応用数値解析特論I

Projection法による混相流体解析

②圧力（非移流相）の解法

Fractional Step法

- Fractional Step法ー物理過程ごとに方程式を分割して計算

時間発展：$\left(u_{ij}^{n}, v_{ij}^{n}, \phi_{ij}^{n}\right) \Rightarrow \left(u_{ij}^{n+1}, v_{ij}^{n+1}, \phi_{ij}^{n+1}\right)$

①：移流相

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = 0$$

を計算して、ϕ_{ij}^{n+1}

と仮の流速 u_{ij}^{*}, v_{ij}^{*}

を計算する

Fractional Step法

時間発展：$\left(u_{ij}^{n}, v_{ij}^{n}, \phi_{ij}^{n}\right) \Rightarrow \left(u_{ij}^{n+1}, v_{ij}^{n+1}, \phi_{ij}^{n+1}\right)$

②：非移流相

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g$$

を計算して、次時刻の流速 $u_{ij}^{n+1}, v_{ij}^{n+1}$ を計算する

非移流相の計算を組み込む！

非移流相の計算手順

１：密度の決定

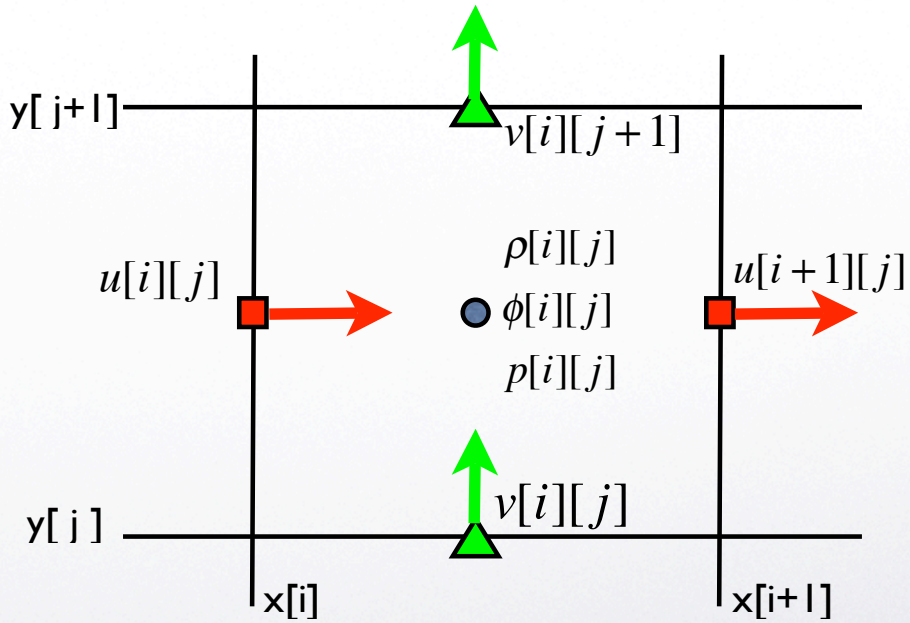
２：圧力をSOR法を用いて解く

３：求められた圧力をもとに速度を修正する。

非移流相の計算手順

1 ：密度の決定

移流相の計算の結果、ColorFuncionは時刻t_{n+1}の値になっている。



計算セルの中心における

密度 $\rho[i][j]$ は、

$$\rho_{ij}^{n+1} = \rho[i][j] = \begin{cases} \rho_{air} & \text{for } \phi_{ij}^{n} = \phi[i][j] < 0.5 \\ \rho_{water} & \text{for } \phi_{ij}^{n} = \phi[i][j] \geq 0.5 \end{cases}$$

$$\rho_{air} = 1.3 kg/m^3, \quad \rho_{water} = 1000 kg/m^3$$

非移流相の計算手順

1：密度の決定

```
////////////////////////////////////////////
// Density is determined from Color Function
////////////////////////////////////////////
rho_water=1.e3;
rho_air=1.3e0;
for(i=0;i<=Nx;i++){
  for(j=0;j<=Ny;j++){
    if(color[i][j]>0.5){
      rho[i][j]=rho_water;
    }else{
      rho[i][j]=rho_air;
    }
}}
```

非移流相の計算手順

２：圧力をSOR法を用いて解く

圧力のPoisson方程式の導出

次時刻の流速 $u_{ij}^{n+1}, v_{ij}^{n+1}$ の値

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g$$

$$u^{n+1} = u^{Adv} - \frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\Delta t$$

$$v^{n+1} = v^{Adv} - \frac{1}{\rho}\frac{\partial p^{n+1}}{\partial y}\Delta t - g\Delta t$$

u^{Adv}, v^{Adv}:移流相の計算結果　p^{n+1} :次時刻での圧力

非移流相の計算手順

連続条件式：非圧縮性が次の時刻でも成り立つとすると

$$\frac{\partial u^{n+1}}{\partial x}+\frac{\partial v^{n+1}}{\partial y}=0$$

前のスライドの計算式を代入して、

$$\frac{\partial}{\partial x}\left(u^{Adv}-\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\Delta t\right)+\frac{\partial}{\partial y}\left(v^{Adv}-\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial y}\Delta t-g\Delta t\right)=0$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial y}\Delta t\right)=\frac{1}{\Delta t}\left(\frac{\partial u^{Adv}}{\partial x}+\frac{\partial v^{Adv}}{\partial y}\right)$$

圧力のPoisson方程式　（⇔非圧縮性）

非移流相の計算手順

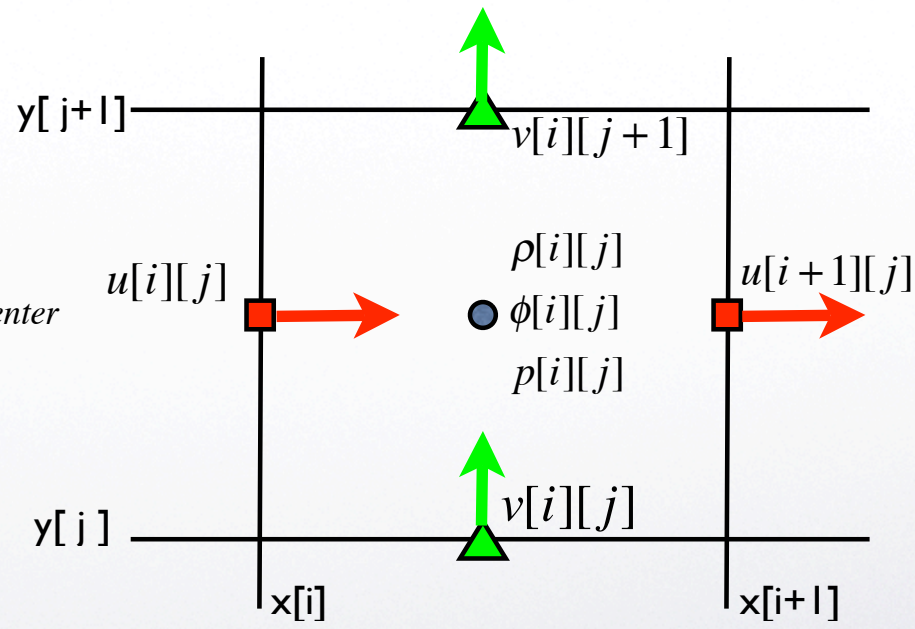
圧力のPoisson方程式の離散化(計算セルの中心で成り立つ)

計算セルの中心： $p_{ij}^{n+1} = p[i][j]$

右辺：

$$(\nabla \cdot \mathbf{u})_{center} = \left(\frac{\partial u^{Adv}}{\partial x}\right)_{center} + \left(\frac{\partial v^{Adv}}{\partial y}\right)_{center}$$

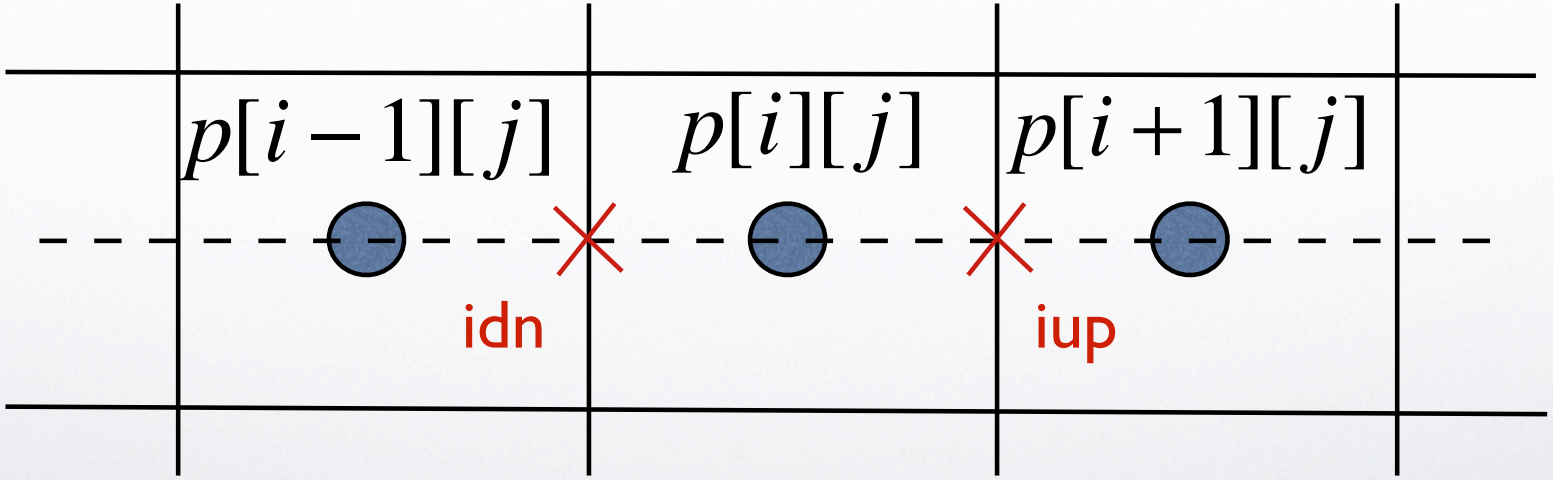
$$= \frac{u^{Adv}[i+1][j] - u^{Adv}[i][j]}{\Delta x} + \frac{v^{Adv}[i][j+1] - v^{Adv}[i][j]}{\Delta y}$$



非移流相の計算手順

圧力のPoisson方程式の離散化(計算セルの中心で成り立つ)

左辺：
$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)_{center} = \frac{1}{\Delta x}\left(\left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)_{iup} - \left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)_{idn}\right)$$

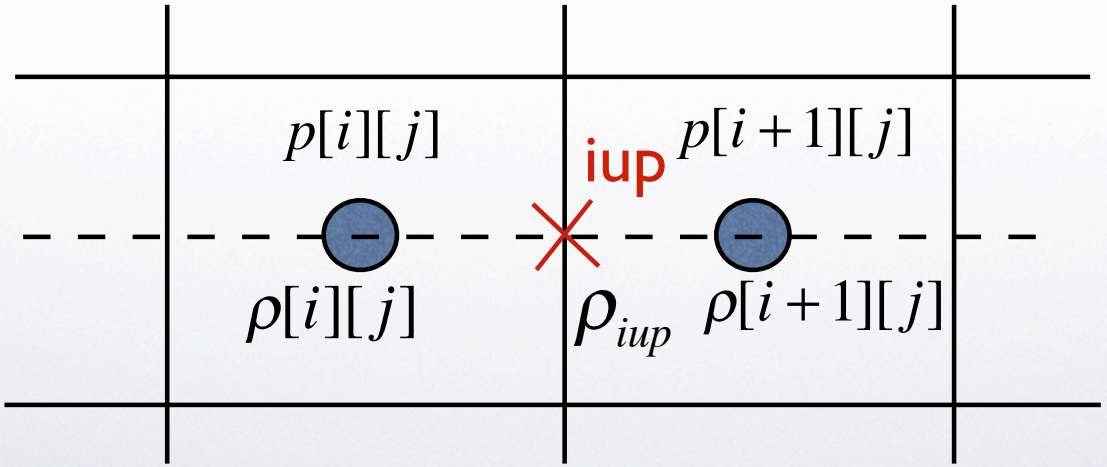


非移流相の計算手順

圧力のPoisson方程式の離散化(計算セルの中心で成り立つ)

さらに：
$$\left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)_{iup} = \frac{1}{\rho_{iup}}\frac{p[i+1][j]-p[i][j]}{\Delta x}$$

ただし、$\rho_{iup} = \frac{1}{2}\left(\rho[i+1][j]+\rho[i][j]\right)$



非移流相の計算手順

圧力のPoisson方程式の離散化(計算セルの中心で成り立つ)

左辺：
$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)_{center} = \frac{1}{\Delta x}\left(\left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)_{iup} - \left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)_{idn}\right)$$

$$= \frac{1}{\Delta x}\left(\frac{1}{\rho_{iup}}\frac{p[i+1][j]-p[i][j]}{\Delta x} - \frac{1}{\rho_{idn}}\frac{p[i][j]-p[i-1][j]}{\Delta x}\right)$$

$$= \frac{1}{\Delta x^2\rho_{iup}}p[i+1][j] - \left(\frac{1}{\Delta x^2\rho_{iup}} + \frac{1}{\Delta x^2\rho_{idn}}\right)p[i][j] + \frac{1}{\Delta x^2\rho_{idn}}p[i-1][j]$$

ただし、

$$\rho_{iup} = \frac{1}{2}\left(\rho[i+1][j] + \rho[i][j]\right)$$

$$\rho_{idn} = \frac{1}{2}\left(\rho[i-1][j] + \rho[i][j]\right)$$

非移流相の計算手順

圧力のPoisson方程式の離散化(計算セルの中心で成り立つ)

y方向にも同様に考えれば、

$$\frac{1}{\Delta x^2 \rho_{iup}} p[i+1][j] + \frac{1}{\Delta x^2 \rho_{idn}} p[i-1][j] + \frac{1}{\Delta y^2 \rho_{jup}} p[i][j+1] + \frac{1}{\Delta y^2 \rho_{jdn}} p[i][j-1]$$

$$-\left(\frac{1}{\Delta x^2 \rho_{iup}} + \frac{1}{\Delta x^2 \rho_{idn}} + \frac{1}{\Delta y^2 \rho_{jup}} + \frac{1}{\Delta y^2 \rho_{jdn}} \right) p[i][j]$$

$$= \frac{1}{\Delta x \Delta t}\left(u^{Adv}[i+1][j] - u^{Adv}[i][j] \right) + \frac{1}{\Delta y \Delta t}\left(v^{Adv}[i][j+1] - v^{Adv}[i][j] \right)$$

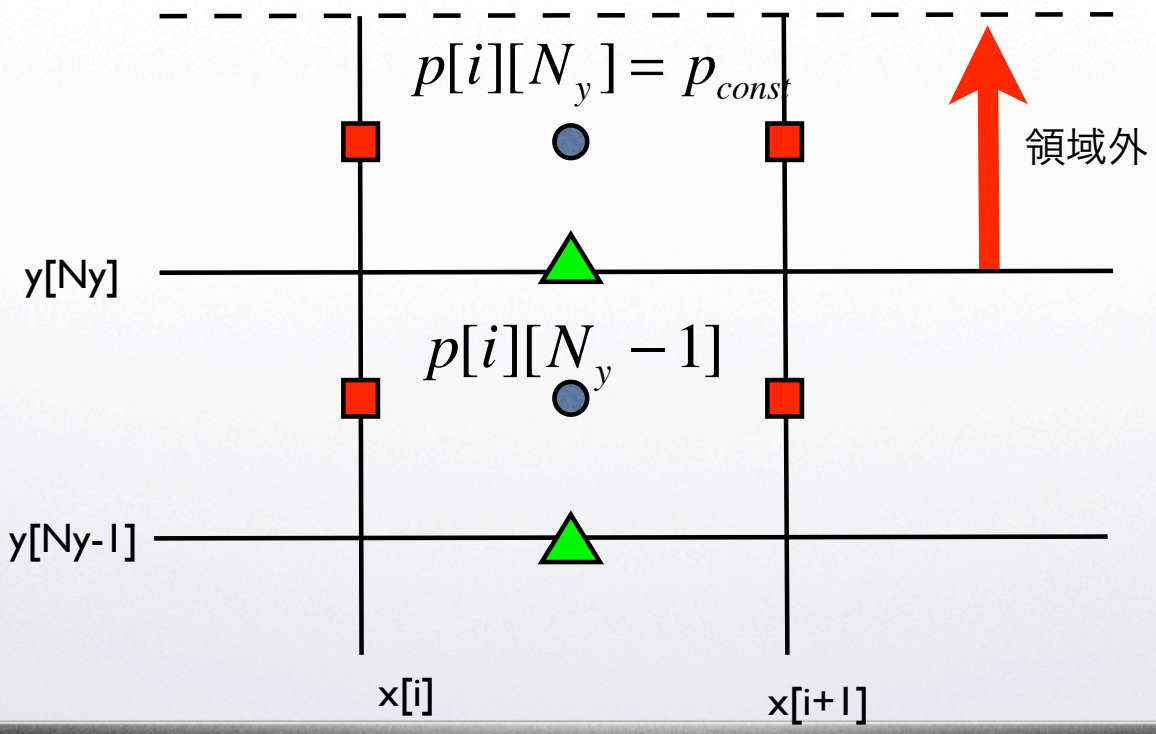
$$\rho_{iup} = \frac{1}{2}\left(\rho[i+1][j] + \rho[i][j] \right), \rho_{idn} = \frac{1}{2}\left(\rho[i-1][j] + \rho[i][j] \right)$$

$$\rho_{jup} = \frac{1}{2}\left(\rho[i][j+1] + \rho[i][j] \right), \rho_{jdn} = \frac{1}{2}\left(\rho[i][j-1] + \rho[i][j] \right)$$

圧力の境界条件

- 上側境界：自由大気（圧力一定）

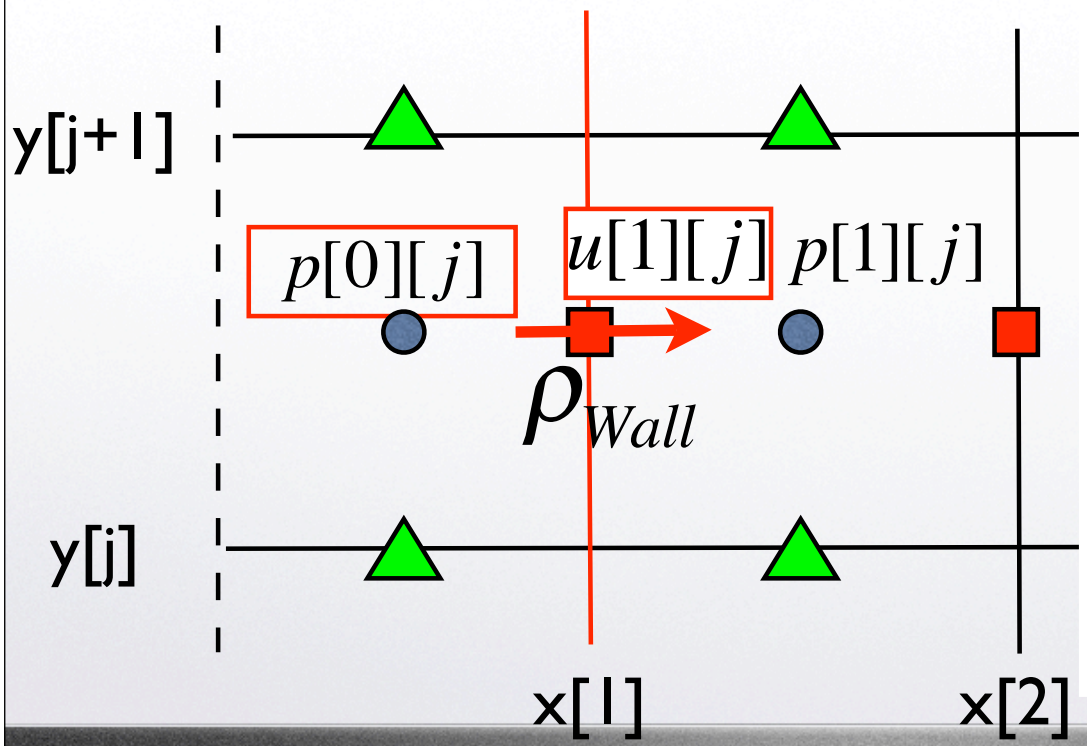
$$p[i][N_y] = p_{const} = 1013[\mathrm{hpa}]$$



圧力の境界条件

- 左側境界：壁

u^{n+1} は壁面上で0 $\Rightarrow$ $\left(u^{n+1}\right)_{Wall} = \left(u^{Adv}\right)_{Wall} - \left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\right)_{Wall} \Delta t = 0$



$$\left(\frac{\partial p^{n+1}}{\partial x}\right)_{Wall} = \frac{\rho_{Wall}}{\Delta t}\left(u^{Adv}\right)_{Wall}$$

$$\frac{p[1][j] - p[0][j]}{\Delta x} = \frac{\rho_{Wall}}{\Delta t} u^{Adv}[1][j]$$

$$p[0][j] = p[1][j] - \frac{\rho_{Wall}\Delta x}{\Delta t} u^{Adv}[1][j]$$

$$\rho_{Wall} = \frac{1}{2}\left(\rho[1][j] + \rho[0][j]\right)$$

圧力の境界条件

- 右側境界：壁

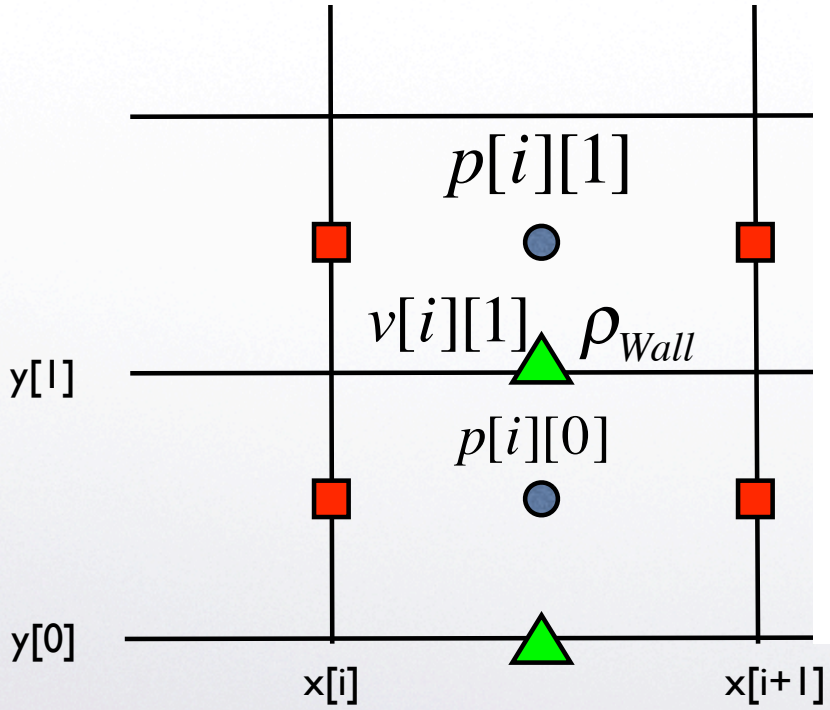
$$p[N_x][j] = p[N_x - 1][j] + \frac{\rho_{Wall}\Delta x}{\Delta t} u^{Adv}[N_x][j]$$

$$\rho_{Wall} = \frac{1}{2}\left(\rho[N_x][j] + \rho[N_x - 1][j]\right)$$

圧力の境界条件

- 下側境界：壁

v^{n+1} は壁面上で0



$$v^{n+1} = v^{Adv} - \frac{1}{\rho}\frac{\partial p^{n+1}}{\partial y}\Delta t - g\Delta t$$

$$v^{n+1}[i][1] = v^{Adv}[i][1] - \left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial y}\right)_{Wall}\Delta t - g\Delta t = 0$$

$$\left(\frac{1}{\rho}\frac{\partial p^{n+1}}{\partial y}\right)_{Wall} = \frac{1}{\Delta t}v^{Adv}[i][1] - g$$

$$\frac{p[i][1] - p[i][0]}{\Delta y} = \frac{\rho_{Wall}}{\Delta t}v^{Adv}[i][1] - \rho_{Wall}g$$

$$p[i][0] = p[i][1] + \rho_{Wall}\Delta y\left(g - v^{Adv}[i][1]/\Delta t\right)$$

非移流相の計算手順

３：求められた圧力をもとに速度を修正する。

$$u^{n+1} = u^{Adv} - \frac{1}{\rho}\frac{\partial p^{n+1}}{\partial x}\Delta t$$

$$v^{n+1} = v^{Adv} - \frac{1}{\rho}\frac{\partial p^{n+1}}{\partial y}\Delta t - g\Delta t$$

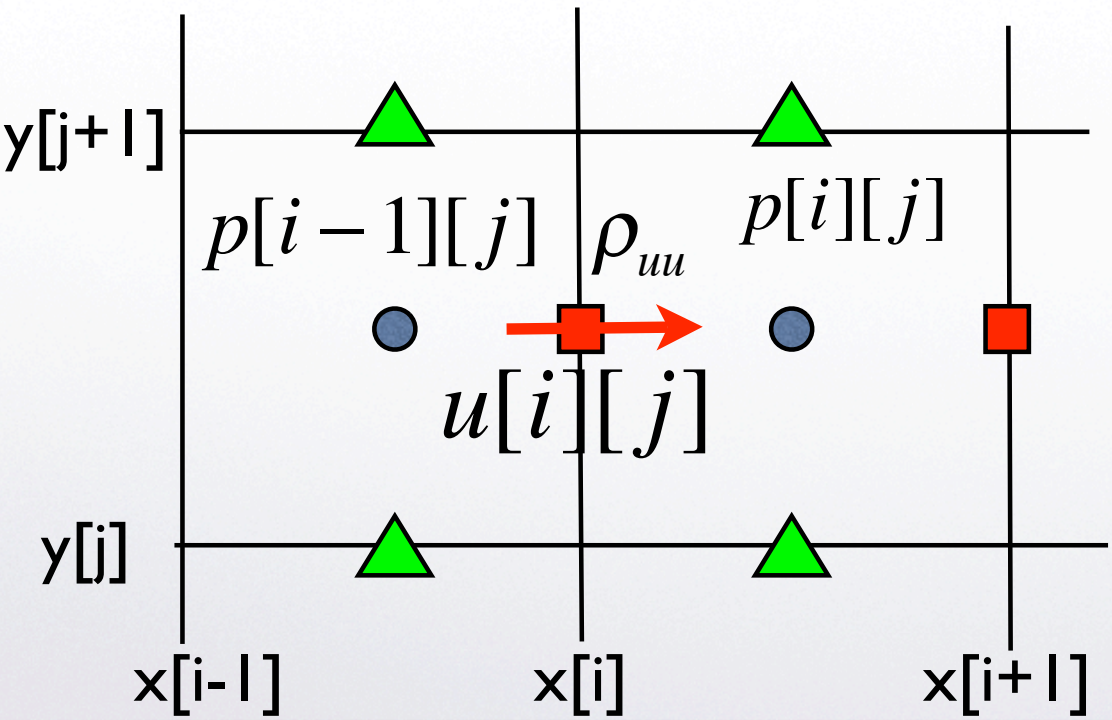
$$u^{n+1}[i][j] = u^{Adv}[i][j] - \frac{1}{\rho_{uu}}\frac{\Delta t}{\Delta x}\left(p[i][j] - p[i-1][j]\right)$$

$$v^{n+1}[i][j] = v^{Adv}[i][j] - \frac{1}{\rho_{vv}}\frac{\Delta t}{\Delta y}\left(p[i][j] - p[i][j-1]\right) - g\Delta t$$

$$\rho_{uu} = \frac{1}{2}\left(\rho[i][j] + \rho[i-1][j]\right), \rho_{vv} = \frac{1}{2}\left(\rho[i][j] + \rho[i][j-1]\right)$$

流速の圧力加速

$$u^{n+1}[i][j] = u^{Adv}[i][j] - \frac{\Delta t}{\rho_{uu}} \frac{1}{\Delta x}\left(p[i][j] - p[i-1][j]\right)$$



$$\left(\frac{\partial p}{\partial x}\right)_{uu} = \frac{p[i][j] - p[i-1][j]}{\Delta x}$$

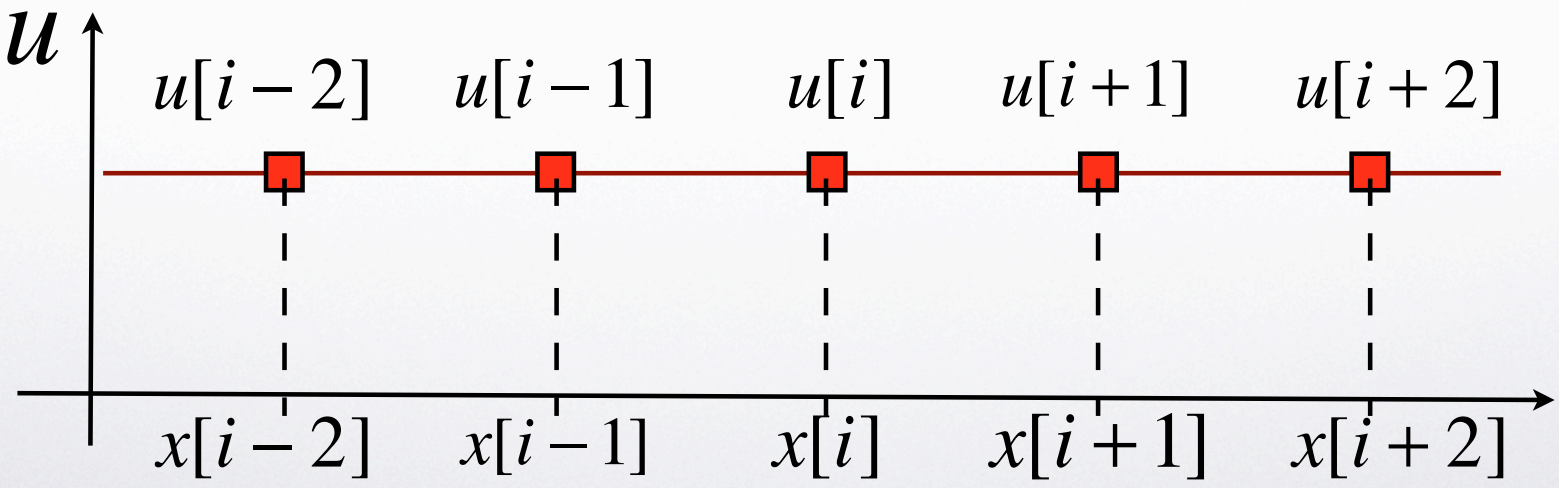
$$\rho_{uu} = \frac{1}{2}\left(\rho[i][j] + \rho[i-1][j]\right)$$

食違い格子の役割

- 数値振動の除去（数値安定性）

１次元での非圧縮性:　$\nabla \cdot \mathbf{u} = \dfrac{\partial u}{\partial x} = 0$

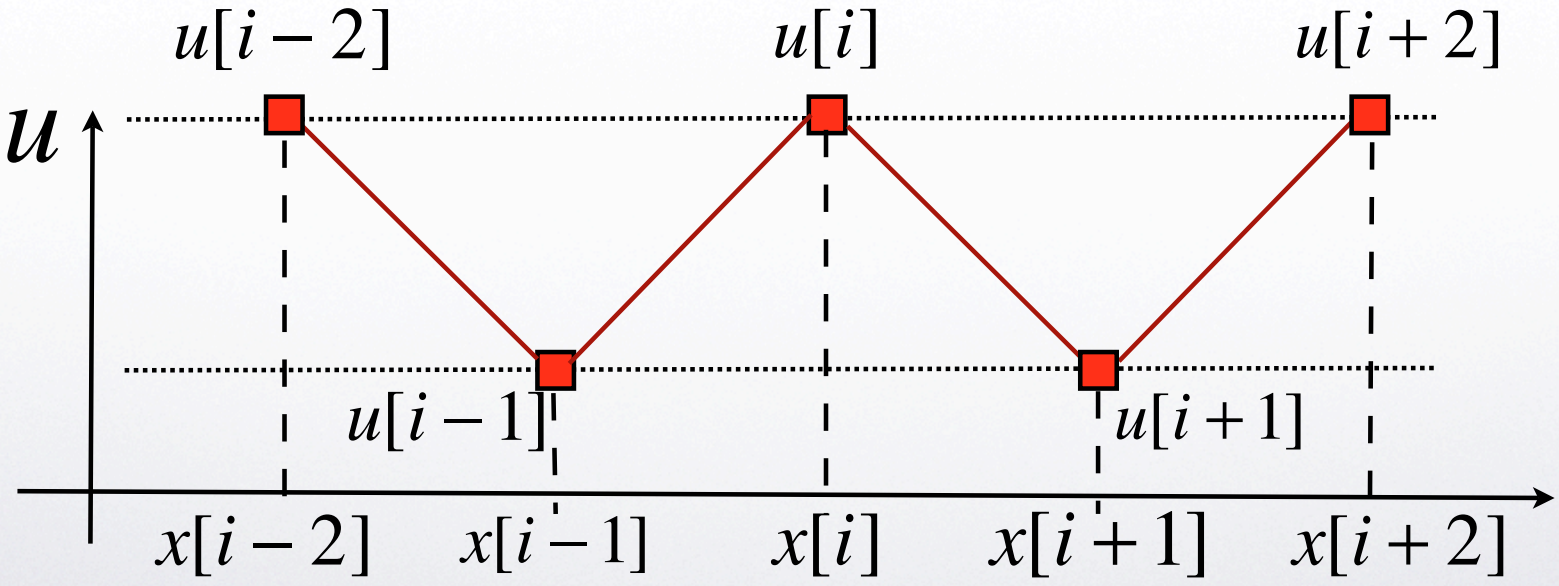
流速uは一定値でなければならない！



食違い格子の役割

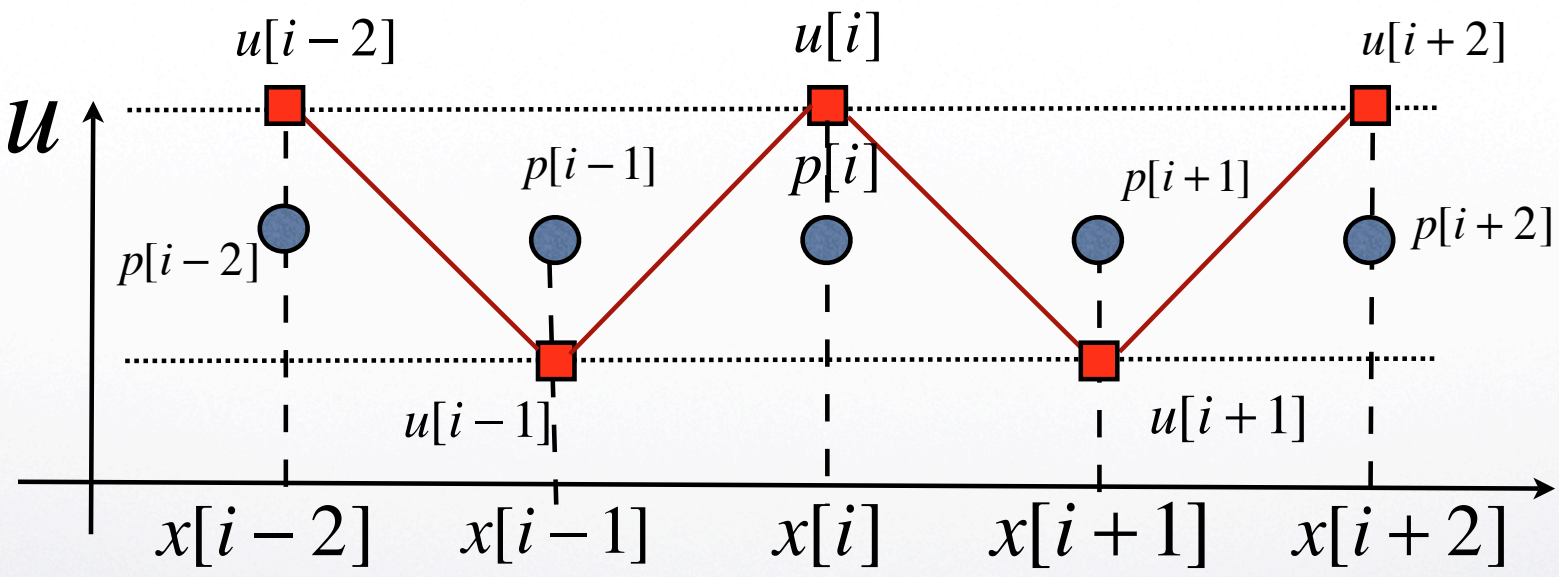
- 実際の計算では様々な要因で数値誤差が生じる

  波長2Δxの振動誤差が生じたとする



食違い格子の役割

- 圧力pがuと同じところに配置されていたとすると集中配置(Collocated Grid;コロケード格子)



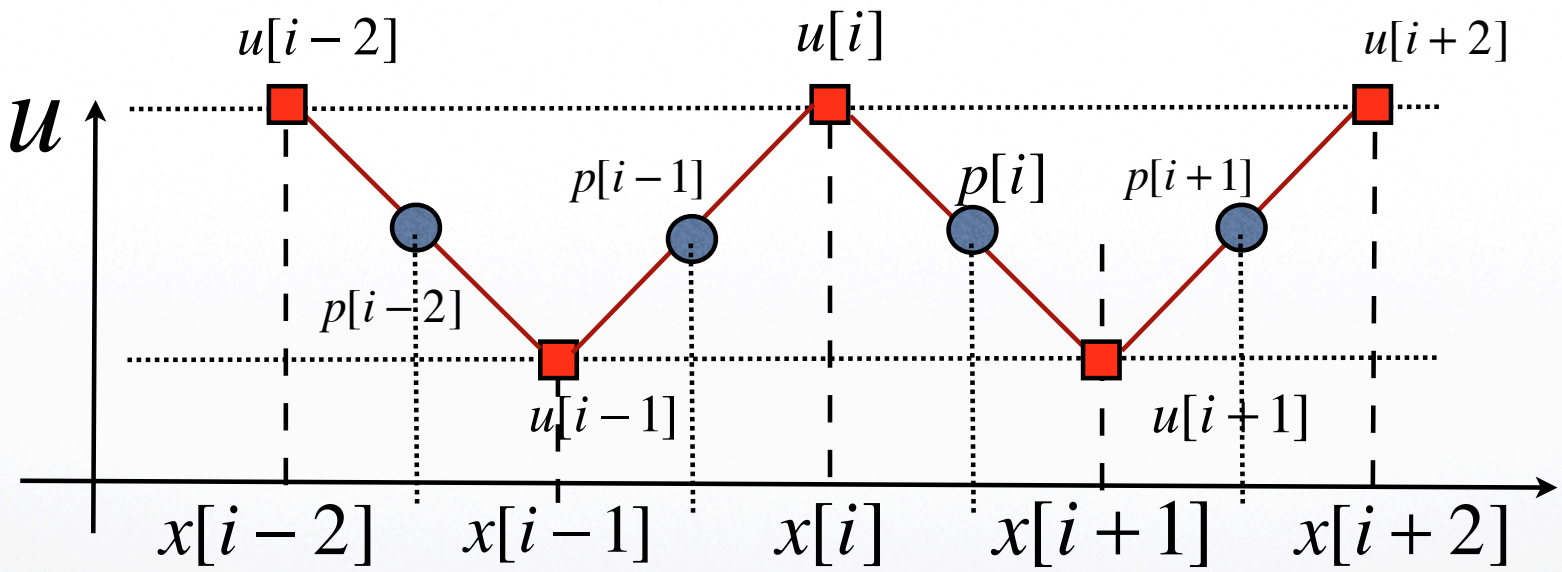
$$\left(\frac{\partial^2 p^{n+1}}{\partial x^2}\right)_i = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right)_i = \frac{\rho}{\Delta t}\left(\frac{u[i+1]-u[i-1]}{2\Delta x}\right) = 0$$

$$\therefore \left(\frac{\partial^2 p^{n+1}}{\partial x^2}\right)_i == 0$$

2Δxの振動を検知できない！

食違い格子の役割

- 食違い格子



$$\left(\frac{\partial^2 p[i+1]}{\partial x^2}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{\Delta t}\left(\frac{u[i+2]-u[i+1]}{\Delta x}\right) > 0 \quad \therefore \left(\frac{\partial^2 p[i]}{\partial x^2}\right) > 0$$

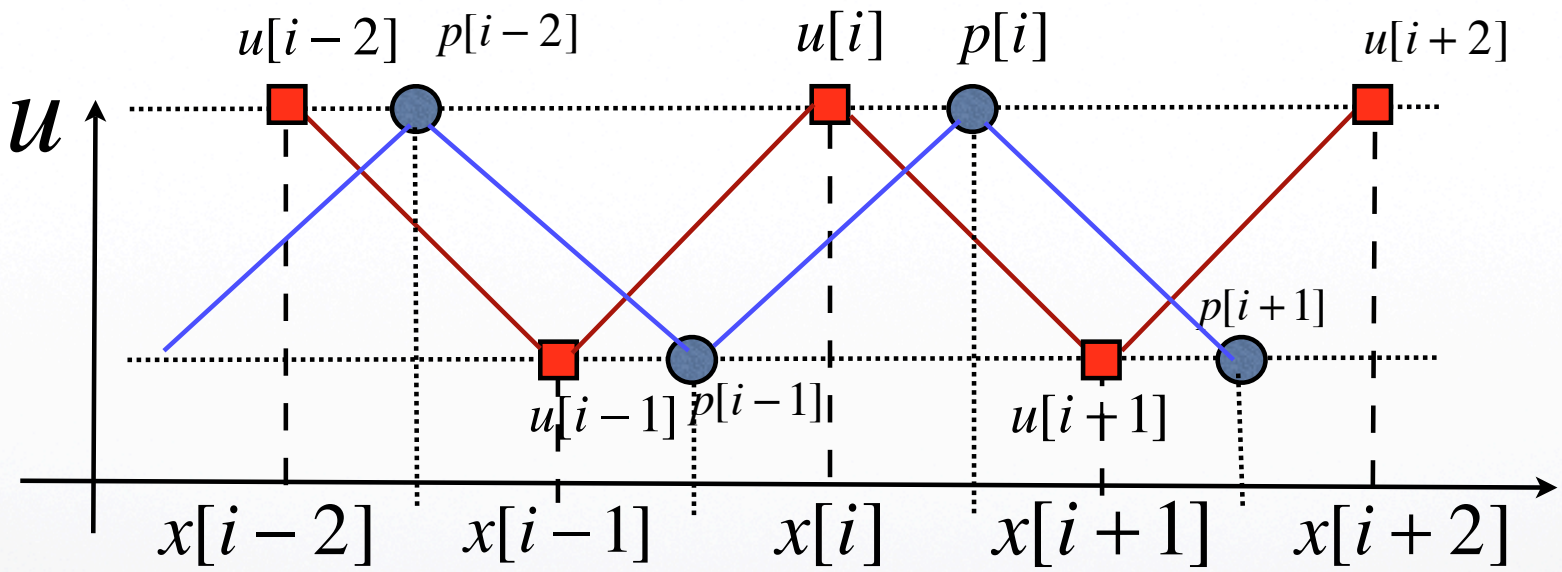
$$\left(\frac{\partial^2 p[i]}{\partial x^2}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{\Delta t}\left(\frac{u[i+1]-u[i]}{\Delta x}\right) < 0 \quad \therefore \left(\frac{\partial^2 p[i]}{\partial x^2}\right) < 0$$

$$\left(\frac{\partial^2 p[i-1]}{\partial x^2}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{\Delta t}\left(\frac{u[i]-u[i-1]}{\Delta x}\right) > 0 \quad \therefore \left(\frac{\partial^2 p[i-1]}{\partial x^2}\right) > 0$$

$$\left(\frac{\partial^2 p[i-2]}{\partial x^2}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{\Delta t}\left(\frac{u[i-1]-u[i-2]}{\Delta x}\right) < 0 \quad \therefore \left(\frac{\partial^2 p[i-2]}{\partial x^2}\right) < 0$$

食違い格子の役割

- 食違い格子



$$\left(\frac{\partial^2 p[i+1]}{\partial x^2}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{\Delta t}\left(\frac{u[i+2]-u[i+1]}{\Delta x}\right) > 0 \quad \therefore \left(\frac{\partial^2 p[i]}{\partial x^2}\right) > 0$$

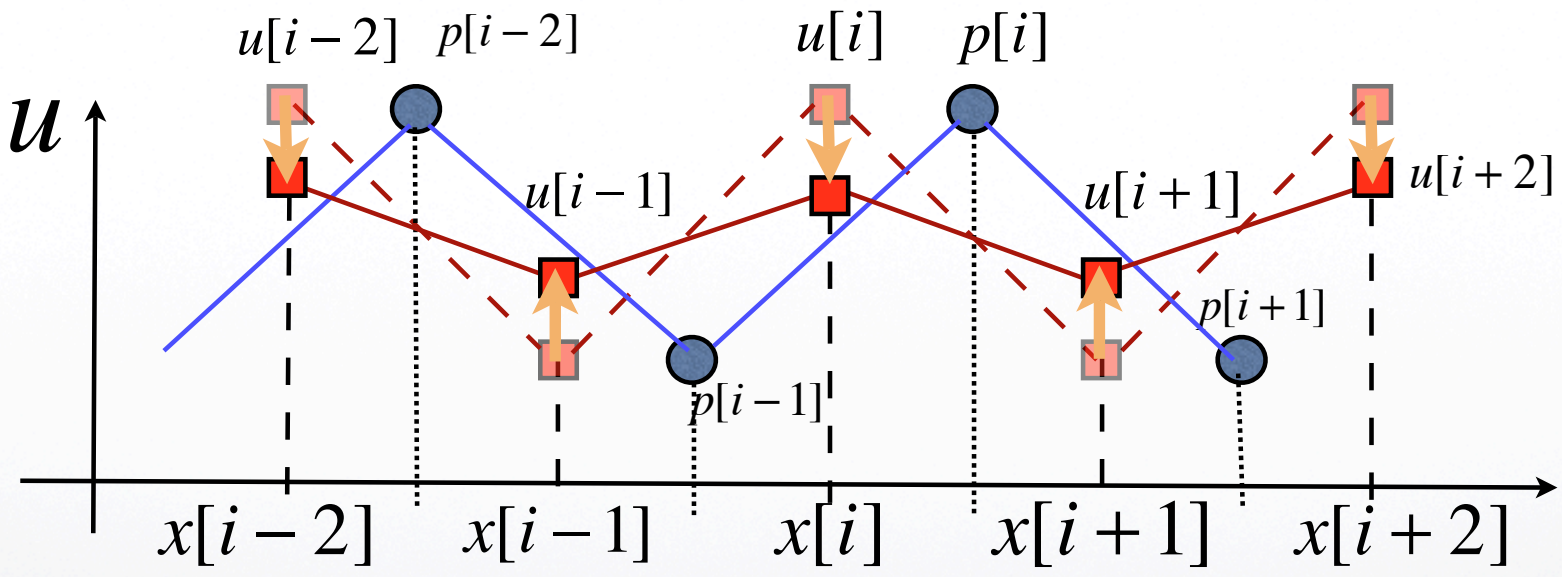
$$\left(\frac{\partial^2 p[i]}{\partial x^2}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{\Delta t}\left(\frac{u[i+1]-u[i]}{\Delta x}\right) < 0 \quad \therefore \left(\frac{\partial^2 p[i]}{\partial x^2}\right) < 0$$

$$\left(\frac{\partial^2 p[i-1]}{\partial x^2}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{\Delta t}\left(\frac{u[i]-u[i-1]}{\Delta x}\right) > 0 \quad \therefore \left(\frac{\partial^2 p[i-1]}{\partial x^2}\right) > 0$$

$$\left(\frac{\partial^2 p[i-2]}{\partial x^2}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{\Delta t}\left(\frac{u[i-1]-u[i-2]}{\Delta x}\right) < 0 \quad \therefore \left(\frac{\partial^2 p[i-2]}{\partial x^2}\right) < 0$$

食違い格子の役割

- 食違い格子



$$\frac{\partial u[i]}{\partial t} = -\frac{1}{\rho}\left(\frac{p[i]-p[i-1]}{\Delta x}\right) < 0$$

$$\frac{\partial u[i-1]}{\partial t} = -\frac{1}{\rho}\left(\frac{p[i-1]-p[i-2]}{\Delta x}\right) > 0$$

2Δxの振動を除去する方向に修正される。