Problem Set 1

1. For the following bargaining games, find the Nash bargaining solution by (a) solving the maximization problem and by (b) using *only* the four axioms.

   (a) R is the closed region inside the triangle with vertices $(0,0),(9,0),(0,6)$ and the disagreement point is $u^0=(0,0)$

   (b) R is the closed region inside the triangle with vertices $(0,0),(9,0),(0,6)$ and the disagreement point is $u^0=(3,2)$

   (c) R is the closed region inside the triangle with vertices $(0,0),(8,0),(0,8)$ and the disagreement point is $u^0=(2,1)$

   (d) R is the closed region inside the quadrilateral with vertices $(0,0),(0,6),(6,3),(8,0)$ and the disagreement point is $(0,0)$

   (e) R is the closed region inside the quadrilateral with vertices $(0,0),(0,6),(6,3),(8,0)$ and the disagreement point is $(2,2)$

2. Proof of the Existence and Uniqueness of the Nash Bargaining Solution
   Nash's Theorem
   There is only one solution $f:B\to\Re^2$ that satisfies Pareto optimality, Symmetry, Preservation under Strictly Increasing Affine Transformation B nd Independence of Irrelevant Alternatives. Moreover, for any (R,u^0), $f(R,u^0)$ solves

$$max\{(u_1-u_1^0)(u_2-u_2^0)|(u_1,u_2)\in R, u_1\geq u_1^0, u_2\geq u_2^0\}$$

   and the solution (u_1,u_2) to the above maximization problem is unique. This f is called the Nash bargaining solution
   Let B be the set of bargaining problems (R,u^0) such that

   - R is a convex and compact subset of $\Re^2$
   - $u^0=(u_1^0,u_2^0)\in R$.
   - There is a $(u_1,u_2)\in R$ such that $u_1>u_1^0, u_2>u_2^0$

   (Proof)

   (a) Let f be a function such that for each (R,u^0), $f(R,u^0)$ is the solution to the maximization problem above. To show that f above is well-defined as a function (i.e. $f(R,u^0)$ is single-valued for each (R,u^0))
   Let $H(u_1,u_2)=(u_1-u_1^0)(u_2-u_2^0)$ and let $R'=\{u\in R|u_1\geq u_1^0,u_2\geq u_2^0\}$
   Because R is compact, R' is also compact
   Because H is a continuous function on R', H attains a maximum on R'
   (Problem) Prove the following statements.

      i. If $s*=(s*_1,s*_2)$ is a maximizer for H on R', then $s*_1>u_1^0$ and $s*_2>u_2^0$
      ii. R' is convex
      iii. There is only one such $s*=(s*_1,s*_2)$; therefore f is a well-defined function
      (Hint)Suppose there is another maximizer $t*=(t*_1,t*_2)$ in R', that is different from $s*$; define $r*=(r*_1,r*_2)=((s*_1+t*_1)/2,(s*_2+t*_2)/2)$
      Show that $H(r*_1,r*_2)>H(s*_1,s*_2)$ and $(r*_1,r*_2)\in R'$, which contradicts the maximality of $s*$

(b) (Problem) Show that f satisfies Pareto optimality, Symmetry, Preservation under Strictly Increasing Affine Transformation B nd Independence of Irrelevant Alternatives.

(c) To show that f is the unique solution that satisfies the four axioms:

Let $g : B \to \Re^2$ be another solution that satisfies Pareto optimality, Symmetry, Preservation under Strictly Increasing Affine Transformation, and Independence of Irrelevant Alternatives.

It is sufficient to show that for each(R, u^0), $f(R, u^0) = g(R, u^0)$

Take any (R, u^0) and let $u* = f(R, u^0)$

i. Consider the following affine transformation and let R' be the set of (u'_1, u'_2) defined below $((u_1, u_2) \in R)$

$$u'_1 = \frac{u_1}{2(u*_1 - u_1^0)} - \frac{u_1^0}{2(u*_1 - u_1^0)}$$
$$u'_2 = \frac{u_2}{2(u*_2 - u_2^0)} - \frac{u_2^0}{2(u*_2 - u_2^0)}$$

ii. (Problem) Show that under the transformation defined above,
- $(u*_1, u*_2)$ is transformed to $(1/2, 1/2)$
- (u_1^0, u_2^0) is transformed to $(0, 0)$

iii. Therefore, $f(R', (0, 0)) = (1/2, 1/2)$ and by axiom 3 (Preservation under Strictly Increasing Affine Transformation), it is sufficient to show $g(R', (0, 0)) = (1/2, 1/2)$

iv. For each $u' = (u'_1, u'_2) \in R'$ it can be shown that $u'_1 + u'_2 \leq 1$ has to hold.
- Suppose $u'_1 + u'_2 > 1$ for some (u'_1, u'_2)
- For a small $\epsilon, 0 \leq \epsilon \leq 1$, consider $(1 - \epsilon)(1/2, 1/2) + \epsilon(u'_1, u'_2)$
- (Problem) Show that this point lies in R'
- (Problem) Show that for sufficiently small ϵ the product of the two coordinates of this point exceed $1/4$
- This contradicts $f(R', u^0) = (1/2, 1/2)$.

v. Let T be any triangle that is symmetric with respect to the 45^0 line and contains R' and that $(1/2, 1/2)$ is Pareto optimal within T. Because R is bounded, such T must exist. By Pareto optimality and symmetry, $g(T, (0, 0)) = (1/2, 1/2)$. $R' \subseteq T$ and $(0, 0), (1/2, 1/2) \in R'$, which implies (by independence of irrelevant alternatives $g(R', (0, 0)) = (1/2, 1/2)$.