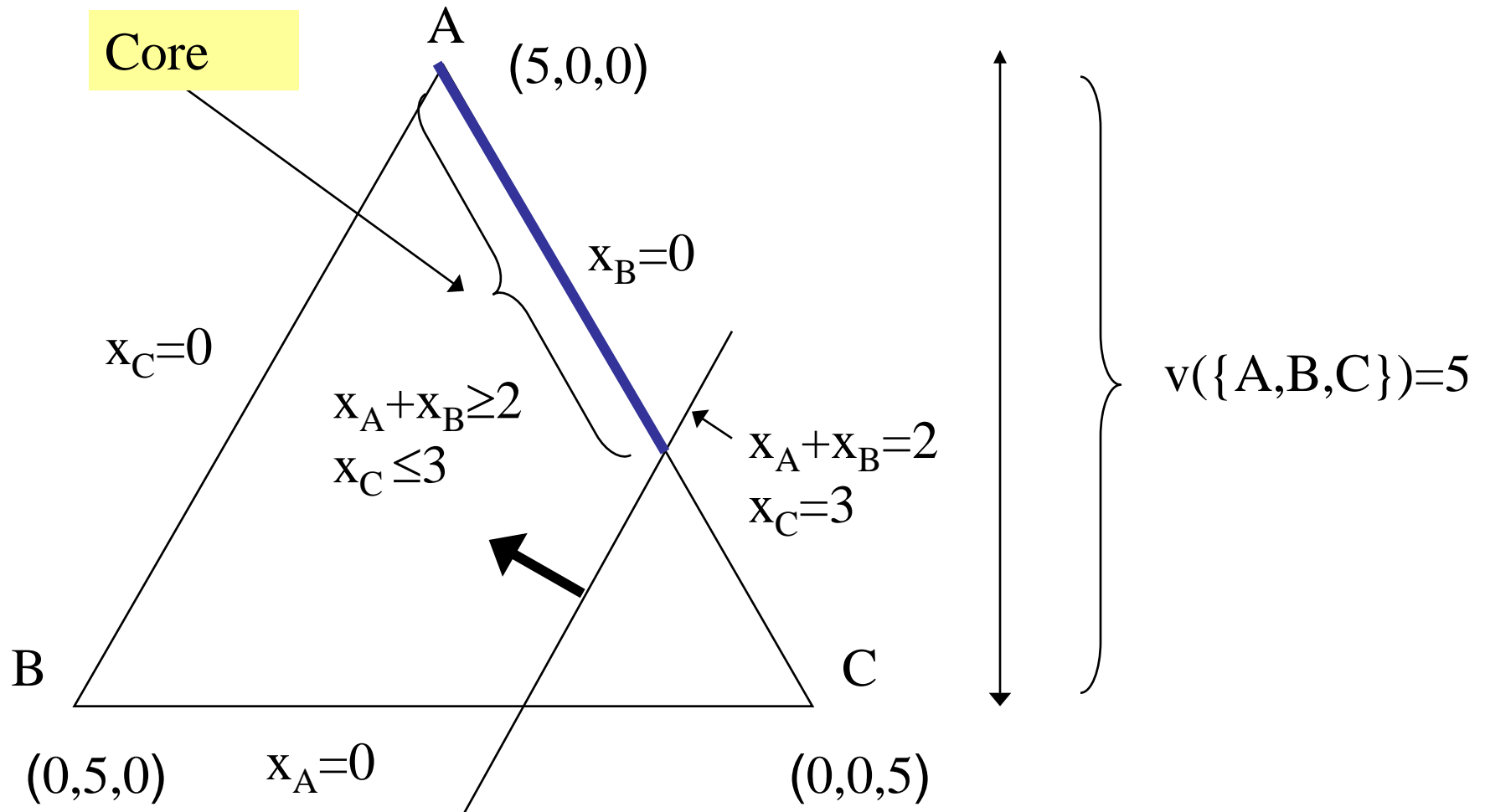


$$x_A + x_C = 5 \quad (x_B = 0), \quad x_A \geq 2$$

$x_B = 0 \rightarrow$ trade between A and C

$x_A \geq 2 \quad (x_C \leq 3) \rightarrow$ sold at a price of at least ¥12 million

Illustration of the Core of Ex. 6-2



Core of Ex 6-3

Characteristic function

$$v(\{A,B,C\}) = 20,$$

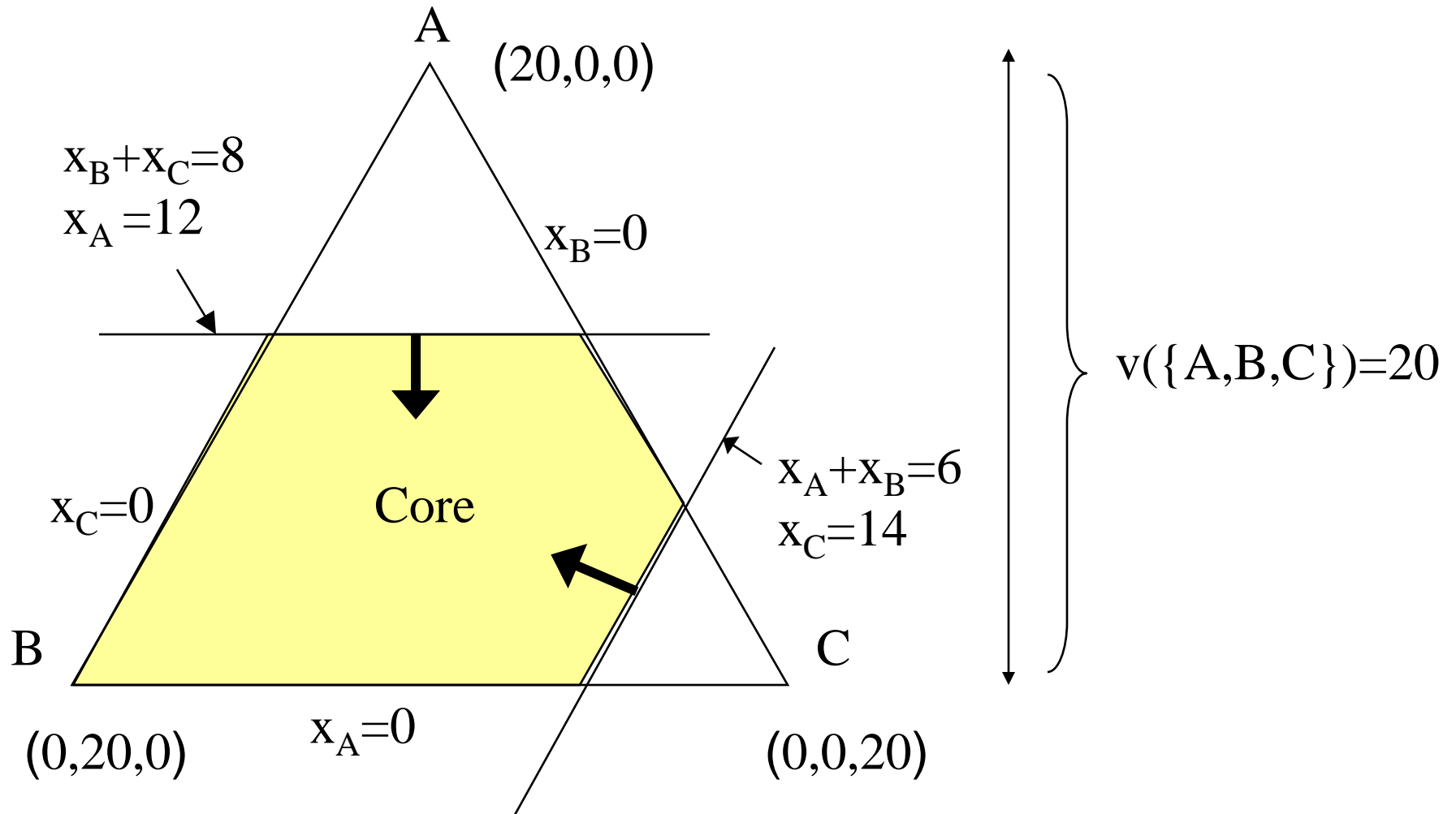
$$v(\{A,B\}) = 6, \quad v(\{A,C\}) = 0, \quad v(\{B,C\}) = 8,$$

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$$

$$\begin{aligned} \text{Core} \quad C = \{ x = (x_A, x_B, x_C) \mid & x_A + x_B + x_C = 20, \quad x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0, \\ & x_A + x_B \geq 6, \quad x_A + x_C \geq 0, \quad x_B + x_C \geq 8 \} \end{aligned}$$

Large Core \rightarrow contains $(0,20,0)$, $(6,0,14)$

Illustration of the Core of Ex 6-3



Strengths and Weaknesses of the Core

Strengths: easy to understand

no coalition has an excess

Weaknesses: could be empty

could be quite large

Using the concept of excess, is there a solution that always exists and is not too large ?

→ **Nucleolus**

Assignment due next lecture

Reading assignment

“Introduction to Game Theory”: pp. 161-179

(“Game Theory”: pp.293 - 308)

Handout: Multi-person cooperative game

Homework

Problem Set 2: #1, 2, 3 (core)

(Use A4-size paper,

and staple on the upper left-hand side)