8.3 Contra-directional mode coupling

EM waves in a corrugated (grating) waveguide

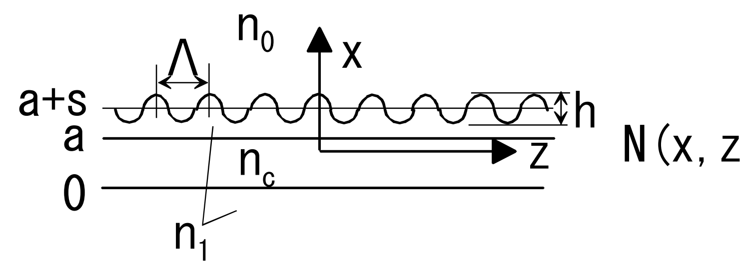
↓

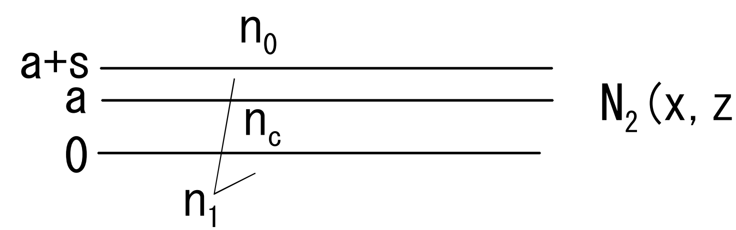
coupling between forward and backward traveling waves

→ same mode,

but opposite propagation direction

$$E_1 = E_2 = E \qquad \beta_1 = -\beta_2 = n_e k_0$$





difference in refractive index

→ periodic perturbation → can be expressed in Fourier series

$$N^2(x,z) - N_2{}^2(x,z) = \begin{cases} 0 & : x < a+s-\frac{h}{2} \\ \sum_{m=-\infty}^{\infty} A_m(x) e^{-j\frac{2\pi}{\Lambda}mz} & : a+s-\frac{h}{2} \le x \le a+s+\frac{h}{2} \\ 0 & : a+s+\frac{h}{2} < x \end{cases}$$

$$A_m(x) = \frac{1}{\Lambda}\int_0^{\Lambda} (N^2(x,z) - N_2{}^2(x,z)) e^{j\frac{2\pi}{\Lambda}mz} dz$$

Coupling in periodic structure : coupling coefficient

$$\kappa_{12}=\frac{\omega\varepsilon_0\iint(\mathrm{N}^2-\mathrm{N}_2{}^2)\mathbf{E}_1{}^*\cdot\mathbf{E}_2dS}{\iint\mathbf{u_z}\cdot(\mathbf{E}_1{}^*\times\mathbf{H}_1+\mathbf{E}_1\times\mathbf{H}_1{}^*)dS}=\frac{\omega\varepsilon_0\int(\sum_m A_m(x)E_1{}^*\cdot E_2)dx\,e^{-j\frac{2\pi}{\Lambda}mz}}{\int_{-\infty}^{\infty}\mathbf{u_z}\cdot(\mathbf{E}_1{}^*\times\mathbf{H}_1+\mathbf{E}_1\times\mathbf{H}_1{}^*)dx}$$

$$=\sum_{m=-\infty}^{\infty}\frac{\omega\varepsilon_0\int A_m(x)E_1{}^*\cdot E_2dx}{\int_{-\infty}^{\infty}\mathbf{u_z}\cdot(\mathbf{E}_1{}^*\times\mathbf{H}_1+\mathbf{E}_1\times\mathbf{H}_1{}^*)dx}e^{-j\frac{2\pi}{\Lambda}mz}=\sum_{m=-\infty}^{\infty}\kappa_{Gm}e^{-j\frac{2\pi}{\Lambda}mz}$$

Example : TE mode in a slab waveguide

$$(\mathbf{E}_1{}^*\times\mathbf{H}_1+\mathbf{E}_1\times\mathbf{H}_1{}^*)\cdot\mathbf{u}_z=\frac{2\beta}{\omega\mu_0}\left|E_y\right|^2$$

$$(\mathbf{E}_2{}^*\times\mathbf{H}_2+\mathbf{E}_2\times\mathbf{H}_2{}^*)\cdot\mathbf{u}_z=-\frac{2\beta}{\omega\mu_0}\left|E_y\right|^2$$

$$\mathbf{E}_1{}^*\cdot\mathbf{E}_2=\mathbf{E}_2{}^*\cdot\mathbf{E}_1=\left|E_y\right|^2$$

$$\kappa_{12}=\frac{\omega\varepsilon_0\iint(N^2-N_2{}^2)\left|E_y\right|^2dS}{\frac{2\beta}{\omega\mu_0}\iint\left|E_y\right|^2dS}=\frac{k_0^2}{2\beta}\frac{\int_{a+s-\frac{h}{2}}^{a+s+\frac{h}{2}}(N^2-N_2{}^2)\left|E_y\right|^2dx}{\int_{-\infty}^{\infty}\left|E_y\right|^2dx}$$

$$=\frac{k_0^2}{2\beta\int_{-\infty}^{\infty}\left|E_y\right|^2dx}\int_{a+s-\frac{h}{2}}^{a+s+\frac{h}{2}}\sum_m A_m(x)e^{-j\frac{2\pi}{\Lambda}mz}\left|E_y\right|^2dx$$

$$=\sum_{m=-\infty}^{\infty}\frac{k_0^2}{2\beta\int_{-\infty}^{\infty}\left|E_y\right|^2dx}\int_{a+s-\frac{h}{2}}^{a+s+\frac{h}{2}}A_m(x)\left|E_y\right|^2dx\times e^{-j\frac{2\pi}{\Lambda}mz}=\sum_{m=-\infty}^{\infty}\kappa_{Gm}e^{-j\frac{2\pi}{\Lambda}mz}$$

Coupling in periodic structure

1-st order coupling (largest coupling coefficient)

$$\kappa_{12} = \kappa_G e^{-j\frac{2\pi}{\Lambda}z}$$

power conservation $\quad \frac{d}{dz}|A|^2 + \frac{d}{d(-z)}|B|^2 = 0$

$$-j(\kappa_{12} + \kappa_{21}^{\ *})A^* B e^{-j2\delta z} + j(\kappa_{12}^{\ *} + \kappa_{21})AB^* e^{j2\delta z} = 0$$

$$\therefore \kappa_{21} = -\kappa_{12}^{\ *} = -\kappa_G e^{j\frac{2\pi}{\Lambda}z}$$

$$\frac{dA}{dz} = (-j\kappa_G e^{-j\frac{2\pi}{\Lambda}z} B e^{-j(\beta_2-\beta_1)z}) = -j\kappa_G B(z) e^{j(\beta_1-\beta_2-\frac{2\pi}{\Lambda})z}$$

$$\frac{dB}{dz} = j\kappa_G A(z) e^{-j(\beta_1-\beta_2-\frac{2\pi}{\Lambda})z}$$

$$\phi = (\beta_1 - \beta_2 - \frac{2\pi}{\Lambda})/2 = \beta - \frac{\pi}{\Lambda}$$

Coupling in periodic structure

A grating is formed in $0 \le z \le L$

Only a forward propagation mode is incident at z=0. $A(0) = A_0 \quad B(L) = 0$

$$\frac{d^2 A}{dz^2} - j2\phi \frac{dA}{dz} - \kappa_G^2 A = 0 \qquad A(z) = e^{\gamma z} \longrightarrow \gamma^2 - j2\phi\gamma - \kappa_G^2 = 0$$

$$\therefore \gamma = j\phi \pm \sqrt{\kappa_G^2 - \phi^2}$$

(1) $|\phi| > \kappa_G \quad \left(\beta > \frac{\pi}{\Lambda} + \kappa_G, \beta < \frac{\pi}{\Lambda} - \kappa_G \right)$

$$\gamma = j(\phi \pm \rho) \quad \text{with} \quad \rho = \sqrt{\phi^2 - \kappa_G^2}$$

$$A(z) = a_1 e^{j(\phi+\rho)z} + a_2 e^{j(\phi-\rho)z}$$

$$B(z) = -\frac{1}{\kappa_G}\left(a_1(\phi+\rho)e^{j(-\phi+\rho)z} + a_2(\phi-\rho)e^{-j(\phi+\rho)z}\right)$$

$$A(z) = A_0 \frac{\rho\cos\rho(z-L) - j\phi\sin\rho(z-L)}{\rho\cos\rho L + j\phi\sin\rho L} e^{j\phi z} \to P_f(z) = \frac{|A(z)|^2}{|A(0)|^2} = \frac{\rho^2 + \kappa_G{}^2 \sin^2\rho(z-L)}{\rho^2 + \kappa_G{}^2\sin^2\rho L}$$

$$B(z) = A_0 \frac{j\kappa_G \sin\rho(z-L)}{\rho\cos\rho L + j\phi\sin\rho L} e^{-j\phi z} \qquad \to P_b(z) = \frac{|B(z)|^2}{|A(0)|^2} = \frac{\kappa_G{}^2\sin^2\rho(z-L)}{\rho^2 + \kappa_G{}^2\sin^2\rho L}$$

reflection at z=0: $R = \dfrac{\kappa_G{}^2\sin^2\rho L}{\rho^2 + \kappa_G{}^2\sin^2\rho L} \qquad \left(\rho = \sqrt{(\beta - \frac{\pi}{\Lambda})^2 - \kappa_G{}^2}\right)$

Coupling in periodic structure

(2) $|\phi| = \kappa_G \quad \left(\beta = \frac{\pi}{\Lambda} \pm \kappa_G \right)$

$\gamma = j\phi$

$$A(z) = a_1 e^{j\phi z} + a_2 z e^{j\phi z}$$

$$B(z) = \frac{e^{-j2\phi z}}{-j\kappa_G}(j\phi a_1 e^{j\phi z} + (1 + j\phi z) a_2 e^{j\phi z})$$

$$A(z) = A_0 \frac{1 - j\phi(z-L)}{1 + j\phi L} e^{j\phi z} \rightarrow P_f(z) = \frac{1 + \phi^2 (z-L)^2}{1 + \phi^2 L^2} = \frac{1 + \kappa_G{}^2 (z-L)^2}{1 + \kappa_G{}^2 L^2}$$

$$B(z) = A_0 \frac{j\kappa_G (z-L)}{1 + j\phi L} e^{-j\phi z} \rightarrow P_b(z) = \frac{\kappa_G{}^2 (z-L)^2}{1 + \kappa_G{}^2 L^2}$$

$$R = \frac{\kappa_G{}^2 L^2}{1 + \kappa_G{}^2 L^2}$$

Coupling in periodic structure (Bragg reflection)

(3) $|\phi| < \kappa_G \quad \left(\frac{\pi}{\Lambda} - \kappa_G < \beta < \frac{\pi}{\Lambda} + \kappa_G \right)$

$$\gamma = j\phi \pm \alpha$$

$$\alpha = \sqrt{\kappa_G^2 - \phi^2}$$

$$A(z) = a_1 e^{(j\phi+\alpha)z} + a_2 e^{(j\phi-\alpha)z} = e^{j\phi z}(a_1 e^{\alpha z} + a_2 e^{-\alpha z})$$

$$B(z) = \frac{e^{-j2\phi z}}{-j\kappa_G}\frac{dA(z)}{dz} = \frac{e^{-j\phi z}}{-j\kappa_G}(a_1(j\phi+\alpha)e^{\alpha z} + a_2(j\phi-\alpha)e^{-\alpha z})$$

$$A(z) = A_0 \frac{\alpha\cosh\alpha(z-L) - j\phi\sinh\alpha(z-L)}{\alpha\cosh\alpha L + j\phi\sinh\alpha L} e^{j\phi z} \quad \to P_f(z) = \frac{\alpha^2 + {\kappa_G}^2 \sinh^2\alpha(z-L)}{\alpha^2 + {\kappa_G}^2\sinh^2\alpha L}$$

$$B(z) = A_0 \frac{j\kappa_G \sinh\alpha(z-L)}{\alpha\cosh\alpha L + j\phi\sinh\alpha L} e^{-j\phi z} \quad \to P_b(z) = \frac{{\kappa_G}^2\sinh^2\alpha(z-L)}{\alpha^2 + {\kappa_G}^2\sinh^2\alpha L}$$

$$R = \frac{{\kappa_G}^2\sinh^2\alpha L}{\alpha^2 + {\kappa_G}^2\sinh^2\alpha L} \quad , \quad \alpha = \sqrt{{\kappa_G}^2 - (\beta - \frac{\pi}{\Lambda})^2}$$

Bragg reflection: $\beta = \frac{\pi}{\Lambda} \quad (\lambda_B = 2\Lambda) \qquad R = \tanh^2\kappa_G L$