Two-person Bargaining Game

1. **Two-person Strategic Form Game**

$$(N = \{1, 2\}, (S_1 = \{s_1, ..., s_m\}, S_2 = \{t_1, ..., t_n\}), (g_1, g_2))$$
$$g_1(s_i, t_j) = a_{ij}, g_2(s_i, t_j) = b_{ij}$$

   (a) Correlated Strategy
   $r = (r_{11}, ..., r_{mn}),\ \sum_{i=1}^m \sum_{j=1}^n r_{ij} = 1,\ r_{ij} \geq 0,\ i = 1, ..., m, j = 1, ..., n$
   r_{ij}: probability that (s_i, t_j) is chosen

   (b) Expected Payoff
   $u_1 = \sum_{i=1}^m \sum_{j=1}^n a_{ij} r_{ij},\ u_2 = \sum_{i=1}^m \sum_{j=1}^n b_{ij} r_{ij}$

   (c) Feasible Set
   $R = \{u = (u_1, u_2) | u_1 = \sum_{i=1}^m \sum_{j=1}^n a_{ij} r_{ij},\ u_2 = \sum_{i=1}^m \sum_{j=1}^n b_{ij} r_{ij}\}$

   (d) Disagreement Point
   $u^0 = (u_1^0, u_2^0)$
   (e.g. maximin value, minimax value, Nash equilibrium outcome)

2. **Bargaining Problem** (R, u^0)

   (a) R: a convex and compact (closed and bounded) subset of $\Re^2$ (two-dimensional Euclidean space)

   (b) $u^0 \in R$

   (c) there is a $u = (u_1, u_2) \in R$ such that $u_1 > u_1^0, u_2 > u_2^0$

   Denote by $\mathcal{B}$ the set of all bargaining problems (R, u^0)

   - R is *convex* $\Leftrightarrow$ for any $u, v \in R$ and for any $\alpha (0 \leq \alpha \leq 1)$, $\alpha u + (1 - \alpha) v \in R$
   - R is *bounded* $\Leftrightarrow$ there exists $M \in \Re_+$ such that for any$u = (u_1, u_2) \in R$, $-M \leq u_1, u_2 \leq M$
   - R is *closed* $\Leftrightarrow$ for any sequence $u^1, u^2, ... \in R$ such that $u^n \to u$, $u \in R$.

3. **Nash Bargaining Solution**

   A function $f : \mathcal{B} \to \Re^2$ that satisfies the following four axioms:

   (a) (Strong) Pareto optimality
   For every $(R, u^0) \in B$
   $f(R, u^0) = (f(R, u^0)_1, f(R, u^0)_2)$ must be a strong Pareto optimal alternative in R.

   (Definition of Strong Pareto Optimality)
   $u = (u_1, u_2)$ is (strong) Pareto optimal in $R \Leftrightarrow$
   if there is a $u' \in R$ with $u'_1 \geq u_1, u'_2 \geq u_2$, then $u' = u$

(b) Symmetry
If (R, u^0) is symmetric then $f(R, u^0)_1 = f(R, u^0)_2$

(Definition of Symmetry for (R, u^0)
(R, u^0) is symmetric $\Leftrightarrow$
(1)if $(u_1, u_2) \in R$, then $(u_2, u_1) \in R$
(2)$u_1^0 = u_2^0$

(c) Independence of Strictly Positive Affine Transformation
For (R, u^0) define (R', u'^0) as follows
$R' = \{u' = (u'_1, u'_2) | u'_1 = \alpha_1 u_1 + \beta_1, u'_2 = \alpha_2 u_2 + \beta_2, u = (u_1, u_2) \in R\}$
$u_1'^0 = \alpha_1 u_1^0 + \beta_1,$
$u_2'^0 = \alpha_2 u_2^0 + \beta_2$
$\alpha_1 > 0, \alpha_2 > 0, \beta_1, \beta_2$ are constants

$f(R', u'^0)_1 = \alpha_1 f(R, u^0)_1 + \beta_1,$
$f(R', u'^0)_2 = \alpha_2 f(R, u^0)_2 + \beta_2$

(d) Independence of Irrelevant Alternatives
For (R, u^0) if there exists $T \subseteq R$ such that $f(R, u^0) \in T, u^0 \in T$, then
$f(T, u^0) = f(R, u^0)$

4. **Existence and Uniqueness of Nash Bargaining Solution**
There exists a unique $f : \mathcal{B} \rightarrow \Re^2$ that satisfies the above four axioms. Moreover, for any bargaining problem $(R, u^0) \in \mathcal{B}$ $f(R, u^0)$ solves

$$max\{(u_1 - u_1^0)(u_2 - u_2^0) | (u_1, u_2) \in R, u_1 \geq u_1^0, u_2 \geq u_2^0\}$$

This f is the Nash bargaining solution.